# TATE CLASSES AND ENDOSCOPY FOR $\mathrm{GSp}_4$ OVER TOTALLY REAL FIELDS

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Abstract. The theory of endoscopy predicts the existence of large families of Tate classes on certain products of Shimura varieties, and it is natural to ask in what cases one can construct algebraic cycles giving rise to these Tate classes. This paper takes up the case of Tate classes arising from the Yoshida lift: these are Tate cycles in middle degree on the Shimura variety for the group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2 \times \mathrm{GSp}_4)$, where $F$ is a totally real field. A special case is the family of Tate classes which reflect the appearance of two-dimensional Galois representations in the middle cohomology of both a modular curve and a Siegel modular threefold. We show that a natural algebraic cycle generates exactly the Tate classes which are associated to *generic* members of the endoscopic $L$-packets on $\mathrm{GSp}_{4,F}$. In the non-generic case, we give an alternate construction, which shows that the predicted Tate classes arise from Hodge cycles.

## Contents



## 1. Introduction

Let $F$ be a totally real number field of degree $d$, and let $G = \mathrm{GSp}_{4,F}$. The unique elliptic endoscopic group for $G$ is $M = (\mathrm{GL}_2 \times \mathrm{GL}_2 / \mathbb{G}_m)_F$, where $\mathbb{G}_m$ is embedded anti-diagonally and the $L$-embedding is induced by

$$\widehat{M} = \mathrm{GL}_2(\mathbb{C}) \times_{\mathbb{C}^\times} \mathrm{GL}_2(\mathbb{C}) \hookrightarrow \mathrm{GSp}_4(\mathbb{C}) = \widehat{G}. \tag{1}$$

The functorial transfer of cuspidal automorphic forms from $M$ to $G$ has been studied by Roberts [31] and Weissauer [38]. For any (unordered) pair of distinct cuspidal automorphic representations $\pi_1, \pi_2$ of $\mathrm{GL}_2(\mathbb{A}_F)$ with the same central character, one obtains an $L$-packet $\Pi(\pi_1, \pi_2)$ of cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$. The members $\Pi_S(\pi_1, \pi_2)$ of this $L$-packet are indexed by finite sets $S$ of places of $F$ at which both $\pi_i$ are discrete series, such that $|S|$ is even. The unique generic member of the $L$-packet $\Pi(\pi_1, \pi_2)$ is $\Pi_\emptyset(\pi_1, \pi_2)$.

Let $\mathbf{GSp}_4 = \mathrm{Res}_{F/\mathbb{Q}} G$ be the restriction of scalars, with the natural Shimura datum, and let

$$S(\mathbf{GSp}_4) := \varprojlim_K S_K(\mathbf{GSp}_4)$$

be the resulting pro-algebraic Shimura variety over $\mathbb{Q}$, where $K$ ranges over compact open subgroups of $\mathbf{GSp}_4(\mathbb{A}_f)$. (For the rest of the introduction, the same notation will apply when $\mathrm{GSp}_4$ is replaced by any $\mathbb{Q}$-group $H$ with a Shimura datum.) If $\pi_1$ and $\pi_2$ correspond to Hilbert modular forms of sufficiently regular

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weights, then the representations $\Pi_S(\pi_1, \pi_2)$ contribute to the interior cohomology of $S(\mathbf{GSp}_4)$ in middle degree $3d$.

As we will recall later in the introduction, the $\Pi_S(\pi_1, \pi_2)_f^\vee \boxtimes \pi_{i,f}$-isotypic part of the étale cohomology of $S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)$ contains Galois-invariant classes, where $i = 1$ or $2$ depending on $S_f$, the set of finite places in $S$. The goal of this paper is to investigate whether these classes have a geometric origin, as suggested by the Tate conjecture in the case of trivial coefficients.

The natural candidate is the sub-Shimura variety $S(\boldsymbol{H}) \subset S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2)$, where:

$$H \coloneqq \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \xhookrightarrow{\iota, p} \mathrm{GSp}_4 \times \mathrm{GL}_2 \,. \tag{2}$$

Here $\iota : H \hookrightarrow \mathrm{GSp}_4$ is the standard inclusion and $p : H \to \mathrm{GL}_2$ is the first projection.

**Theorem A** (Theorem 7.2.5)**.** *Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to Hilbert modular forms of weights $\boldsymbol{m}_1 = (m_{1,v})_{v|\infty}$ and $\boldsymbol{m}_2 = (m_{2,v})_{v|\infty}$ with $m_{1,v} \geq m_{2,v} + 2 \geq 4$ for all $v$ and all $m_{i,v}$ of the same parity, and suppose $\pi_1$ and $\pi_2$ have equal central character. Then the $\Pi_S(\pi_1, \pi_2)_f^\vee \boxtimes \pi_{2,f}$-isotypic component of*

$$[S(\boldsymbol{H})] \in H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{V}^\vee_{(\boldsymbol{\ell}_1 - 3, \boldsymbol{\ell}_2 - 3), E} \boxtimes \mathcal{V}_{\boldsymbol{m}_2 - 2, E}) \quad \textit{(see below)}$$

*is nontrivial if and only if $S_f = \emptyset$.*

We now explain the conditions on the weights in Theorem A and the meaning of the class $[S(\boldsymbol{H})]$. The condition for $\Pi_S(\pi_1, \pi_2)$ to appear cohomology is that $m_{1,v}$, $m_{2,v}$, and $|m_{1,v} - m_{2,v}|$ are all at least 2 for each $v|\infty$, and all $m_{i,v}$ have the same parity; in this case the local component at $v$ of $\Pi_S(\pi_1, \pi_2)$ belongs to a discrete series $L$-packet of weight $(\ell_{1,v}, \ell_{2,v})$ for $\mathrm{GSp}_4(F_v)$, where

$$\ell_{1,v} = \frac{m_{1,v} + m_{2,v}}{2}, \;\; \ell_{2,v} = \frac{|m_{1,v} - m_{2,v}| + 4}{2}.$$

If $E$ is a coefficient field for $\pi_1$, $\pi_2$, and $\Pi_S(\pi_1, \pi_2)$ containing the field $\mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)$ of (5.3.1), then the weights $(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = (\ell_{1,v}, \ell_{2,v})_{v|\infty}$ and $\boldsymbol{m}_i$ determine local systems of $E$-vector spaces $\mathcal{V}_{(\boldsymbol{\ell}_1 - 3, \boldsymbol{\ell}_2 - 3), E}$ and $\mathcal{V}_{\boldsymbol{m}_i - 2, E}$ on $S(\mathbf{GSp}_4)$ and $S(\mathbf{GL}_2)$, respectively; the notation is explained in (3.6.3). Note that $\mathcal{V}^\vee_{(\boldsymbol{\ell}_1 - 3, \boldsymbol{\ell}_2 - 3), E} \boxtimes \mathcal{V}_{\boldsymbol{m}_2 - 2, E}$ is trivial if and only if $\pi_1$ and $\pi_2$ have parallel weights 4 and 2, respectively, in which case $[S(\boldsymbol{H})]$ is just the algebraic cycle class.

More generally, the pullback $(\iota, p)^* \mathcal{V}^\vee_{(\boldsymbol{\ell}_1 - 3, \boldsymbol{\ell}_2 - 3), E} \boxtimes \mathcal{V}_{\boldsymbol{m}_2 - 2, E}$ contains the constant local system on $S(\boldsymbol{H})$ (with multiplicity one) if and only if $m_{1,v} > m_{2,v}$ for all $v$, in which case the cycle class $[S(\boldsymbol{H})]$ appearing in Theorem A is defined using the adjunction map $(\iota, p)_* \underline{E}_{S(\boldsymbol{H})} \to \mathcal{V}^\vee_{(\boldsymbol{\ell}_1 - 3, \boldsymbol{\ell}_2 - 3), E} \boxtimes \mathcal{V}_{\boldsymbol{m}_2 - 2, E}$. So the conditions on $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ in Theorem A are the minimum required to formulate the statement.

Under these conditions, Galois-invariant classes appear in the $\Pi_S(\pi_1, \pi_2)_f^\vee \boxtimes \pi_{2,f}$-isotypic part of étale cohomology if and only if $|S_f|$ is even. However, Theorem A asserts that only in the case $S_f = \emptyset$ (corresponding to the unique generic member of the $L$-packet $\Pi(\pi_1, \pi_2)$) do these classes arise from the special cycle $S(\boldsymbol{H})$. For the case $S_f \neq \emptyset$, we are not able to produce any nontrivial algebraic cycle classes. However, we give an alternative construction that shows the Galois-invariant classes arise from Hodge cycles.

**Theorem B** (Theorem 10.2.4)**.** *Let $\pi_1$ and $\pi_2$ be as in Theorem A, and let $S$ be a set of places of $F$ at which both $\pi_i$ are discrete series, such that $|S_f| \geq 2$ is even. Then there exists a Hodge class*

$$0 \neq \xi \in H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{\ell}_1 - 3, \boldsymbol{\ell}_2 - 3), E} \boxtimes \mathcal{V}_{\boldsymbol{m}_2 - 2, E}(2d))[\Pi_S(\pi_1, \pi_2)_f^\vee \boxtimes \pi_{2,f}]$$

*such that, for all finite places $\lambda$ of $E$, the image of $\xi$ in $\lambda$-adic étale cohomology is $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$-invariant.*

In fact, Theorem 10.2.4 in the text produces a cohomology class defined over the subfield $\mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2) \subset E$; for instance, in the case of trivial coefficients, the $\xi$ in Theorem B is the $\Pi_S(\pi_1, \pi_2)_f^\vee \boxtimes \pi_{2,f}$-isotypic component of a Hodge-Tate class defined over $\mathbb{Q}$.

**Expected Galois representations.** For the reader's convenience, we recall the expectations of Kottwitz's conjectures [19] for the Galois representations in étale cohomology associated to $\pi_1$, $\pi_2$, and $\Pi_S(\pi_1, \pi_2)$. Suppose the weights of $\pi_1$ and $\pi_2$ are $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, where $m_{1,v}$, $m_{2,v}$, and $|m_{1,v} - m_{2,v}|$ are all at least 2 for all $v|\infty$ and all $m_{i,v}$ have the same parity. We normalize the $\pi_i$ so that their common central character $\omega$ has the infinity type $\omega_{\boldsymbol{m}_i}$ in the notation of (2.1.4). Let $\lambda$ be a finite places of the coefficient field $E$, and set $\rho_\Pi = \rho_{\pi_1} \oplus \rho_{\pi_2}$, where $\rho_{\pi_i}$ are the usual $\lambda$-adic Galois representations associated to Hilbert modular forms, normalized to have determinant $\chi^{-1}\omega$ with $\chi$ the cyclotomic character. Then let $(\widetilde{\rho}_\Pi, V)$ and $(\widetilde{\rho}_i, V_i)$ be the tensor induction of $\rho_\Pi$ and $\rho_{\pi_i}$, respectively, from $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We have natural inclusions $V_i \hookrightarrow V$.

Consider the involution $s \in \mathrm{End}(\widetilde{\rho}_\Pi, V)$ such that, in the $v$th factor of the decomposition

$$V = \otimes_{v|\infty} \rho_\Pi, \tag{3}$$

$s$ acts as $-1$ on $\rho_{\pi_1}$ and 1 on $\rho_{\pi_2}$ if $m_{1,v} > m_{2,v}$, and vice versa if $m_{1,v} < m_{2,v}$. Taking $s$-eigenspaces induces a decomposition $V = V^+ \oplus V^-$. The predicted contributions to étale cohomology are:

$$\begin{gathered}\mathrm{Hom}\left(\Pi_S(\pi_1,\pi_2)_f, H^{3d}_{\text{ét},!}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),E_\lambda})\right) = \begin{cases} V^+(-d), & |S_f| \text{ even},\\ V^-(-d), & |S_f| \text{ odd}.\end{cases}\\ \mathrm{Hom}\left(\pi_{i,f}, H^{d}_{\text{ét},!}(S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{V}_{\boldsymbol{m}_i-2,E_\lambda})\right) = V_i.\end{gathered} \tag{4}$$

If $m_{1,v} > m_{2,v}$ for all $v$, then we have $V_2 \subset V^+$, hence there should exist nontrivial maps of Galois representations

$$H^{3d}_{\text{ét},!}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),E_\lambda}(d))[\Pi_S(\pi_1,\pi_2)_f] \to H^{d}_{\text{ét},!}(S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{V}_{\boldsymbol{m}_2-2,E_\lambda})[\pi_{2,f}] \tag{5}$$

whenever $|S_f|$ is even. Theorems A and B yield, by Poincaré duality, a geometric construction of nontrivial maps (5). In the text (Theorems 7.2.5 and 10.2.4), we actually show that the maps we construct are non-degenerate in the sense that their images generate the $\mathrm{GL}_2(\mathbb{A}_{F,f})$-module $H^{d}_{\text{ét},!}(S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{V}_{\boldsymbol{m}_2-2,E_\lambda})[\pi_{2,f}]$; this is equivalent to nontriviality only if $V_2$ is irreducible.

One could also ask for an analogue of Theorem B that uses $\pi_1$, the higher-weight representation, in the place of $\pi_2$, or relaxes the condition that $m_{1,v} > m_{2,v}$ for all places $v$. Our construction does not appear to yield any results in this direction.

**Comparison with previous work.** In the case when $F = \mathbb{Q}$, $S = \emptyset$, and $\pi_1$ and $\pi_2$ correspond to classical modular forms of weights 4 and 2, Theorem A was proven by Lemma [23], using a different method.

In the setting of Jacquet-Langlands transfers for cohomological representations of inner forms of $\mathrm{GL}_{2,F}$, an analogue of Theorem B was proven by Ichino and Prasanna [14]. For the transfer between quaternion algebras $B_1$ and $B_2$ which are split at exactly one archimedean place, the Shimura varieties associated to $B_1^\times$ and $B_2^\times$ are curves. The resulting Tate classes are known to arise from cycles by Faltings's isogeny theorem [3], but no more explicit construction of these algebraic cycles is known. When the relevant Shimura varieties have higher dimension, Ichino and Prasanna showed that the Jacquet-Langlands transfers (for general cohomological weights) are induced by Hodge cycles. Their construction is similar to the one used to prove Theorem B. However, in the Jacquet-Langlands setting there is no natural algebraic cycle such as $S(\boldsymbol{H})$, so there is no analogue of Theorem A.

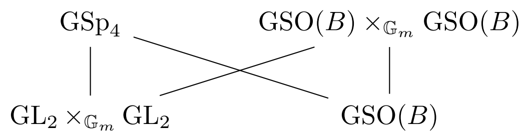
In an earlier version of this paper, Theorem B was stated conditionally on Arthur's conjectures; the result is now unconditional.

**Overview of the proofs.** Both Theorem A and Theorem B rely on the explicit realization of $\Pi_S(\pi_1,\pi_2)$ as a theta lift from a four-dimensional orthogonal group, cf. [31, 38]. Indeed, if $|S|$ is even, then there is a quaternion algebra $B$ over $F$ ramified exactly at the places in $S$, and the orthogonal group $\mathrm{GSO}(B) \simeq B^\times \times B^\times/\mathbb{G}_m$ is an inner form of $M$. The automorphic representation $\Pi_S(\pi_1,\pi_2)$ is the theta lift of $\pi_1^B \boxtimes \pi_2^B$ from $\mathrm{GSO}(B)$ to $\mathrm{GSp}_{4,F}$, where $\pi_i^B$ is the Jacquet-Langlands transfer of $\pi_i$ to $B^\times$. This is crucial because it allows for the calculation of period integrals involving $\Pi_S(\pi_1,\pi_2)$.

*Proof of Theorem A.* Since the non-vanishing of $[S(\boldsymbol{H})]$ may be detected in de Rham cohomology, the theorem is essentially a statement about periods of $\Pi_S(\pi_1,\pi_2) \boxtimes \pi_2^\vee$ along the subgroup $H \subset \mathrm{GSp}_4 \times \mathrm{GL}_2$. That is, we must compute integrals of the form

$$\mathcal{P}_S(\gamma,\beta) := \int_{Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \gamma(\iota(h))\beta(p(h))\,\mathrm{d}h, \;\; \gamma \in \Pi_S(\pi_1,\pi_2),\; \beta \in \pi. \tag{6}$$

Because $\Pi_S(\pi_1,\pi_2)$ is a theta lift from $\mathrm{GSO}(B)$, we can compute (6) using the seesaw diagram:




Here the vertical lines are inclusions and the diagonals are dual reductive pairs inside $\mathrm{GSp}_{16}$. Formally, the seesaw identity would read:

$$\mathcal{P}_S(\theta(\alpha), \beta) = \int_{[\mathrm{PGSO}(B)]} \theta(\beta)(g)\theta(\mathbb{1})(g)\alpha(g)\,\mathrm{d}g, \quad \alpha \in \pi_1 \otimes \pi_2, \quad \beta \in \pi_2^\vee, \tag{7}$$

where the theta lifts on the right are from $\mathrm{GL}_2$ to $\mathrm{GSO}(B)$, and the theta lifts on both sides depend on choices of Schwartz functions which must be made compatibly. The integral defining $\theta(\mathbb{1})$ is divergent, so a regularization step is necessary to interpret (7). However, after regularization, $\theta(\mathbb{1})$ can be recognized as 0 if $B$ is not split (i.e. if $S \neq \emptyset$), and as a certain Eisenstein series on $\mathrm{GSO}(B)$ if $B$ is split. The integral (7) then unfolds to an Euler product which allows us to evaluate it explicitly. The result of the calculation is:

**Theorem C** (Theorems 6.2.2, 6.5.2)**.** *Let $\pi_1$, $\pi_2$, and $\pi$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\pi_i$ and $\pi^\vee$ have the same central character, and let $S$ be a finite set of (possibly archimedean) places of $F$ at which both $\pi_i$ are discrete series, such that $|S|$ is even. Consider the period pairing*

$$\mathcal{P}_S(\gamma, \beta) := \int_{Z_H(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} \gamma(\iota(h))\beta(p(h))\,\mathrm{d}h, \quad \gamma \in \Pi_S(\pi_1, \pi_2), \; \beta \in \pi, \tag{8}$$

*where* $\mathrm{d}h$ *is normalized as in (6.1.1).*

*(1) If $\mathcal{P}_S(\gamma, \beta) \neq 0$, then $S = \emptyset$, i.e. $\Pi_S(\pi_1, \pi_2)$ is generic, and $\pi$ is isomorphic to either $\pi_1^\vee$ or $\pi_2^\vee$.*

*(2) Suppose given factorizable Schwartz functions*

$$\phi_i = \otimes_v \phi_{i,v} \in \mathcal{S}(M_2(\mathbb{A}_F)), \quad i = 1, 2$$

*and factorizable vectors*

$$\alpha = \otimes_v \alpha_v \in \pi_1 \otimes \pi_2, \quad \beta = \otimes_v \beta_v \in \pi_2^\vee.$$

*Then the theta lift $\theta_{\phi_1 \otimes \phi_2}(\alpha)$ lies in $\Pi_\emptyset(\pi_1, \pi_2)$ and, for a sufficiently large finite set $S$ of places of $F$,*

$$\mathcal{P}_\emptyset(\theta_{\phi_1\otimes\phi_2}(\alpha), \beta) = 2|D_F|^{1/2} \cdot \pi^{-d} \frac{L^S(1, \pi_1 \times \pi_2^\vee) L^S(1, \mathrm{Ad}\,\pi_2)}{\zeta_F^S(2)} \prod_{v \in S} \frac{\mathcal{Z}_v(\phi_{1,v}, \phi_{2,v}, \alpha_v, \beta_v)}{1 - q_v^{-1}}.$$

*Here $\mathcal{Z}_v(\phi_{1,v}, \phi_{2,v}, \alpha_v, \beta_v)$ is an explicit local zeta integral which is nonzero for appropriate choices of test data; $\phi_1 \otimes \phi_2$ is the tensor product Schwartz function in $\mathcal{S}(M_2(\mathbb{A}_F)^2)$; the theta lift $\theta_{\phi_1\otimes\phi_2}(\alpha)$ is defined in §4; and the other notations are introduced in (2.1.1).*

**Remark.** The $L$-values appearing in Theorem C are nonzero by the classical result of Shahidi [33].

In fact, Theorem C amounts to a special case of the nontempered Gan-Gross-Prasad conjectures in [6]: if $\pi_1$ and $\pi_2$ have trivial central character, then $\Pi_S(\pi_1, \pi_2)$ descends to $\mathrm{PGSp}_4 = \mathrm{SO}_5$, and the period (6) reduces to a period for the split GGP pair $\mathrm{SO}_4 \subset \mathrm{SO}_5$. Although $\Pi_S(\pi_1, \pi_2)$ is tempered, the automorphic representation of $\mathrm{SO}_4$ corresponding to the forms $\beta(p(h))$ on $H$ is not, and so this period falls outside the scope of the usual GGP conjecture.

To deduce Theorem A from Theorem C, one additional ingredient is needed. In the period integrals (6), one really wants to consider only vectors $\gamma$ and $\beta$ that contribute to cohomology, which in our case is equivalent to generating a certain $K$-type at archimedean places. The most delicate part is to write such a vector $\gamma$ as a theta lift $\theta_\phi(\alpha)$, which requires a careful choice of archimedean component for the Schwartz function $\phi$ (determined using local Howe duality). Once we know which $\phi$ to consider, we can evaluate the relevant archimedean zeta integrals to show that the periods (6) are nontrivial.

*Proof of Theorem B.* For simplicity, assume $\pi_1$ and $\pi_2$ have parallel weights 4 and 2, respectively, so all coefficients are trivial. The main difficulty in the proof of Theorem B is to find a nontrivial family of Hodge classes on $S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2)$ (besides the ones coming from the algebraic cycle $S(\boldsymbol{H})$). Once we have a good supply of Hodge classes, the proof that they are nontrivial uses similar methods to the proof of Theorem A. To construct this family of Hodge classes, we use certain nontempered, cohomological automorphic representations of $\mathrm{GSp}_6(\mathbb{A}_F)$ which contribute to cohomology of $S(\mathbf{GSp}_6)$ in degree $4d$.

More precisely, let $S = S_f \sqcup S_\infty$ with $|S_f|$ even, and let $B$ be the quaternion algebra over $F$ which is ramified exactly at $S_f$. Assume $S_f \neq \emptyset$, i.e. $B$ is nonsplit. Then for any auxiliary automorphic representation $\pi$ of $PB(\mathbb{A}_F)^\times$ of parallel weight 6, we consider $\Theta(\pi\boxtimes\mathbb{1})$, the theta lift from $\mathrm{GSO}(B)$ to $\mathrm{GSp}_6$ of the automorphic representation $\pi\boxtimes\mathbb{1}$ of $\mathrm{GSO}(B)\simeq B^\times\times B^\times/\mathbb{G}_m$. Let $S(\pi)$ be the set of automorphic representations of $\mathrm{GSp}_6(\mathbb{A}_F)$ which are nearly equivalent to a constituent of $\Theta(\pi\boxtimes\mathbb{1})$. To connect these automorphic representations to the cohomology of $S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2)$, we use the correspondence of Shimura varieties

$$S(\mathbf{GSp}_6) \xleftarrow{\iota_1} S(\widetilde{\boldsymbol{H}}) \xrightarrow{\iota_2} S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),$$

where $\widetilde{H} \coloneqq \mathrm{GSp}_4\times_{\mathbb{G}_m}\mathrm{GL}_2 \subset \mathrm{GSp}_6$, and $\iota_2$ is open and closed. Thus we obtain a well-defined map

$$\begin{aligned} H^*(S(\mathbf{GSp}_6),E) &\xrightarrow{\iota_{2,*}\circ\iota_1^*} H^*(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),E) \\ &\twoheadrightarrow H^*(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),E)[\Pi_S(\pi_1,\pi_2)_f^\vee\boxtimes\pi_{2,f}]. \end{aligned} \tag{9}$$

For any $\widetilde{\Pi}\in S(\pi)$, let

$$\mathrm{Hdg}(\widetilde{\Pi}) = \mathrm{im}\left(H^{4d}(S(\mathbf{GSp}_6),\overline{\mathbb{Q}})[\widetilde{\Pi}_f] \xrightarrow{(9)} H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GSp}_2),\overline{\mathbb{Q}})[\Pi_S(\pi_1,\pi_2)_f^\vee\boxtimes\pi_{2,f}]\right),$$

$$\mathrm{Hdg}(\pi)_{\overline{\mathbb{Q}}} = \sum_{\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})}\sum_{\widetilde{\Pi}\in S(\pi^\sigma)}\mathrm{Hdg}(\widetilde{\Pi}).$$

Since $\mathrm{Hdg}(\pi)_{\overline{\mathbb{Q}}}$ is stable under the action of $\mathrm{Aut}(\overline{\mathbb{Q}}/E)$ on the coefficients, we can descend to a subspace

$$\mathrm{Hdg}(\pi)\subset H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GSp}_2),E)[\Pi_S(\pi_1,\pi_2)_f^\vee\boxtimes\pi_{2,f}].$$

We prove that:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F^c) \text{ acts trivially on } \mathrm{Hdg}(\pi)(2d)\otimes_E\overline{\mathbb{Q}}_\ell \subset H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}\times S(\mathbf{GSp}_2)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell(2d)) \tag{10}$$

(where the inclusion comes from the Betti-étale comparison theorem) and

$$\mathrm{Hdg}(\pi) \text{ is a trivial sub-Hodge-structure of } H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GSp}_2),E). \tag{11}$$

In fact, (11) follows from (10) by the étale-de Rham comparison − proved for automorphic local systems on general Shimura varieties in [2] − and its compatibility with the map (9). To prove (10), it suffices to consider

$$\mathrm{Hdg}(\widetilde{\Pi})\otimes_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}_\ell\subset H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}\times S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell)[\Pi_S(\pi_1,\pi_2)_f^\vee\boxtimes\pi_{2,f}],$$

for any $\pi$ as above, any $\widetilde{\Pi}\in S(\pi)$, and any embedding $\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_\ell$. Suppose that $p\neq\ell$ splits completely in $F$ and that $\widetilde{\Pi}_v$ is spherical for all $v|p$. Then the generalized Eichler-Shimura relation proven by Lee [21, 22] provides a polynomial $P(X)$ such that $P(\mathrm{Frob}_p)=0$ on $H^*(S(\mathbf{GSp}_6),\overline{\mathbb{Q}}_\ell)[\widetilde{\Pi}_f]$. The coefficients of $P(X)$ depend on the Satake parameters of $\widetilde{\Pi}_v$ for $v|p$, which in turn are determined by those of $\pi_v$ via the spherical theta correspondence for orthogonal-symplectic similitude pairs (Proposition 4.3.3). It turns out that $P(X)$ has a unique root of weight $4d$, which is $p^{-2d}$. On the other hand, $H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}\times S(\mathbf{GSp}_2)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell)[\Pi_S(\pi_1,\pi_2)_f^\vee\boxtimes\pi_{2,f}]$ is pure of weight $4d$, because $\Pi_S(\pi_1,\pi_2)$ and $\pi_2$ are both tempered. (By contrast, the cohomology $H^{4d}_{\text{ét}}(S(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell)[\widetilde{\Pi}_f]$ need not be pure, because $\widetilde{\Pi}_f$ can be non-tempered and even non-cuspidal.) In particular, the purity of $H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}}\times S(\mathbf{GSp}_2)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell)[\Pi_S(\pi_1,\pi_2)_f^\vee\boxtimes\pi_{2,f}]$ implies that $\mathrm{Frob}_p = p^{-2d}$ on $\mathrm{Hdg}(\widetilde{\Pi})\otimes_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}_\ell$ for all $p$ as above, which shows (10) by the Chebotarev density theorem.

It remains to show that some element $\xi\in\mathrm{Hdg}(\pi)$ induces a nonzero map as claimed in Theorem B. Similarly to the proof of Theorem A, we reduce this question to showing that the triple product period integral

$$\int_{[Z_{\widetilde{H}}\backslash\widetilde{H}]}\theta(\alpha)(h,h')\beta(h)\gamma(h')\,\mathrm{d}(h,h'),\ \ \alpha\in\pi\boxtimes\mathbb{1},\ \beta\in\Pi_S(\pi_1,\pi_2),\ \gamma\in\pi_2^\vee \tag{12}$$

is nonzero for some choice of $\pi$ and some choice of test vectors $\alpha$, $\beta$, and $\gamma$. Here $\widetilde{H}$ is parametrized by pairs $(h, h') \in \mathrm{GSp}_4 \times \mathrm{GL}_2$, and the theta lift, which again depends on a choice of Schwartz function, is from $\mathrm{GSO}(B)$ to $\mathrm{GSp}_6$. The relevant seesaw diagram for this period is:

$$\begin{array}{ccc} \mathrm{GSp}_6 & & \mathrm{GSO}(B) \times_{\mathbb{G}_m} \mathrm{GSO}(B) \\ | & \times & | \\ \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2 & & \mathrm{GSO}(B) \end{array}$$

The seesaw identity reduces (12) to

$$\int_{[\mathrm{PGSO}(B)]} \alpha(g)\theta(\beta)(g)\theta(\gamma)(g)\,\mathrm{d}g, \tag{13}$$

where the theta lifts are now from $\mathrm{GSp}_4$ and $\mathrm{GL}_2$ to $\mathrm{GSO}(B)$. (Under the assumption that $B$ is nonsplit, all the integrals involved in the seesaw identity converge absolutely.) The theta lift $\theta(\gamma)$ runs over $(\pi_2^B)^\vee \boxtimes (\pi_2^B)^\vee$ as $\gamma$ varies, and the image of the theta lift $\theta(\beta)$ includes $\pi_1^B \boxtimes \pi_2^B$ as $\beta$ varies. We choose $\alpha$ to be a Hilbert modular eigenform on $PB^\times(\mathbb{A}_F)$ such that $\langle f_1^B \cdot f_2^B, \alpha\rangle_{\mathrm{Pet}} \neq 0$, where $f_1^B \in \pi_1^B$ and $f_2^B \in (\pi_2^B)^\vee$ are holomorphic newforms, and let $\pi$ be the automorphic representation generated by $\alpha$. Having made this choice of $\pi$ and $\alpha$, it follows that (13) is nonzero for appropriate choices of $\beta$ and $\gamma$.

**Arithmetic implications.** This work was originally motivated by a question of Weissauer in [37], which can be paraphrased as follows: if $F = \mathbb{Q}$ and $\pi_2$ is the automorphic representation associated to an elliptic curve $E/\mathbb{Q}$, then the motive associated to $E$ appears attached to members of the $L$-packet $\Pi(\pi_1, \pi_2)$ in the cohomology of $S(\mathrm{GSp}_4)$. Can we then use Shimura curves on $S(\mathrm{GSp}_4)$ to construct interesting Selmer classes for $E$ in the spirit of Heegner points? Theorem A implies that, when applied to quaternionic Shimura curves and a generic representation $\Pi_\emptyset(\pi_1, \pi_2)$, this construction would recover the Heegner points on $E$. Indeed, all appearances of the motive of $E$ attached to generic representations $\Pi_\emptyset(\pi_1, \pi_2)$ are fully accounted for by Hecke translates of the correspondence from $S(\mathrm{GSp}_4)$ to the modular curve $S(\mathrm{GL}_2)$ induced by (2), and nonsplit quaternionic Shimura curves on $S(\mathrm{GSp}_4)$ are necessarily sent to CM divisors on $S(\mathrm{GL}_2)$ under this correspondence. It is an intriguing question whether Weissauer's construction yields interesting Selmer classes when applied to quaternionic Shimura curves and the non-generic members of the $L$-packets $\Pi(\pi_1, \pi_2)$.

**Organization of the paper.** In §2, we give some basic notations and conventions. In §3, we recall the plectic version of Matsushima's formula and its relation to vector-valued automorphic forms. In §4, we give notations and conventions for similitude theta lifts. This section also contains a proof of the $L$-functoriality for similitude theta lifts of certain spherical representations from orthogonal to symplectic groups (Proposition 4.3.3); this is well-known to experts but we were not able to find a suitable reference. In §5, we recall the construction of the Yoshida lift $L$-packets via theta lifts, and compute the plectic Hodge structures associated to $\Pi_S(\pi_1, \pi_2)_f$. The material up to this point is necessary for all the main results. However, the proofs of Theorems C and A, which are given in §6 and §7, respectively, are logically independent of the proof of Theorem B. The only exceptions are some results on the archimedean theta correspondence in §7.1. In §8, we study the nontempered representations used for the construction of Hodge classes. In §9, we compute the vector-valued triple product periods that are necessary for the nonvanishing of the Hodge classes. The proof of Theorem B is completed in §10.

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## 2. Preliminaries

### 2.1. Basic notations.

2.1.1. Throughout this article, $F$ is a fixed totally real number field of degree $d$ and discriminant $D_F$, $\mathcal{O}_F$ is its ring of integers, and $\mathbb{A}_F$ is its ring of adèles. For each place $v$ of $F$, denote by $F_v$ the completion; if $v$ is non-archimedean, $\mathcal{O}_v$ is the valuation ring of $F_v$, $\varpi_v \in \mathcal{O}_v$ is the uniformizer, and $q_v = \#\mathcal{O}_v/\varpi_v$. For archimedean $v$, $q_v = 1$. The Haar measure on the additive group $\mathbb{A}_F$ is the product measure $\mathrm{d}a = \prod_v \mathrm{d}a_v$, where $\mathrm{d}a_v$ is the Haar measure on $F_v$ such that $\mathcal{O}_v$ has volume 1 if $v$ is nonarchimedean, and $\mathrm{d}a_v$ is the standard measure on $F_v \cong \mathbb{R}$ if $v$ is archimedean.

2.1.2. If $G$ is an algebraic group over $F$, $[G]$ denotes the adelic quotient $G(F)\backslash G(\mathbb{A}_F)$. If $\mathrm{d}g$ denotes a Haar measure on $G(\mathbb{A}_F)$, then we write $\mathrm{d}g$ as well for the quotient Haar measure on $[G]$ (where $G(F)$ is given the counting measure).

2.1.3. We fix the additive character $\psi = \psi_0 \circ \mathrm{tr}$ of $F\backslash\mathbb{A}_F$, where $\psi_0 : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}$ is the unique unramified character such that $\psi_0(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$.

2.1.4. For any $m$, let $\omega_m : \mathbb{R}^\times \to \mathbb{R}^\times$ be the character

$$t \mapsto t^{m-2\cdot\lfloor \frac{m}{2} \rfloor}.$$

If $\boldsymbol{m} = (m_v)_{v|\infty}$, let $\omega_{\boldsymbol{m}} : (F \otimes \mathbb{R})^\times \to \mathbb{R}^\times$ be the character $\otimes_{v|\infty}\omega_{m_v}$. These characters will be used as the central characters for "nearly unitary" normalizations of automorphic forms appearing in cohomology.

2.1.5. If $V$ is a vector space over a local field $k$ (either Archimedean or non-Archimedean), then $\mathcal{S}_k(V)$ is the Schwartz space of functions on $V$. If $V$ is a vector space over $F$ and $v$ is a place of $F$, then $\mathcal{S}_{F_v}(V)$ denotes the space of Schwartz functions on $V \otimes_F F_v$. Likewise, we write $\mathcal{S}_{F\otimes\mathbb{R}}(V)$ for the tensor product of the Schwartz spaces $\mathcal{S}_{F_v}(V)$ as $v$ ranges over archimedean places of $F$.

## 2.2. Conventions for $\mathrm{GL}_2$ and $\mathrm{SL}_2$.

2.2.1. The standard Borel and unipotent subgroups of $\mathrm{GL}_2$ are denoted $B$ and $N$, respectively; $\overline{B}$ denotes the image of $B$ in $\mathrm{PGL}_2$ . We shall abbreviate by $c \mapsto h_c$ the section of $\det : \mathrm{GL}_2 \to \mathbb{G}_m$ given by $h_c = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$.

2.2.2. For each non-archimedean place $v$ of $F$, we normalize the Haar measure $\mathrm{d}g_v$ on $\mathrm{PGL}_2(F_v)$ to assign volume 1 to $\mathrm{PGL}_2(\mathcal{O}_v)$, and likewise for $\mathrm{SL}_2(F_v)$. For non-archimedean $v$, we choose the Haar measure $\mathrm{d}g_v$ on $\mathrm{PGL}_2(F_v) \cong \mathrm{PGL}_2(\mathbb{R})$ given by:

$$\mathrm{d}g_v = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{\pi t^2}, \quad g_v = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a \in \mathbb{R}, t \in \mathbb{R}^\times, \theta \in [0,\pi). \tag{14}$$

On $\mathrm{SL}_2(F_v) \cong \mathrm{SL}_2(\mathbb{R})$, we choose the Haar measure $\mathrm{d}g_v$ given by:

$$\mathrm{d}g_v = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{2\pi t^2}, \quad g_v = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a \in \mathbb{R}, t \in \mathbb{R}_{>0}, \theta \in [0,2\pi). \tag{15}$$

2.2.3. For the standard compact subgroup $\mathrm{SO}(2)$ of $\mathrm{SL}_2(\mathbb{R})$, we denote by $\chi_m : \mathrm{SO}(2) \mapsto \mathbb{C}^\times$ the character

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto (\cos\theta + i\sin\theta)^m.$$

## 2.3. Conventions for symplectic groups.

2.3.1. Let $J$ be the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, for any field $k$, the block-diagonal matrix $\begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$ defines a symplectic pairing on the $k$-space $W_{2n,k} = \langle e_1, \cdots, e_{2n} \rangle$ such that

$$W_{2n,k} = \langle e_1, e_3, \cdots, e_{2n-1} \rangle \oplus \langle e_2, e_4, \cdots, e_{2n} \rangle$$

is a decomposition into maximal isotropic subspaces; we refer to $W_{2n,k}$ as the standard symplectic space of dimension $2n$. The symplectic group $\mathrm{Sp}_{2n,k}$ and the general symplectic group $\mathrm{GSp}_{2n,k}$ are the isometry and similitude groups, respectively, of $W_{2n,k}$. When not otherwise specified, $k = F$.

2.3.2. The maximal compact-modulo-center subgroup of the symplectic group $\mathrm{GSp}_{2n,\mathbb{R}}$ is $K_n \simeq (U(n) \times \mathbb{R}^\times)/\{\pm 1\}$, consisting of the matrices whose $2 \times 2$ blocks commute with $J$. When $K_n$ is viewed as a subgroup of $\mathrm{GSp}_{2n}(F_v)$ we write it $K_{n,v}$. There is a maximal compact torus $T \subset U(n)$ such that

$$\mathfrak{t} = \begin{pmatrix} \alpha_1 J & & \\ & \ddots & \\ & & \alpha_n J \end{pmatrix}, \quad \alpha_i \in \mathbb{R}.$$

We parameterize the weights of $U(n)$ by tuples of integers $(m_1, \cdots, m_n)$, corresponding to the character

$$\begin{pmatrix} \alpha_1 J & & \\ & \ddots & \\ & & \alpha_n J \end{pmatrix} \mapsto m_1\alpha_1 + \cdots + m_n\alpha_n.$$

When $n = 1$, the character $\chi_m \boxtimes \omega_m^{-1}$ on $U(1) \times Z_{\mathrm{GL}_2}$ descends to a character of $K_1$, which we will also denote by $\chi_m$; we hope that this will cause no confusion.

## 3. Cohomology of Shimura varieties

### 3.1. Shimura varieties and local systems.

3.1.1. Let $(G, X)$ be a Shimura datum with reflex field $E_0$, and let $\mu : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,C} \to G_\mathbb{R}$ be the corresponding cocharacter. Given a neat compact open subgroup $K \subset G(\mathbb{A}_{\mathbb{Q},f})$, we have a smooth algebraic Shimura variety $S_K(G, X)$ defined over $E_0$, such that

$$S_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q},f}) \times X/K.$$

We will usually drop $X$ from the notation for $S_K(G, X)$, and write

$$S(G) := \varprojlim_K S_K(G) \tag{16}$$

for the pro-algebraic Shimura variety.

3.1.2. Following the convention of [2, §5.1], let $Z(G) \subset G$ be the center, with neutral connected component $Z(G)^\circ \subset Z(G)$. Define $Z_s(G) \subset Z(G)^\circ$ to be the smallest subtorus such that $Z(G)^\circ/Z_s(G)$ has the same $\mathbb{Q}$-split and $\mathbb{R}$-split ranks, and let $G^c = G/Z_s(G)$.

Let $E \subset \mathbb{C}$ be any subfield, and let $\rho$ be an algebraic representation of $G^c$ on an $E$-vector space $V$; then for each level subgroup $K$ as above we have the Betti local system $\mathcal{V}_K$ on $S_K(G)_\mathbb{C}$ whose total space over $\mathbb{C}$ is

$$G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q},f}) \times X \times V/K. \tag{17}$$

By the discussion in [2, p. 535], to $V$ is also associated an algebraic vector bundle $\mathcal{V}_{\mathbb{C},\mathrm{dR},K}$ over $S_K(G)_\mathbb{C}$ equipped with a connection and a canonical filtration, whose complex analytification is associated under the Riemann-Hilbert correspondence to the complex local system $\mathcal{V}_{K,\mathbb{C}} := \mathcal{V}_K \otimes_E \mathbb{C}$.

Finally, if $E$ is a number field and $\lambda$ is a finite prime of $E$, write $\mathcal{V}_{\lambda,K}$ for the étale local system on $S_K(G)$ whose base change to $S_K(G)_\mathbb{C}$ coincides with $\mathcal{V}_K \otimes_E E_\lambda$, and $\mathcal{V}_{\lambda,K,\overline{\mathbb{Q}}_\ell}$ for $\mathcal{V}_{\lambda,K} \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$. A choice of prime $\ell$ and isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ determines a prime $\lambda$ of $E$, and in this context we omit $\lambda$ from the notation and write simply $\mathcal{V}_{K,\overline{\mathbb{Q}}_\ell}$.

Each of the constructions described above is compatible with the maps $S_K(G) \to S_{K'}(G)$ for neat compact open subgroups $K \subset K' \subset G(\mathbb{A}_{\mathbb{Q},f})$, and we drop the subscript $K$ to denote the corresponding objects for the pro-algebraic variety $S(G)$. We abbreviate the Betti cohomology of $\mathcal{V}$ by

$$H^i(S(G), \mathcal{V}) := H^i(S(G)(\mathbb{C}), \mathcal{V}).$$

**Theorem 3.1.3.** *Fix a prime $\ell$, an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, and an algebraic representation $\rho$ of $G^c$ on an $E$-vector space $V$, for a number field $E \subset \mathbb{C}$ containing $E_0$. Let $\lambda$ be the prime of $E$ induced by $\iota$. Then for each $i$, there is a canonical isomorphism of $G(\mathbb{A}_{\mathbb{Q},f})$-modules, compatible with the canonical filtrations and $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/E_\lambda)$-actions on both sides:*

$$H^i_{\text{ét}}(S(G)_{\overline{\mathbb{Q}}_\ell}, \mathcal{V}_\lambda) \otimes_{E_\lambda} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(S(G)_\mathbb{C}, \mathcal{V}_{\mathbb{C},\mathrm{dR}}) \otimes_{\mathbb{C},\iota_\ell^{-1}} B_{\mathrm{dR}}.$$

*In particular, the 0th graded piece is an isomorphism*

$$H^i_{\text{ét}}(S(G)_{\overline{\mathbb{Q}}_\ell}, \mathcal{V}_\lambda) \otimes_{E_\lambda} \widehat{\overline{\mathbb{Q}}}_\ell \cong \oplus_j \operatorname{gr}^j H^i_{\mathrm{dR}}(S(G)_{\mathbb{C}}, \mathcal{V}_{\mathbb{C},\mathrm{dR}}) \otimes_{\mathbb{C},\iota_\ell^{-1}} \widehat{\overline{\mathbb{Q}}}_\ell(-j).$$

*Moreover, these isomorphisms are functorial in $V$ and compatible with pullbacks induced by maps of Shimura data.*

*Proof.* The stated isomorphism and the functoriality in $V$ are immediate from [2, Theorem 1.1, Theorem 5.3.1]. It remains to check that Theorem 1.1 of *op. cit.* is compatible with pullback.[1] For this, let $X$ be a smooth algebraic variety over an $\ell$-adic field $k$, with a compactification $\overline{X}$ such that the complement of $X$ is a normal crossing divisor, and let $D^{\mathrm{alg}}_{\mathrm{dR},X}$ be the functor of *loc. cit.*. Then for any $\mathbb{Q}_\ell$-étale local system $\mathbb{L}$ on $X$, the rigid analytification of $D^{\mathrm{alg}}_{\mathrm{dR},X}(\mathbb{L})$ is the restriction of a filtered vector bundle with log connection $D_{\mathrm{dR},\log,X}(\overline{\mathbb{L}})$ on the rigid analytification of $\overline{X}$, where $\overline{\mathbb{L}}$ is as in [2, Corollary 3.2.10], cf. [2, §4.1]. Now suppose $Y$ is another smooth algebraic variety over $k$ with a map $f : Y \to X$, and choose a compactification $\overline{Y}$ of $Y$ as above so that $f$ extends to $\overline{f} : \overline{Y} \to \overline{X}$. We claim that there is a canonical isomorphism

$$f^* D^{\mathrm{alg}}_{\mathrm{dR},X}(\mathbb{L}) \xrightarrow{\sim} D^{\mathrm{alg}}_{\mathrm{dR},Y}(f^{-1}\mathbb{L}). \tag{18}$$

Indeed, by the canonical adjunction morphism in [2, Lemma 3.5.3], we have a morphism of filtered vector bundles with log connection $\overline{f}^* D_{\mathrm{dR},\log,\overline{X}}(\overline{\mathbb{L}}) \to D_{\mathrm{dR},\log,\overline{Y}}(\overline{f}^{-1}\overline{\mathbb{L}})$, which is algebraizable by the rigid analytic GAGA theorem of [18]. The restriction of the resulting morphism gives the desired map (18), which is an isomorphism because its analytification is so, cf. [24, Theorem 3.8].

Once we have (18), it suffices to show that the following diagram commutes, where the horizontal arrows are provided by [2, Theorem 1.1]:

$$\begin{array}{ccc}
H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{L}) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}} & \xrightarrow{\sim} & H^i_{\mathrm{dR}}(X, D^{\mathrm{alg}}_{\mathrm{dR},X}(\mathbb{L})) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}} \\
\Big\downarrow & & \Big\downarrow \\
 & & H^i_{\mathrm{dR}}(Y, f^* D^{\mathrm{alg}}_{\mathrm{dR},X}(\mathbb{L})) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}} \\
 & & \Big\| {\scriptstyle (18)} \\
H^i_{\text{ét}}(Y_{\overline{\mathbb{Q}}_\ell}, f^{-1}\mathbb{L}) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}} & \xrightarrow{\sim} & H^i_{\mathrm{dR}}(Y, D^{\mathrm{alg}}_{\mathrm{dR},Y}(f^{-1}\mathbb{L})) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{dR}}
\end{array}$$

This commutativity can be checked by hand by tracing through the construction of the horizontal arrows in [2, §3]. □

3.2. **The structure of cohomology as a $G(\mathbb{A}_{\mathbb{Q},f})$-module.** We continue to fix an $E$-linear algebraic representation $(\rho, V)$ of $G^c$ as in (3.1.2).

3.2.1. If $\Pi_f$ is a $\mathbb{C}[G(\mathbb{A}_{F,f})]$-module, $\Pi_f$ is *defined* over $E \subset \mathbb{C}$ if there exists a $E[G(\mathbb{A}_{F,f})]$-module $\Pi_f^E$ such that $\Pi_f^E \otimes_E \mathbb{C} \simeq \Pi_f$. In this case, we write:

$$H^*_?(S(\boldsymbol{G}), \mathcal{V})_{\Pi_f} := \operatorname{Hom}_{E[G(\mathbb{A}_{F,f})]}(\Pi_f^E, H^*_?(S(\boldsymbol{G}), \mathcal{V}), \tag{19}$$

where $H^*_?$ denotes compactly supported, inner, or singular cohomology as $? = c$, $!$, or $\emptyset$. The maximal $\Pi_f^E$-isotypic component, which we write as

$$H^*_?(S(\boldsymbol{G}), \mathcal{V})[\Pi_f],$$

is then isomorphic to

$$H^*_?(S(\boldsymbol{G}), \mathcal{V})_{\Pi_{S_f}} \otimes_E \Pi^E_{S_f}.$$

[1] We thank Kai-Wen Lan for explaining the following argument.

3.2.2. An irreducible admissible complex $G(\mathbb{A}_{\mathbb{Q},f})$-representation $\Pi_f$ is said to be *Eisenstein* if $\Pi_f$ is a subquotient of a parabolic induction $\mathrm{Ind}_{P(\mathbb{A}_{\mathbb{Q},f})}^{G(\mathbb{A}_{\mathbb{Q},f})}\pi_f$, for a parabolic subgroup $P = MN$ of $G$ and a cuspidal automorphic representation $\pi$ of $M(\mathbb{A}_{\mathbb{Q}})$. For any admissible $E[G(\mathbb{A}_{\mathbb{Q},f})]$-module $H$, we say $H$ is Eisenstein if all irreducible constituents of $H \otimes_E \mathbb{C}$ are.

If $\Pi$ is an automorphic representation of $G(\mathbb{A}_{\mathbb{Q},f})$ with $\Pi_f$ non-Eisenstein and defined over $E$, then for all $i \geq 0$, it follows from Franke's proof of the Borel conjecture [5, Theorem 18] that the maximal $\Pi_f$-isotypic submodule

$$H^i(S(G),\mathcal{V})[\Pi_f] \subset H^i(S(G),\mathcal{V}) \tag{20}$$

is a direct summand.

**Lemma 3.2.3.** *Let $\mathcal{V}$ be the automorphic local system on $S(G)$ associated to an $E$-linear $G^c$-representation $(V,\rho)$. Then:*

1. *The $E[G(\mathbb{A}_{\mathbb{Q},f})]$-module*
$$H^i(S(G),\mathcal{V})/H^i_!(S(G),\mathcal{V})$$
*is Eisenstein.*
2. *There exists an $E[G(\mathbb{A}_{\mathbb{Q},f})]$-stable direct summand $H^i(S(G),\mathcal{V})_0 \subset H^i(S(G),\mathcal{V})$ such that*
$$H^i(S(G),\mathcal{V})_0 \subset H^i_!(S(G),\mathcal{V})$$
*and*
$$H^i(S(G),\mathcal{V})_0 \otimes_E \mathbb{C} = \bigoplus_{\substack{\Pi_f \text{non-} \\ \text{Eisenstein}}} H^i(S(G),\mathcal{V}_{\mathbb{C}})[\Pi_f],$$
*where $\Pi_f$ runs over the finite parts of automorphic representations of $G(\mathbb{A}_{\mathbb{Q},f})$.*

*Proof.* Part (1) is well-known; a lucid exposition may be found in the preprint [9, Chapter 9]. For (2), by (1) it suffices to note that the property of being non-Eisenstein is stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. □

### 3.3. **Mixed Hodge structures.**

3.3.1. With notation as in (3.1.2), consider the following conditions on the representation $(\rho, V)$:

(1) $E$ is a number field contained in $\mathbb{R}$.
(2) The composite map
$$\mathbb{G}_{m,\mathbb{R}} \to \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu} G_{\mathbb{R}} \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{R}) \to \mathrm{GL}(V \otimes_E \mathbb{R})$$
is given by $z \mapsto z^m$ for some integer $m$.

Under these conditions, the canonical filtration on $\mathcal{V}_{\mathbb{C},\mathrm{dR},K}$ makes $\mathcal{V}_K$ into a polarizable variation of Hodge structures of weight $m$ with coefficients in $E$. Both conditions are satisfied for the local systems on symplectic Shimura varieties defined in §3.6 below.

3.3.2. By Saito's theory of mixed Hodge modules [32], under conditions (1) and (2) above, the Betti cohomology $H^i(S(G),\mathcal{V})$ is a mixed Hodge structure with coefficients in $E$. We write $\mathcal{W}_\bullet H^i(S(G),\mathcal{V})$ for the weight filtration and $\mathcal{F}^\bullet H^i(S(G),\mathcal{V}_{\mathbb{C}})$ for the Hodge filtration. By definition, a Hodge class in $H^i(S(G),\mathcal{V})$ is a Hodge class for the pure Hodge structure $\mathcal{W}_{i+m}H^i(S(G),\mathcal{V})$.

### 3.4. **Plectic Hodge structures.**

3.4.1. We now let $G$ be a reductive group over $F$, and set $\boldsymbol{G} = \mathrm{Res}_{F/\mathbb{Q}}\,G_F$. Since

$$\boldsymbol{G}(\mathbb{R}) = \prod_{v|\infty} G(F_v) = \prod_{v|\infty} G_v(\mathbb{R}), \tag{21}$$

a Shimura datum $(\boldsymbol{G},\boldsymbol{X})$ is necessarily a product $\boldsymbol{X} \simeq \prod_{v|\infty} X_v$. If $K_v \subset G_v(\mathbb{R})$ denotes the stabilizer of a distinguished point $h_v \in X_v$, then the stabilizer of the corresponding point $\boldsymbol{h} \in \boldsymbol{X}$ is

$$\boldsymbol{K}_\infty = \prod_{v|\infty} K_v.$$

3.4.2. Let $(\rho, V)$ be an algebraic representation of $\boldsymbol{G}^c$ as in (3.1.2). Matsushima's formula for the $L^2$ cohomology of $S(\boldsymbol{G})$ is:

$$H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}) \cong \bigoplus_{\pi=\pi_f\otimes\pi_\infty} m_{\mathrm{disc}}(\pi)\cdot \pi_f \otimes H^*(\mathrm{Lie}\ \boldsymbol{G}; \boldsymbol{K}_\infty, \pi_\infty^{\mathrm{sm}} \otimes V_{\mathbb{C}})). \tag{22}$$

Here $\pi$ runs over cuspidal automorphic representations of $G(\mathbb{A})$, $m_{\mathrm{disc}}(\pi)$ refers to the multiplicity in the discrete spectrum, and $\pi_\infty^{\mathrm{sm}}$ is the dense subspace of smooth vectors. Moreover (22) is equivariant for the natural actions of $G(\mathbb{A}_{F,f})$ on both sides. Suppose $V_{\mathbb{C}} = \otimes_v V_v$, where $V_v$ are $\mathbb{C}$-vector spaces equipped with algebraic representations $\rho_v$ of $G_v(\mathbb{R})$, such that $\rho$ factors as

$$\rho : G(F) \hookrightarrow G(F\otimes\mathbb{R}) \simeq \prod_v G_v(\mathbb{R}) \xrightarrow{\otimes\rho_v} \prod_v \mathrm{Aut}(V_v). \tag{23}$$

Since the Lie algebra of $\boldsymbol{G}$ is $\prod_{v|\infty}\mathfrak{g}_v$, the right hand side of (22) has a decomposition (cf. [26, §16]):

$$\bigoplus_{\boldsymbol{p},\boldsymbol{q}} \left( \bigoplus_{\pi_f\otimes\pi_\infty} m_{\mathrm{disc}}(\pi)\cdot\pi_f \otimes \bigotimes_{v|\infty} H^{p_v,q_v}(\mathfrak{g}_v, K_v, \pi_v^{\mathrm{sm}}\otimes V_v)\right). \tag{24}$$

Here $\boldsymbol{p}$ and $\boldsymbol{q}$ are plectic Hodge types, i.e. tuples of positive integers $(p_v)_{v|\infty}$ and $(q_v)_{v|\infty}$. Then (22) induces a plectic Hodge decomposition on $H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}})$, written:

$$H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}) = \bigoplus_{\boldsymbol{p},\boldsymbol{q}} H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}). \tag{25}$$

**Remark 3.4.3.** Because this decomposition does not take into account any variation of Hodge structures on $\mathcal{V}_{\mathbb{C}}$, it does not compare with the canonical mixed Hodge structure on $H^*(S(\boldsymbol{G}), \mathcal{V})$ recalled in §3.3 above. For this reason, (25) should be viewed more as a computational tool then as a suitable definition of "the" plectic Hodge structure on $H^*_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}})$.

### 3.5. **Realizing automorphic forms in cohomology.**

3.5.1. The complex structure on $X_v$ induces a decomposition

$$\mathfrak{g}_{v,\mathbb{C}} = \mathfrak{k}_\infty \oplus \mathfrak{p}_{v,+} \oplus \mathfrak{p}_{v,-}.$$

We define

$$\wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^* \coloneqq \otimes_{v|\infty}(\wedge^{p_v}\mathfrak{p}_{v,+}^* \otimes \wedge^{q_v}\mathfrak{p}_{v,-}^*), \tag{26}$$

and let $(\sigma^{\boldsymbol{p},\boldsymbol{q}}, \wedge^{\boldsymbol{p},\boldsymbol{q}})$ be the corresponding natural representation of $\boldsymbol{K}_\infty$. The smooth vector bundle $\Omega^*$ of differential forms on $S(\boldsymbol{G})$ has a decomposition

$$\Omega^* = \oplus_{\boldsymbol{p},\boldsymbol{q}}\Omega^{\boldsymbol{p},\boldsymbol{q}},$$

where the vector bundle $\Omega^{\boldsymbol{p},\boldsymbol{q}}$ of $(\boldsymbol{p},\boldsymbol{q})$-forms on $S(\boldsymbol{G})$ corresponds to the local system whose complex points are:

$$G(F)\backslash G(\mathbb{A}_{F,f}) \times \boldsymbol{G}(\mathbb{R}) \times \wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^*/\boldsymbol{K}_\infty.$$

In particular, the space $\Gamma_{(2)}(\Omega^{\boldsymbol{p},\boldsymbol{q}}\otimes\mathcal{V}_{\mathbb{C}})$ of $L^2$ global $(\boldsymbol{p},\boldsymbol{q})$-forms with coefficients in $\mathcal{V}_{\mathbb{C}}$ is identified with:

$$\left\{ f\in C^\infty_{(2)}(G(\mathbb{A}_F))\otimes V_{\mathbb{C}}\otimes\wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^* \,:\, f(\gamma g k) = \rho(\gamma)\sigma^{\boldsymbol{p},\boldsymbol{q}}(k^{-1})f(g),\ \forall\gamma\in G(F),\ k\in\boldsymbol{K}_\infty\right\}. \tag{27}$$

Here $C^\infty_{(2)}(G(\mathbb{A}_F))$ is the space of smooth $L^2$ functions on $G(\mathbb{A}_F)$; by definition, we have:

$$\Gamma_{(2)}(\Omega^{\boldsymbol{p},\boldsymbol{q}}\otimes\mathcal{V}_{\mathbb{C}}) \twoheadrightarrow H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}). \tag{28}$$

Finally, we remark that there is a canonical isomorphism:

$$\begin{aligned}(\mathcal{A}_{(2)}(G(\mathbb{A}_F))\otimes V_{\mathbb{C}}|_{\boldsymbol{K}_\infty}\otimes\wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^*)^{\boldsymbol{K}_\infty} &\xrightarrow{\sim} \Gamma_{(2)}(\Omega^{\boldsymbol{p},\boldsymbol{q}}\otimes\mathcal{V}_{\mathbb{C}})\\ \phi &\mapsto f_\phi,\quad f_\phi(g) = \rho(g_\infty)\phi(g).\end{aligned} \tag{29}$$

Here $\rho(g_\infty)$ is defined via the decomposition (23). By composing with (28), we obtain a realization of vector-valued automorphic forms in cohomology:

$$(\mathcal{A}_{(2)}(G(\mathbb{A}_F))\otimes V_{\mathbb{C}}|_{\boldsymbol{K}_\infty}\otimes\wedge^{\boldsymbol{p},\boldsymbol{q}}\mathfrak{p}_G^*)^{\boldsymbol{K}_\infty} \twoheadrightarrow H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\boldsymbol{G}), \mathcal{V}_{\mathbb{C}}). \tag{30}$$

3.6. **Symplectic Shimura varieties.**

3.6.1. When $G = \mathrm{GSp}_{2n}$, equipped with its usual Shimura datum, the subgroup $\boldsymbol{K}_\infty$ is just

$$\boldsymbol{K}_n \coloneqq \prod_v K_{n,v} \subset \mathrm{GSp}_{2n}(F \otimes \mathbb{R}). \tag{31}$$

We establish some notation for local systems on $S(\mathbf{GSp}_{2n})$. Suppose given a tuple $\boldsymbol{\lambda} = (\lambda_v)_{v|\infty}$, where $\lambda_v = (m_{1,v}, \cdots, m_{n,v})$ is a dominant weight of $\mathrm{Sp}_{2n,\mathbb{R}}$. We define $(\rho_{\lambda_v}, V_{\lambda_v})$ to be the unique irreducible $\mathbb{C}$-linear algebraic representation of $\mathrm{GSp}_{2n}$ whose restriction to $\mathrm{Sp}_{2n}$ has weight $\lambda_v$ and whose central character is $\omega^{-1}_{m_{1,v}+\cdots+m_{n,v}}$ in the notation of (2.1.4). This defines a representation $(\rho_{\boldsymbol{\lambda}}, V_{\boldsymbol{\lambda}})$ of $\mathrm{GSp}_{2n}$ according to (23), which clearly descends to $F^c$.

**Proposition 3.6.2.** *The representation $(\rho_{\boldsymbol{\lambda}}, V_{\boldsymbol{\lambda}})$ descends to a $\mathbb{Q}(\boldsymbol{\lambda})$-linear representation of $\mathbf{GSp}_{2n}$, where $\mathbb{Q}(\boldsymbol{\lambda})$ is the fixed field of*

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \,:\, \lambda_{\sigma\cdot v} = \lambda_v \ \forall v|\infty\}.$$

*Proof.* The proof of [36, Proposition I.3] applies unchanged. □

3.6.3. When the parity of $\sum_{i=1}^n m_{i,v}$ is independent of $v$, then $(\rho_{\boldsymbol{\lambda}}, V_{\boldsymbol{\lambda}})$ descends to a $\mathbb{Q}(\boldsymbol{\lambda})$-linear representation of the quotient $\mathbf{GSp}^c_{2n}$ of $\mathbf{GSp}_{2n}$, cf. (3.1.2). We then obtain a $\mathbb{Q}(\boldsymbol{\lambda})$-local system $\mathcal{V}_{\boldsymbol{\lambda}}$ on $S(\mathbf{GSp}_{2n})$ such that $\mathcal{V}_{\boldsymbol{\lambda},\mathbb{C}}$ arises from the tuple of representations $(\rho_{\lambda_v}, V_{\lambda_v})$ of $\mathrm{GSp}_{2n}(F_v)$ according to (23).

3.7. **The case $G = \mathrm{GL}_2$.**

3.7.1. We recall some basic results on the cohomology of $S(\boldsymbol{G})$ in the simplest case, $G = \mathrm{GL}_2 = \mathrm{GSp}_2$. For a tuple of integers $\boldsymbol{m} = (m_v)_{v|\infty}$ with $m_v \geq 2$ and all $m_v$ of the same parity, define $\mathbb{Q}(\boldsymbol{m})$ to be the fixed field of

$$\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \,:\, m_{\sigma\cdot v} = m_v \ \forall v|\infty\}. \tag{32}$$

We then obtain from §3.6 a $\mathbb{Q}(\boldsymbol{m})$-local system $\mathcal{V}_{\boldsymbol{m}-2}$ on $S(\mathbf{GL}_2)$, where $\boldsymbol{m}-2 = (m_v-2)_{v|\infty}$.

3.7.2. Let $(p(+), q(+)) = (1,0)$ and $(p(-), q(-)) = (0,1)$, and define $(\boldsymbol{p}(\boldsymbol{\epsilon}), \boldsymbol{q}(\boldsymbol{\epsilon}))$ to be the plectic Hodge type $(p_v(\epsilon_v), q_v(\epsilon_v))_{v|\infty}$, for any choice of signs $\boldsymbol{\epsilon} = (\epsilon_v)_{v|\infty}$. Let $\boldsymbol{\chi}_{\boldsymbol{\epsilon m}}$ be the character of $\boldsymbol{K}_1$ from (2.3.2). Then we have:

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_{\boldsymbol{K}_1}\left(\boldsymbol{\chi}_{-\boldsymbol{\epsilon m}}, \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon})}\mathfrak{p}^*_{\mathrm{GL}_2} \otimes V_{\boldsymbol{m}-2,\mathbb{C}}\right) &= 1,\\ \dim_{\mathbb{C}} \mathrm{Hom}_{\boldsymbol{K}_1}\left(\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}}, \wedge^{1-\boldsymbol{p}(\boldsymbol{\epsilon}),1-\boldsymbol{q}(\boldsymbol{\epsilon})}\mathfrak{p}^*_{\mathrm{GL}_2} \otimes V^\vee_{\boldsymbol{m}-2,\mathbb{C}}\right) &= 1. \end{aligned} \tag{33}$$

Let $\pi$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\boldsymbol{m}$, whose central character has infinity type $\omega_{\boldsymbol{m}}$. Then combining (30) with (33) yields maps, well-defined up to scalars:

$$\begin{aligned} \mathrm{cl}_{\boldsymbol{\epsilon}} &: (\pi \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1} \to H^{\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi_f]\\ \mathrm{cl}'_{\boldsymbol{\epsilon}} &: (\pi^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon m}})^{\boldsymbol{K}_1} \to H^{1-\boldsymbol{p}(\boldsymbol{\epsilon}),1-\boldsymbol{q}(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}})[\pi^\vee_f] \end{aligned} \tag{34}$$

Here $(1-\boldsymbol{p}(\boldsymbol{\epsilon}), 1-\boldsymbol{q}(\boldsymbol{\epsilon})) = (\boldsymbol{p}(-\boldsymbol{\epsilon}), \boldsymbol{q}(-\boldsymbol{\epsilon}))$ is the plectic Hodge type $(1-p(\epsilon_v), 1-q(\epsilon_v))_{v|\infty}$. The following is well-known:

**Proposition 3.7.3.** *For each $\boldsymbol{\epsilon}$, the maps in (34) are isomorphisms, and*

$$H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi_f] = H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}-2,\mathbb{C}})[\pi^\vee_f] = 0$$

*if $(\boldsymbol{p}, \boldsymbol{q})$ is not of the form $(\boldsymbol{p}(\boldsymbol{\epsilon}), \boldsymbol{q}(\boldsymbol{\epsilon}))$ for some $\boldsymbol{\epsilon}$. Moreover, if $\pi_f$ is defined over $E \supset \mathbb{Q}(\boldsymbol{m})$, then there are $\mathrm{GL}_2(\mathbb{A}_{F,f})$-equivariant isomorphisms*

$$H^*_!(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,E})[\pi_f] \otimes_E \mathbb{C} \simeq H^*_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,\mathbb{C}})[\pi_f]$$

*and*

$$H^*_c(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,E})[\pi_f] \simeq H^*(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}-2,E})[\pi_f] \simeq H^*_!(S(\boldsymbol{G}), \mathcal{V}_{\boldsymbol{m}-2})[\pi_f],$$

*and similarly for $\mathcal{V}^\vee_{\boldsymbol{m}-2,E}$ and $\pi^\vee_f$.*

□

## 4. Similitude theta lifting

### 4.1. **Local Weil representation and local theta lift.**

4.1.1. Let $\epsilon = \pm 1$, and let $V$, $W$ be vector spaces over a field $k$ equipped with nondegenerate $\epsilon$-symmetric and $(-\epsilon)$-symmetric pairings, respectively. We assume $\dim W = 2n$ and $\dim V = 2m$ are even, and that $W$ is equipped with a complete polarization

$$W = W_1 \oplus W_2, \quad W_2 = W_1^*. \tag{35}$$

*For simplicity, assume as well that the discriminant character of $V$ is trivial (as will be the case in our applications).* Let $G_1 = G_1(V)$, $G = G(V)$ be the connected isometry and similitude groups, respectively, of $V$, and likewise $H_1 = H_1(W)$ and $H = H(W)$. Let $P = P(W_1) \subset H(W)$ be the parabolic subgroup stabilizing $W_1$, $P_1$ its intersection with $H_1$, and $N \subset P_1$ its unipotent radical. Also set

$$R_0 = \{(h, g) \in H \times G \, : \, \nu_H(h) = \nu_G(g)\}, \tag{36}$$

where $\nu_G : G \to \mathbb{G}_m$ and $\nu_H : H \to \mathbb{G}_m$ are the similitude characters.

4.1.2. Assume that $k$ is a local field. Then, for any nontrivial additive character $\psi_k$ of $k$, the Weil representation $\omega = \omega_{W,V,\psi_k}$ of $H_1(k) \times G_1(k)$ is realized on the Schwartz space $\mathcal{S}_k(W_2 \otimes V)$; in this model, the action of the parabolic $P_1 \times G_1 \subset H_1 \times G_1$ stabilizing $W_1 \times V$ is described as follows.

$$\begin{cases} \omega(1, g)\phi(x) = \phi(g^{-1}x), & g \in G_1(k), \\ \omega(n, 1)\phi(x) = \psi\left(\frac{1}{2}\langle n(x), x\rangle\right) \cdot \phi(x), & n \in N(k) \subset \mathrm{Hom}(W_2, W_1), \\ \omega(h(a), 1)\phi(x) = |\det(a)|^m \phi(a^t x), & a \in \mathrm{GL}(W_1)(k) \subset P_1(k), \end{cases} \tag{37}$$

where $GL(W_1)$ is viewed as the Levi factor of $P_1$ by the standard embedding

$$a \mapsto h(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix} \in P_1. \tag{38}$$

The representation $\omega$ extends naturally to $R_0(k)$ by defining

$$\omega\left(\begin{pmatrix} 1 & 0 \\ 0 & \nu_G(g) \end{pmatrix}, g\right)\phi(x) = |\nu_G(g)|^{-mn/2}\phi(g^{-1}x) \tag{39}$$

for all $g \in G(k)$, cf. [30, §3]. Note that $\omega$ is trivial on the center $\{(\lambda, \lambda)\} \subset R_0$.

4.1.3. Suppose that $V = V_1 \oplus V_2$ is also split; then the preceding construction also defines an action of $R_0(k)$ on $\mathcal{S}_k(W \otimes V_2)$ by interchanging the roles of $V$ and $W$. These two representations are isomorphic via the partial Fourier transform. More precisely, consider the map $\mathcal{F} : \mathcal{S}_k(W_2 \otimes V) \to \mathcal{S}_k(W \otimes V_2)$ defined by

$$\phi \mapsto \widehat{\phi}, \quad \widehat{\phi}(x_1, x_2) = \int_{W_2 \otimes V_1} \phi(z, x_2)\psi(\langle z, x_1\rangle)\, \mathrm{d}z, \tag{40}$$

where $x_1 \in W_1 \otimes V_2$, $x_2 \in W_2 \otimes V_2$, and $\mathrm{d}z$ is the self-dual Haar measure with respect to $\psi_k$. Then it is well-known that $\mathcal{F}$ intertwines the actions of $H_1(k) \times G_1(k)$ on both sides, and it is immediate to check that it intertwines the actions of

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\right) \in R_0(k)$$

according to the definition (39); so $\mathcal{F}$ is equivariant for all of $R_0(k)$.

4.1.4. If $\pi$ is an irreducible admissible representation of $H(k)$, then the local theta lift $\Theta(\pi) = \Theta_{W,V}(\pi)$ is the largest semisimple representation of $G(k)$ such that there is a surjection

$$\omega_{W,V,\psi_k} \twoheadrightarrow \pi^\vee \boxtimes \Theta(\pi)$$

of admissible $R_0(k)$-representations. Symmetrically, if $\sigma$ is an irreducible admissible representation of $G(k)$, then the local theta lift $\Theta(\sigma) = \Theta_{V,W}(\sigma)$ is the largest semisimple representation of $H(k)$ admitting a surjection $\omega_{W,V,\psi_k} \twoheadrightarrow \Theta(\sigma) \boxtimes \sigma^\vee$. The theta lift does not depend on $\psi_k$ by [31, Proposition 1.9].

4.1.5. We remark that the Weil representation extends naturally to the full isometry groups of $V$ and $W$, not just the neutral connected components; we denote these by $G_1' = G_1'(V)$ and $H_1' = H_1'(W)$. Similarly, if $G'$ and $H'$ denote the full similitude groups, then the Weil representation extends to the subgroup

$$R_0' \coloneqq \{(h,g) \in H' \times G' \,:\, \nu_{H'}(h) = \nu_{G'}(g)\}\,.$$

The theta lift is usually defined in the literature using $G'$, $H'$, and $R_0'$. The drawback of working with neutral connected components of similitude groups is that we no longer have Howe duality, and in particular the local theta lift may be reducible. However, using connected similitude groups is more convenient for our global calculations.

## 4.2. **Global Weil representation and global theta lifts.**

4.2.1. Now turning to the global situation, assume $k = F$ in (4.1.1). Fix $\mathcal{O}_F$-lattices $\mathcal{W}_i \subset W_i$, for $i = 1$, 2, and $\mathcal{V} \subset V$. The adelic Schwartz space $\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$ is the restricted tensor product of the local Schwartz spaces $\mathcal{S}_{F_v}(W_2 \otimes V)$ with respect to the indicator function of $(\mathcal{W}_2 \otimes \mathcal{V}) \otimes_{\mathcal{O}_F} \mathcal{O}_v$. The global Weil representation of $R_0(\mathbb{A}_F)$, realized on $\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$, is defined as the restricted tensor product of the local Weil representations (using the characters $\psi_{F_v}$ determined by the fixed global character $\psi$). Recall the automorphic realization of $\omega$, given by the theta kernel:

$$\theta(h,g;\phi) = \sum_{x \in W_2(F) \otimes V(F)} \omega(h,g)\phi(x), \quad (h,g) \in R_0(\mathbb{A}_F), \;\; \phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V). \tag{41}$$

If $V$ is also split, then we again have the alternate model $\mathcal{S}_{\mathbb{A}_F}(W \otimes V_2)$, related to $\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$ by the adelic partial Fourier transform. Note that

$$\theta(h,g;\phi) = \theta(h,g;\widehat{\phi}) = \sum_{x \in W \otimes V_2} \omega(h,g)\widehat{\phi}(x)$$

by Poisson summation.

4.2.2. Let $f \in \mathcal{A}_0(H(\mathbb{A}_F))$ be an automorphic cusp form and choose any $\phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$. Then, fixing a Haar measure $\mathrm{d}h_1$ on $H_1(\mathbb{A}_F)$, the similitude theta lift $\theta_\phi(f)$ to $G$ is the automorphic function

$$g \mapsto \int_{[H_1]} \theta(h_1h_0, g;\phi) f(h_1h_0)\,\mathrm{d}h_1, \quad g \in G(\mathbb{A}_F), \tag{42}$$

where $h_0 \in H(\mathbb{A}_F)$ is any element such that $\nu_H(h_0) = \nu_G(g)$.

Likewise, if $f \in \mathcal{A}_0(G(\mathbb{A}_F))$ is an automorphic cusp form and $\mathrm{d}g_1$ is a Haar measure on $G_1(\mathbb{A})$, then the similitude theta lift $\theta_\phi(f)$ to $H$ is the automorphic function

$$h \mapsto \int_{[G_1]} \theta(h, g_1g_0;\phi) f(g_1g_0)\,\mathrm{d}g_1, \quad h \in H(\mathbb{A}_F),$$

where $g_0 \in G(\mathbb{A}_F)$ is any element such that $\nu_G(g_0) = \nu_H(h)$.

If $\pi$ is a cuspidal automorphic representation of $H(\mathbb{A}_F)$, then the similitude theta lift $\Theta(\pi) = \Theta_{W,V}(\pi)$ is the subspace of $\mathcal{A}(G(\mathbb{A}_F))$ spanned by the theta lifts $\theta_\phi(f)$ for $f \in \pi$ and $\phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$; if $\pi$ is a cuspidal automorphic representation of $G(\mathbb{A}_F)$, we similarly define $\Theta(\pi) = \Theta_{V,W}(\pi)$ to be the subspace of $\mathcal{A}(H(\mathbb{A}_F))$ spanned by the theta lifts $\theta_\phi(f)$ for $f \in \pi$ and $\phi \in \mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V)$. A key property of the global theta lift is its compatibility with the local theta lift. Although this is well-known, we include a proof for the reader's convenience.

**Proposition 4.2.3.** *Let $\pi$ be a cuspidal automorphic representation of either $G(\mathbb{A}_F)$ or $H(\mathbb{A}_F)$, and suppose that $\Theta(\pi)$ lies in the $L^2$ subspace. Then for any automorphic representation $\sigma = \otimes_v' \sigma_v \subset \Theta(\pi)$, $\sigma_v$ is a constituent of $\Theta(\pi_v)$ for all $v$.*

*Proof.* Suppose $\pi$ is a representation of $G(\mathbb{A}_F)$; the other case is the same. We consider the map of $R_0(\mathbb{A}_F)$-representations:

$$\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V) \otimes \pi \otimes \sigma^\vee \to \mathbb{C}$$
$$\phi \otimes f \otimes f' \mapsto \int_{[Z_H \backslash H]} \theta_\phi(f)(h) f'(h)\,\mathrm{d}h.$$

This map is well-defined and nontrivial by assumption. By duality, it also gives a nontrivial map

$$\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V) \twoheadrightarrow \pi^\vee \boxtimes \sigma,$$

which is evidently a restricted tensor product. This implies the proposition. □

4.2.4. The theta lift defined in (4.2.2) generalizes readily to vector-valued automorphic forms. Suppose $K \subset G(F \otimes \mathbb{R})$ and $L \subset H(F \otimes \mathbb{R})$ are subgroups which are compact modulo center, and let

$$(L \times K)_0 \coloneqq (L \times K) \cap R_0(F \otimes \mathbb{R}).$$

Suppose given finite-dimensional representations $\sigma$ and $\tau$ of $L$ and $K$, and let $f \in (\mathcal{A}_0(H(\mathbb{A}_F)) \otimes \sigma)^L$ be a vector-valued automorphic form. Then for a vector-valued Schwartz function

$$\varphi \in (\mathcal{S}_{F\otimes\mathbb{R}}(W_2 \otimes V) \otimes \sigma^\vee \otimes \tau)^{(L\times K)_0},$$

and a Schwartz function

$$\phi_f \in \mathcal{S}_{\mathbb{A}_{F,f}}(W_2 \otimes V) \coloneqq \otimes'_{v\nmid\infty} \mathcal{S}_{F_v}(W_2 \otimes V),$$

we may define

$$\theta_{\phi_f \otimes \varphi}(f) \in (\mathcal{A}(G(\mathbb{A}_F)) \otimes \tau)^K$$

by the same formula (42) as for the scalar-valued theta lift. The vector-valued theta lift from $G$ to $H$ is defined in the same way.

### 4.3. **Spherical theta correspondence for similitudes.**

4.3.1. We shall require an explicit description of the spherical similitude theta correspondence in certain cases. Continuing the notation of (4.1.1), assume $k$ is a nonarchimedean local field, that $\psi_k$ is unramified, and that $V = V_1 \oplus V_2$ is a split orthogonal space (so that $\epsilon = +$). For this subsection, we will need to work with the disconnected isometry and similitude groups $G_1' = O(V)$ and $G' = \mathrm{GO}(V)$.

Now choose bases $\{e_1, \cdots, e_m\}$ and $\{f_1, \cdots, f_n\}$ of $V_1$ and $W_1$, respectively, and let $\{e_1^*, \cdots, e_m^*\}$ and $\{f_1^*, \cdots, f_n^*\}$ be the dual bases of $V_2$ and $W_2$. Let $T_{G_1} \subset \mathrm{GL}(V_1) \subset G_1$ and $T_{H_1} \subset \mathrm{GL}(W_1) \subset H_1$ be the standard diagonal tori; then we choose the maximal tori for $G$, $H$, and $R_0$ given (with respect to the bases $\{e_1, \cdots, e_m, e_1^*, \cdots, e_m^*\}$ and $\{f_1, \cdots, f_n, f_1^*, \cdots, f_n^*\}$) by:

$$\begin{aligned} T_G &= T_{G_1} \times \mathbb{G}_m = \left\{\mathrm{diag}(x_1, \cdots, x_m, \lambda x_1^{-1}, \cdots, \lambda x_m^{-1})\right\} \\ T_H &= T_{H_1} \times \mathbb{G}_m = \left\{\mathrm{diag}(y_1, \cdots, y_n, \kappa y_1^{-1}, \cdots, \kappa y_n^{-1})\right\} \\ T_{R_0} &= T_H \times_{\mathbb{G}_m} T_G \simeq T_{G_1} \times T_{H_1} \times \mathbb{G}_m \end{aligned} \tag{43}$$

4.3.2. To fix notation, we recall the unramified principal series of $G$ and $H$. The unramified characters of $T_{G_1}(k)$ are parameterized by tuples $\chi_1 = (\alpha_1, \cdots, \alpha_m) \in (\mathbb{C}^\times)^m$, where

$$\chi_1(\mathrm{diag}(x_1, \cdots, x_m, x_1^{-1}, \cdots, x_n^{-1})) = \prod_{i=1}^m \alpha_i^{\mathrm{ord}\, x_i}.$$

The unramified characters of $T_G(k)$ are parameterized by $\chi = (\alpha_1, \cdots, \alpha_m, s) \in (\mathbb{C}^\times)^{m+1}$, where

$$\chi(\mathrm{diag}(x_1, \cdots, x_m, \lambda x_1^{-1}, \cdots, \lambda x_m^{-1})) = s^{\mathrm{ord}\,\lambda} \prod_{i=1}^m \alpha_i^{\mathrm{ord}\, x_i}.$$

Similarly, the unramified characters of $T_{H_1}(k)$ (resp. $T_H(k)$) are parametrized by $\mu_1 = (\beta_1, \cdots, \beta_n) \in (\mathbb{C}^\times)^n$ (resp. $\mu = (\beta_1, \cdots, \beta_n, t) \in (\mathbb{C}^\times)^{n+1}$), and the unramified characters of $T_{R_0}(k)$ are parameterized by

$$\eta = (\beta_1, \cdots, \beta_n, \alpha_1, \cdots, \alpha_m, u) \in (\mathbb{C}^\times)^{n+m+1}.$$

Note that the character $\mu \boxtimes \chi$ of $T_H(k) \times T_G(k)$ pulls back to the character

$$\mu \cdot \chi \coloneqq (\beta_1, \cdots, \beta_n, \alpha_1, \cdots, \alpha_m, st)$$

of $T_{R_0}(k)$ under the inclusion $T_{R_0} \subset T_H \times T_G$.

For Borel subgroups $B_G = T_G N_G \subset G$ and $B_H = T_H N_H \subset H$, the (normalized) principal series representations $\mathrm{Ind}_{B_G(k)}^{G(k)} \chi$ and $\mathrm{Ind}_{B_H(k)}^{H(k)} \mu$ possess unique irreducible spherical subquotients denoted $\pi_\chi$ and $\sigma_\mu$, respectively; note $\pi_\chi$ and $\sigma_\mu$ depend only on the Weyl orbits of $\chi$ and $\mu$.

**Proposition 4.3.3.** *Suppose $m \leq n$, $\epsilon = +$, and that the residue field of $k$ has odd cardinality $q$.*

*(1) Let $\pi_\chi$ be the spherical representation of $G(k)$ associated to $\chi = (\alpha_1, \cdots, \alpha_m, s)$, and suppose that $\pi'_\chi := \mathrm{Ind}_{G(k)}^{G'(k)} \pi_\chi$ is irreducible with multiplicity-free restriction to $G'_1(k)$. Then if $\Theta(\pi_\chi) \neq 0$, $\Theta(\pi_\chi)$ is the spherical representation $\sigma_\mu$ of $H(k)$ for*

$$\mu = (\alpha_1, \cdots, \alpha_m, q, q^2, \cdots, q^{n-m}, sq^{-(m^2-m)/4-(n^2+n)/4+nm/2}).$$

*(2) If $m \leq 3$ and $\pi'_\chi$ is irreducible, then $\pi'_\chi|_{G'_1(k)}$ is multiplicity-free.*

*Proof.* We first show (1). By [30, Theorem 4.4] and [31, Proposition 1.11], our assumptions on $\pi_\chi$ imply that $\Theta(\pi_\chi)$ is irreducible and spherical if it is nonzero; then $\Theta(\pi_\chi) = \sigma_\mu$ for some $\mu$, and it remains to determine $\mu$.

As in [28, §4], let $\sigma = (\sigma_1, \cdots, \sigma_m) \in \mathbb{C}^m$, and consider for all $\Re(\sigma_i) \gg 0$ the family of integrals:

$$I(\sigma, \phi) = \int \phi\left(\sum_{i=1}^m a_{ii} f_i^* \otimes e_i + \sum_{1\leq i<j\leq m} z_{ij} f_i^* \otimes e_j\right) \prod_{i=1}^m |a_{ii}|^{\sigma_i+i-m} \,\mathrm{d}^\times a_{ii} \prod_{i<j} \mathrm{d}z_{ij}, \tag{44}$$

where $\phi \in \mathcal{S}_k(W_2 \otimes V)$. Let $B_G$ be the unique Borel subgroup that stabilizes the complete isotropic flag $\langle e_m\rangle \subset \langle e_m, e_{m-1}\rangle \subset \cdots \subset V_1$; and similarly for $B_H$. A direct calculation shows that, if $\Re(\sigma_i) \gg 0$ for all $i$,

$$Z_\sigma(\phi)(h, g) := I(\sigma, \omega(h, g)\phi)$$

defines an $R'_0(k)$-intertwining map from $\omega$ to the induced representation

$$\begin{aligned} I_\sigma &:= \mathrm{Ind}_{T_{R_0}(k)\cdot(N_H\times N_G)(k)}^{R'_0(k)} \eta(\sigma), \\ \eta(\sigma) &= (q^{\sigma_1+1-m}, \cdots, q^{\sigma_i+i-m}, \cdots, q^{\sigma_m}, q, q^2, \cdots, q^{n-m}, \\ &\quad q^{m-\sigma_1-1}, \cdots, q^{m-\sigma_i-i}, \cdots, q^{-\sigma_m}, q^{-(m^2-m)/4-(n^2+n)/4+nm/2}) \in (\mathbb{C}^\times)^{n+m+1}. \end{aligned}$$

Now choose a hyperspecial subgroup $K_{R'_0}$ of $R'_0(k)$ (arising from choices of self-dual lattices in $W$ and $V$), and let $\mathcal{H}$ be the Hecke algebra of $\mathbb{C}$-valued, $K_{R'_0}$-biinvariant functions on $R'_0$. For all $\sigma$ as in the claim, the Hecke action on the unique spherical vector in $I_\sigma$ defines an algebra morphism $z_\sigma : \mathcal{H} \to \mathbb{C}$. It follows from the discussion in [28, p. 493] that the support of the $\mathcal{H}$-module

$$\mathcal{S}_k(W_2 \otimes V)^{K_{R'_0}}$$

is contained in the Zariski closure of the points $z_\sigma$ of $\operatorname{Spec}\mathcal{H}$. On the other hand, the Satake isomorphism identifies complex points of $\operatorname{Spec}\mathcal{H}$ with $R'_0$-Weyl orbits of parameters $\eta = (\beta_1, \cdots, \beta_m, \alpha_1, \cdots, \alpha_n, u)$ as above. By assumption, there is a surjection

$$\mathcal{S}_k(W_2 \otimes V) \twoheadrightarrow \pi_\chi^\vee \boxtimes \sigma_\mu,$$

and hence the character $\chi^{-1} \cdot \mu$ lies in the Zariski closure of the Weyl orbit of the parameters $\eta(\sigma)$ in the claim. However, the $\mu$ listed in the proposition is the only one (up to $H$-Weyl action) satisfying this property. This proves (1).

For (2), by [8, Lemma 2.1] it suffices to show that

$$\#\left\{\rho : G'(k) \to \mathbb{C}^\times \;:\; \rho|_{G'_1(k)} = 1,\, \pi_\chi \otimes \rho \cong \pi_\chi\right\} < 4.$$

So suppose given such a $\rho$. Since $\pi'_\chi$ is the unique spherical constituent of $\mathrm{Ind}_{B_G(k)}^{G'(k)} \chi$, $\pi'_\chi \otimes \rho$ is a constituent of $\mathrm{Ind}_{B_G(k)}^{G'(k)} \chi \otimes \rho$. For this induced representation to contain a spherical vector, $\rho$ must be unramified. In particular it is of the form $\rho_0 \circ \nu_{G'}$ for an unramified character $\rho_0$ of $k^\times$. By considering central characters of $\pi_\chi$ and $\pi_\chi \otimes \rho$, we also have $\rho_0^m = 1$; hence there are at most $m$ choices of $\rho$, and $m < 4$ by assumption. □

## 5. Yoshida lifts on $\mathrm{GSp}_4$

### 5.1. **Some four-dimensional orthogonal spaces.**

5.1.1. Let $B$ be a quaternion algebra, possibly split, over a field $k$. Then $B$ comes equipped with a norm $N : B \to k$ and an involution $b \mapsto b^*$ such that $bb^* = N(b)$ for all $b \in B$. The $k$-orthogonal space $V_B$ associated to $B$ is isomorphic to $B$ as a vector space, with the inner product defined by

$$(b_1, b_2) \coloneqq \mathrm{tr}(b_1 b_2^*) = b_1 b_2^* + b_2 b_1^*. \tag{45}$$

When $B$ is split, we often drop the subscript and abbreviate $V = V_{M_2}$.

5.1.2. One has a map of algebraic groups over $k$:

$$\boldsymbol{p}_Z : B^\times \times B^\times \to \mathrm{GO}(V_B) \tag{46}$$

defined by

$$\boldsymbol{p}_Z(b_1, b_2) \cdot x = b_1 x b_2^*, \quad x \in V_B.$$

The kernel of $\boldsymbol{p}_Z$ is the antidiagonally embedded $\mathbb{G}_m$, and $\boldsymbol{p}_Z$ is a surjection onto the connected similitude group $\mathrm{GSO}(V_B)$.

If $k$ is a local field, then irreducible admissible representations of $\mathrm{GSO}(V_B)(k)$ are all of the form $\pi_1 \boxtimes \pi_2$, where $\pi_i$ are irreducible admissible representations of $B^\times$ of the same central character; if $k = F$, the same is true of automorphic representations of $\mathrm{GSO}(V_B)(\mathbb{A}_F)$.

## 5.2. **Elliptic endoscopic $L$-parameters.**

5.2.1. The unique elliptic endoscopic group of $\mathrm{GSp}_{4,F}$ is $\mathrm{GSO}(V)$, equipped with the $L$-embedding:

$$ {}^L\mathrm{GSO}(V) = (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) \hookrightarrow \mathrm{GSp}_4(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) = {}^L\mathrm{GSp}_4. \tag{47}$$

The Langlands functoriality principle for the map (47) then suggests that, to an automorphic representation $\pi = \pi_1 \boxtimes \pi_2$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$, one can associate an $L$-packet of automorphic representations $\Pi(\pi_1, \pi_2)$ of $\mathrm{GSp}_4(\mathbb{A}_F)$. These $L$-packets and their local analogues are constructed via similitude theta lifting in [31, 38]. More precisely, for each place $v$ of $F$ and each irreducible admissible representation $\pi_{1,v} \boxtimes \pi_{2,v}$ of $\mathrm{GSO}(V)(F_v)$, one associates a local $L$-packet

$$\left\{\Pi^+(\pi_{1,v}, \pi_{2,v}), \Pi^-(\pi_{1,v}, \pi_{2,v})\right\}, \tag{48}$$

where by convention $\Pi^-(\pi_{1,v}, \pi_{2,v}) = 0$ unless both $\pi_{i,v}$ are discrete series. For all $v$, $\Pi^+(\pi_{1,v}, \pi_{2,v})$ is the unique generic member of the $L$-packet, and is explicitly given by the (nonzero, irreducible) local similitude theta lift:

$$\Pi^+(\pi_{1,v}, \pi_{2,v}) \coloneqq \Theta_{V, W_4}(\pi_{1,v} \boxtimes \pi_{2,v}). \tag{49}$$

If $\pi_{i,v}$ are both discrete series, then they admit Jacquet-Langlands transfers $\pi_{i,v}^B$ to $B^\times$, where $B$ is the non-split quaternion algebra over $F_v$. In this case, we have

$$\Pi^-(\pi_{1,v}, \pi_{2,v}) \coloneqq \Theta_{V_B, W_4}(\pi_{1,v}^B, \pi_{2,v}^B), \tag{50}$$

a nonzero irreducible representation. We remark that the central character of $\Pi^\pm(\pi_{1,v}, \pi_{2,v})$ is the common central character of $\pi_{i,v}$ (since the central character of the Weil representation is trivial). The $L$-packets associated to $\pi_v$ and $\pi_v' = \pi_{2,v} \boxtimes \pi_{1,v}$ coincide, but otherwise are all disjoint. Globally, given a cuspidal automorphic representation $\pi_1 \boxtimes \pi_2$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$ and a finite set $S$ of places where $\pi_i$ are both discrete series, we form the adelic representation

$$\Pi_S(\pi_1, \pi_2) \coloneqq \sideset{}{'}\bigotimes_{v \notin S} \Pi^+(\pi_{1,v}, \pi_{2,v}) \otimes \bigotimes_{v \in S} \Pi^-(\pi_{1,v}, \pi_{2,v}). \tag{51}$$

**Theorem 5.2.2.** *Let $\pi_1 \boxtimes \pi_2$ be a cuspidal automorphic representation of $\mathrm{GSO}(V)(\mathbb{A}_F)$, where $\pi_1 \not\cong \pi_2$. Then the automorphic multiplicity of $\Pi_S(\pi_1, \pi_2)$ is given by:*

$$m_{\mathrm{disc}}(\Pi_S(\pi_1, \pi_2)) = m_{\mathrm{cusp}}(\Pi_S(\pi_1, \pi_2)) = \begin{cases} 1, & \text{if } |S| \text{ is even,} \\ 0, & \text{if } |S| \text{ is odd.} \end{cases}$$

*The representations $\Pi_S(\pi_1, \pi_2)$ constitute a full near equivalence class in the discrete spectrum of $\mathcal{A}_{(2)}(\mathrm{GSp}_4(\mathbb{A}_F))$. They are not CAP, and are generic if and only if $S = \emptyset$. Moreoever, if $|S|$ is even,*

$$\Pi_S(\pi_1, \pi_2) = \Theta_{V_B, W_4}(\pi_1^B \boxtimes \pi_2^B),$$

*where $B$ is the unique $F$-quaternion algebra ramified at the set of primes $S$ and $\pi_i^B$ are the Jacquet-Langlands transfers of $\pi_i$ to $B^\times(\mathbb{A}_F)$.*

*Proof.* That $\Pi_S(\pi_1,\pi_2)$ is not CAP is [38, Lemma 5.2]. The multiplicity formula, and the fact that $\Pi_S(\pi_1,\pi_2)$ is a full near equivalence class, is [38, Theorem 5.2]. The genericity assertion follows from [38, Theorems 4.1 and 4.5(c), Corollary 4.16]. The nonvanishing of the global theta lift is [38, Corollary 5.5]; note that, given $\Theta_{V_B,W_4}(\pi_1^B\boxtimes\pi_2^B)\neq 0$, it is cuspidal if $\pi_1^B\neq\pi_2^B$ by [38, Theorem 4.3], and hence abstractly isomorphic to $\Pi_S(\pi_1,\pi_2)$ by Proposition 4.2.3. □

### 5.3. **Yoshida lifts in cohomology.**

5.3.1. Fix tuples $\boldsymbol{m}_1=(m_{1,v})_{v|\infty}$ and $\boldsymbol{m}_2=(m_{2,v})_{v|\infty}$ of integers such that $m_{1,v}\geq m_{2,v}+2\geq 4$ for all $v$ and all $m_{i,v}$ have the same parity. Let $\pi_1,\pi_2$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ of weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, with equal central characters of infinity type $\omega_{\boldsymbol{m}_i}$. For $v|\infty$, the Yoshida lifts $\Pi^\pm(\pi_{1,v},\pi_{2,v})$ of (5.2.1) form the discrete series $L$-packet of weight $(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)$ on $\mathrm{GSp}_4$, where

$$\begin{aligned}\ell_{1,v}&=\frac{m_{1,v}+m_{2,v}}{2},\\ \ell_{2,v}&=\frac{m_{1,v}-m_{2,v}+4}{2},\end{aligned}\tag{52}$$

and $\boldsymbol{\ell}_i=(\ell_{i,v})_{v|\infty}$. More precisely, the restriction of $\Pi^-(\pi_{1,v},\pi_{2,v})$ to $\mathrm{Sp}_4(F_v)$ is the direct sum of the holomorphic and anti-holomorphic discrete series with Harish-Chandra parameters $\lambda+\rho=(\ell_{1,v}-1,\ell_{2,v}-2)$ and $(2-\ell_{2,v},1-\ell_{1,v})$. (Here, $\rho=(2,1)$ is the half-sum of the standard choice of positive roots for $\mathrm{Sp}_4$.) The restriction of $\Pi^+(\pi_{1,v},\pi_{2,v})$ to $\mathrm{Sp}_4(F_v)$ is the direct sum of two generic discrete series representations with Harish-Chandra parameters $\lambda+\rho=(\ell_{1,v}-1,2-\ell_{2,v})$ and $(\ell_{2,v}-2,1-\ell_{1,v})$.

For a set $S_f$ of *finite* places of $F$ at which $\pi_i$ are both discrete series, set

$$\Pi_{S_f}=\bigotimes_{\substack{v\notin S_f\\ v\nmid\infty}}{}' \Pi^+(\pi_{1,v},\pi_{2,v})\otimes\bigotimes_{v\in S_f}\Pi^-(\pi_{1,v},\pi_{2,v}).$$

We consider the local system $\mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)}$ of $\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)$-vector spaces on $S(\mathbf{GSp_4})$ according to the conventions of §3.6 (the fields $\mathbb{Q}(\boldsymbol{m}_i)$ are defined in (3.7.1), and $\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)$ is the compositum).

For each $v|\infty$, let $\tau^+_{\ell_{1,v},\ell_{2,v}}$, resp. $\tau^-_{\ell_{1,v},\ell_{2,v}}$, be the unique irreducible representation of $K_{2,v}$ of central character $\omega^{-1}_{m_{i,v}}=\omega^{-1}_{\ell_{i,v}}$ whose restriction to $U(2)$ has highest weight $(\ell_{2,v}-2,-\ell_{1,v})$, resp. $(\ell_{1,v},2-\ell_{2,v})$. Similarly, let $\sigma^+_{\ell_{1,v},\ell_{2,v}}$, resp. $\sigma^-_{\ell_{1,v},\ell_{2,v}}$, be the unique irreducible representation of $K_{2,v}$ of central character $\omega^{-1}_{m_{i,v}}$ whose restriction to $U(2)$ has highest weight $(-\ell_{2,v},-\ell_{1,v})$, resp. $(\ell_{1,v},\ell_{2,v})$. Note that the duals of the $K_{2,v}$-types $\tau^\pm_{\ell_{1,v},\ell_{2,v}}$ appear with multiplicity one in $\Pi^+(\pi_{1,v},\pi_{2,v})$, and the duals of the $K_{2,v}$-types $\sigma^\pm_{\ell_{1,v},\ell_{2,v}}$ appear with multiplicity one in $\Pi^-(\pi_{1,v},\pi_{2,v})$, cf. [10, Table 2.2.1].

For a subset $S_\infty\subset\{v|\infty\}$ and a collection of signs $\boldsymbol{\epsilon}=\{\epsilon_v\}_{v|\infty}$, define the $\boldsymbol{K}_2$-representation

$$\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty}:=\bigotimes_{v\in S_\infty}\sigma^{\epsilon_v}_{\ell_{1,v},\ell_{2,v}}\otimes\bigotimes_{\substack{v\notin S_\infty\\ v|\infty}}\tau^{\epsilon_v}_{\ell_{1,v},\ell_{2,v}}.\tag{53}$$

Thus $\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty}$ is dual to a $\boldsymbol{K}_2$-type of $\Pi_S(\pi_1,\pi_2)$ with multiplicity one, if $S_\infty$ is the set of archimedean places in $S$.

5.3.2. Now let $(\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty))$ be the plectic Hodge type determined by:

$$(p_v(\boldsymbol{\epsilon},S_\infty),q_v(\boldsymbol{\epsilon},S_\infty))=\begin{cases}(3,0), & \epsilon_v=+,v\in S_\infty,\\ (2,1), & \epsilon_v=+,v\notin S_\infty,\\ (1,2), & \epsilon_v=-,v\notin S_\infty,\\ (0,3), & \epsilon_v=-,v\in S_\infty.\end{cases}\tag{54}$$

Thus $(\boldsymbol{p}(\boldsymbol{\epsilon},\emptyset),\boldsymbol{q}(\boldsymbol{\epsilon},\emptyset))=(\boldsymbol{p}(\boldsymbol{\epsilon})+1,\boldsymbol{q}(\boldsymbol{\epsilon})+1)$ in the notation of (3.7.1). An easy calculation shows that

$$\dim\mathrm{Hom}_{\boldsymbol{K}_2}\left(\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty},V_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}}\otimes\wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}\mathfrak{p}^*_{\mathrm{GSp}_4}\right)=1.\tag{55}$$

Hence, if $S = S_f \sqcup S_\infty$ is a finite set of places of $F$ with $|S|$ even, combining (30) and (55) yields a map (well-defined up to a scalar):

$$\mathrm{cl}_S^{\boldsymbol{\epsilon}} : \left(\Pi_S(\pi_1,\pi_2) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty}\right)^{\boldsymbol{K}_2} \to H^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}]. \tag{56}$$

**Proposition 5.3.3.** *The map* $\mathrm{cl}_S^{\boldsymbol{\epsilon}}$ *is an isomorphism of* $G(\mathbb{A}_{F,f})$*-representations, and moreover*

$$H^{\boldsymbol{p},\boldsymbol{q}}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}] = 0$$

*if* $(\boldsymbol{p},\boldsymbol{q})$ *is not of the form* $(\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty))$ *for some* $S_\infty$ *such that* $|S_f \sqcup S_\infty|$ *is even.*

*Proof.* That $\mathrm{cl}_S^{\boldsymbol{\epsilon}}$ is an injection follows from [35, Proposition 5.4] and the calculation of Casimir operators for $G$, cf. [10, p. 67]. The surjectivity and the vanishing of other plectic Hodge types follows from (24), Theorem 5.2.2, and the calculation of the nonvanishing $(\mathfrak{g}, K_2)$ cohomology groups:

$$\dim H^{3,0}(\mathfrak{g},K_2;\Pi_v^-(\pi_{1,v},\pi_{2,v})\otimes V_{(\ell_{1,v}-3,\ell_{2,v}-3),\mathbb{C}}) = \dim H^{0,3}(\mathfrak{g},K_2;\Pi_v^-(\pi_{1,v},\pi_{2,v})\otimes V_{(\ell_{1,v}-3,\ell_{2,v}-3),\mathbb{C}}) = 1,$$
$$\dim H^{2,1}(\mathfrak{g},K_2;\Pi_v^+(\pi_{1,v},\pi_{2,v})\otimes V_{(\ell_{1,v}-3,\ell_{2,v}-3),\mathbb{C}}) = \dim H^{1,2}(\mathfrak{g},K_2;\Pi_v^+(\pi_{1,v},\pi_{2,v})\otimes V_{(\ell_{1,v}-3,\ell_{2,v}-3),\mathbb{C}}) = 1.$$

The dimensions of these cohomology groups can be calculated from the main results of [35], and are also recalled in [34, §1]. □

5.3.4. Finally, we relate the $\Pi_{S_f}$-isotypic parts of the $L^2$ and singular cohomology.

**Proposition 5.3.5.** *Assume* $\Pi$ *is defined over* $E$*, where* $\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2) \subset E \subset \mathbb{C}$*. Then there exist* $\mathrm{GSp}_4(\mathbb{A}_{F,f})$*-equivariant isomorphisms*

$$H^*_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),E})[\Pi_{S_f}] \otimes_E \mathbb{C} \simeq H^*_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}]$$

*and*

$$\begin{aligned} H^*_c(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),E})[\Pi_{S_f}] &\simeq H^*(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),E})[\Pi_{S_f}] \\ &\simeq H^*_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),E})[\Pi_{S_f}]. \end{aligned}$$

*Proof.* By Theorem 5.2.2,

$$H^*_{\mathrm{cusp}}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}] \simeq H^*_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}],$$

and the first statement follows by the discussion in [34, p. 293]. The second assertion is an immediate consequence of Lemma 3.2.3 (and Poincaré duality), since $\Pi_{S_f}$ is not Eisenstein. □

## 6. Periods of Yoshida lifts

### 6.1. The period problem.

6.1.1. Let $\pi_1 \boxtimes \pi_2$ be a cuspidal automorphic representation of $\mathrm{GSO}(V)(\mathbb{A}_F)$, and let $\pi$ be an auxiliary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\pi^\vee$ and $\pi_i$ have the same central character. Consider the subgroup

$$H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_4$$

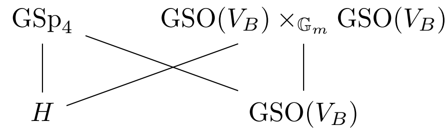
and the period integral $\mathcal{P}_{S,\pi_1,\pi_2,\pi} : \Pi_S(\pi_1,\pi_2)\otimes\pi \to \mathbb{C}$ defined by

$$\mathcal{P}_{S,\pi_1,\pi_2,\pi}(\alpha,\beta) = \int_{[Z_H\backslash H]} \alpha(h,h')\cdot\beta(h)\,\mathrm{d}(h,h'), \tag{57}$$

where $H(\mathbb{A}_F) \subset \mathrm{GSp}_4(\mathbb{A}_F)$ is parameterized by pairs $(h,h') \in \mathrm{GL}_2(\mathbb{A}_F)\times\mathrm{GL}_2(\mathbb{A}_F)$ such that $\det(h) = \det(h')$. When $\pi_1$, $\pi_2$, and $\pi$ are clear from context, we drop them from the notation $\mathcal{P}_{S,\pi_1,\pi_2,\pi}$. The goal of this section is to calculate $\mathcal{P}_{S,\pi_1,\pi_2,\pi}$ explicitly (Theorems 6.2.2 and 6.5.2). The result is applied to the cohomology of Shimura varieties in the next section.

6.1.2. Of course, we must specify a Haar measure on $[Z_H\backslash H]$ for (57) to be well-defined. Let $\mathcal{C} = \mathbb{A}_F^{\times,2}F^\times\backslash\mathbb{A}_F^\times$, and let $\mathrm{d}c$ be the Haar measure on $\mathcal{C}$ assigning volume 1 to the image of $\widehat{\mathcal{O}}_F$. As the measure on $[Z_H\backslash H]$, we take the measure induced by pullback from the surjection $[\mathrm{SL}_2]\times[\mathrm{SL}_2]\times\mathcal{C} \twoheadrightarrow [Z_H\backslash H]$. The Haar measure on $\mathrm{SL}_2$ is described in (2.2.2).

6.1.3. Before we begin the calculation of (57), we explain the seesaw diagram that lies behind it:




Here $B$ is the quaternion algebra ramified at $S$, the vertical lines are inclusions, and the diagonals are similitude dual pairs inside $\mathrm{GSp}_8$; the diagram corresponds to the two decompositions

$$W_4 \otimes V_B = W_2 \otimes V_B \oplus W_2 \otimes V_B$$

of $W_{16}$. Since $\Pi_S(\pi_1, \pi_2)$ is spanned by theta lifts $\theta_\phi(f_1 \otimes f_2)$ for $f_i \in \pi_i^B$, we wish to apply the formal seesaw identity:

$$\langle \theta_\phi(f_1 \otimes f_2)|_H, \beta \otimes \mathbb{1} \rangle_H = \langle f_1 \otimes f_2, \theta_\phi(\beta \otimes \mathbb{1})|_{\mathrm{GSO}(V_B)} \rangle_{\mathrm{GSO}(V_B)}, \tag{58}$$

Here $\beta \otimes \mathbb{1}$ is the automorphic form $(h, h') \mapsto \beta(h)$ on $H$. Now, the theta lift from $H$ to $\mathrm{GSO}(V_B) \times_{\mathbb{G}_m} \mathrm{GSO}(V_B)$ is simply two copies of the theta lift from $\mathrm{GL}_2$ to $\mathrm{GSO}(V_B)$; restriction to the diagonal amounts to multiplying the theta lifts of $\beta$ and $\mathbb{1}$ on $\mathrm{GSO}(V_B)$. The theta lift of $\beta$ to $\mathrm{GSO}(V_B)$ will be a vector in $\pi^B \boxtimes \pi^B$, where $\pi^B$ is the Jacquet-Langlands transfer. However, the theta lift of the constant function is formally divergent; to regularize it, we need a certain second-term Siegel-Weil formula. Ignoring this technicality, the theta lift $\theta_\phi(\beta \otimes \mathbb{1})$ restricted to the diagonal $\mathrm{GSO}(V_B)$ should be the product of a vector in $\pi^B \boxtimes \pi^B$ and an Eisenstein series on $\mathrm{GSO}(V_B)$. Of course, the Eisenstein series can only exist when $B$ is split, so (57) should vanish identically unless $S = \emptyset$. But when $S = \emptyset$, integrating $\theta_\phi(\beta \otimes 1)$ against the form $f_1 \otimes f_2$ gives a Rankin-Selberg integral that unfolds to an Euler product and ultimately an $L$-function.

Thus to compute $\mathcal{P}_{S,\pi_1,\pi_2,\pi}$, we first must dispatch the trivial case $S \neq \emptyset$, and then study the theta lift of both cusp forms and constant functions from $\mathrm{GL}_2$ to $\mathrm{GSO}(V)$. This is the content of the next three subsections.

## 6.2. Calculation of period integral: trivial case.

6.2.1. The trivial case $S \neq \emptyset$ can be handled easily:

**Theorem 6.2.2.** *If $S \neq \emptyset$, then $\mathcal{P}_S$ is identically zero.*

*Proof.* Let $B$ be the quaternion algebra over $F$ ramified exactly at $S$ (recall $|S|$ is even). By Theorem 5.2.2, it suffices to show the vanishing of all integrals of the form

$$I(\phi, g, f) = \int_{[Z_H \backslash H]} \theta_\phi(g)(h, h') \cdot f(h) \, \mathrm{d}(h, h'),$$

for $\phi \in \mathcal{S}_{\mathbb{A}}(\langle e_2, e_4 \rangle \otimes B)$ and $g \in \pi_1^B \boxtimes \pi_2^B$. Let us fix a place $v$ at which $B$ ramifies, and a Schwartz function $\phi^v \in \mathcal{S}_{\mathbb{A}_F{}^v}(W_2 \otimes B)$. Then, holding the other data $f, g$ fixed as well, consider the linear map $I_v : \mathcal{S}_{F_v}(W_2 \otimes B) \to \mathbb{C}$ defined by

$$\phi_v \mapsto I(\phi^v \otimes \phi_v, f, g). \tag{59}$$

Now $I_v$ clearly factors through the maximal quotient $Q$ of $\mathcal{S}_{F_v}(W_2 \otimes B) = \mathcal{S}_{F_v}(B \oplus B)$ on which $\{1\} \times \mathrm{SL}_2(F_v) \subset H(F_v) \subset \mathrm{GSp}_4(F_v)$ acts trivially. We claim this quotient is trivial. Indeed, the action of the Borel subgroup of $\{1\} \times \mathrm{SL}_2(F_v)$ is explicitly described by:

$$\begin{aligned} \omega\left(1 \times \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1\right) \phi_v(b_1, b_2) &= \psi\left(\frac{1}{2} n N(b_2)\right) \phi_v(b_1, b_2) \\ \omega\left(1 \times \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1\right) \phi_v &= |a|^2 \phi_v(b_1, a b_2). \end{aligned} \tag{60}$$

Since $B_v$ is anisotropic, it follows from the first equation that $\mathcal{S}_{F_v}(W_2 \otimes B) \to Q$ factors through $\phi_v \mapsto \phi_v(b_1, 0)$; then the second equation implies $Q = 0$. Therefore $I_v$ is identically zero for all choices of $(\phi^v, f, g)$, and in particular (since the adelic Schwartz space is generated by factorizable Schwartz functions) all the period integrals $I(\phi, f, g)$ vanish as well. ☐

## 6.3. Lifts of cuspidal representations from $\mathrm{GL}_2$ to $\mathrm{GSO}(V)$.

6.3.1. Since $V$ is split, the Weil representation for the pair $(W_2, V)$ has the alternate model given by the complete polarization $V = V_1 \oplus V_2$, where

$$V_1 = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \ V_2 = \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}. \tag{61}$$

6.3.2. Let $\pi$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. It is well-known that the theta lift $\Theta(\pi) \subset \mathcal{A}_0(\mathrm{GSO}(V)(\mathbb{A}_F))$ is isomorphic to the automorphic representation $\pi \boxtimes \pi$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$; for instance, this follows from strong multiplicity one for $\mathrm{GL}_2$ and a calculation of local Langlands parameters analogous to Proposition 4.3.3(1). To obtain our ultimate period formula, we will require the following calculation:

**Lemma 6.3.3.** *Let $\phi = \otimes_v \phi_v \in \mathcal{S}_{\mathbb{A}}(\langle e_2 \rangle \otimes W)$ and $f = \otimes_v f_v \in \pi$ be factorizable vectors, and choose a factorization*

$$W_{\psi,f}(h) = \prod_v W_{f,v}(h_v), \ \ h = (h_v) \in \mathrm{GL}_2(\mathbb{A}_F)$$

*of the global Whittaker function of $f$ (so that $W_f(h_v)(1) = 1$ for almost all $v$). Then the Whittaker coefficient of $\theta_\phi(f)$ along the standard unipotent subgroup $N \times N \subset \boldsymbol{p}_Z(\mathrm{GL}_2 \times \mathrm{GL}_2)$ is given by:*

$$\theta_\phi(f)(g)_{N\times N, \psi^{-1}\times\psi^{-1}} = \prod_v \left( \int_{\mathrm{SL}_2(F_v)} W_{f,v}(h_v h_{c_v}) \omega(h_v h_{c_v}, g)\widehat{\phi}(1,0,0,-1)\,\mathrm{d}h_v \right),$$
$$c = (c_v) = \det(g).$$

*Proof.* We compute in two steps. First, for $(h, g) \in R_0(\mathbb{A}_F)$,

$$\begin{aligned}
\theta(h, g; \phi)_{N\times 1, \psi^{-1}\times 1} &= \int_{[N]} \sum_{x \in W\otimes V_2} \omega(h, ng)\widehat{\phi}(x)\psi(n)\,\mathrm{d}n \\
&= \int_{F\backslash \mathbb{A}_F} \sum_{(z_1,w_1,z_2,w_2)} \psi(a(w_2 z_1 - z_2 w_1))\omega(h,g)\widehat{\phi}(z_1,w_1,z_2,w_2)\psi(a)\,\mathrm{d}a \\
&= \sum_{\substack{(z_1,w_1,z_2,w_2) \\ z_1 w_2 - w_1 z_2 = -1}} \omega(h,g)\widehat{\phi}(x) \\
&= \sum_{\gamma \in \mathrm{SL}_2(F)} \omega(\gamma h, g)\widehat{\phi}(1,0,0,-1).
\end{aligned}$$

Here $\mathrm{d}n$ is the Haar measure on $N$ such that $[N]$ has volume 1. Now, using the identity

$$\omega(nh, g)\widehat{\phi}(1,0,0,-1) = \omega(h, \boldsymbol{p}_Z(1,n)g)\widehat{\phi}(1,0,0,-1), \ \ (g,h) \in R_0, \ n \in N(\mathbb{A}),$$

we obtain:

$$\begin{aligned}
\theta_\phi(f)(g)_{N\times N, \psi^{-1}\times\psi^{-1}} &= \int_{[N]} \int_{[\mathrm{SL}_2]} \theta(hh_c, \boldsymbol{p}_Z(1,n)g;\phi)_{N\times 1,\psi^{-1}\times 1}\psi(n) f(hh_c)\,\mathrm{d}h\,\mathrm{d}n \\
&= \int_{[N]} \int_{\mathrm{SL}_2(\mathbb{A})} \omega(hh_c, \boldsymbol{p}_Z(1,n)g)\widehat{\phi}(1,0,0,-1)\psi(n) f(hh_c)\,\mathrm{d}h\,\mathrm{d}n \\
&= \int_{[N]} \int_{\mathrm{SL}_2(\mathbb{A})} \omega(hh_c, g)\widehat{\phi}(1,0,0,-1)\psi(n) f(n^{-1}hh_c)\,\mathrm{d}n\,\mathrm{d}h \\
&= \int_{\mathrm{SL}_2(\mathbb{A})} \omega(hh_c, g)\widehat{\phi}(1,0,0,-1) W_{\psi,f}(hh_c)\,\mathrm{d}h,
\end{aligned}$$

which gives the lemma. □

### 6.4. **A Siegel-Weil identity for** $\mathrm{GSO}(V)$.

6.4.1. *Degenerate principal series for* $\mathrm{GSO}(V)$. The maximal isotropic subspace $V_1 \subset V$ of (61) has stabilizer

$$P = \boldsymbol{p}_Z(B \times \mathrm{GL}_2) \subset \mathrm{GSO}(V). \tag{62}$$

Let $v$ be a place of $F$, and consider the (normalized) induced representation

$$\boldsymbol{I}_v(s) = \mathrm{Ind}_{P(F_v)}^{\mathrm{GO}(V)(F_v)} \delta_{P_1}^s.$$

We also consider the induced representations $I_v(s) = \mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_2(F_v)} \delta_B^s$. Let $\tau \in \mathrm{GO}(V)(F_v)$ be an element such that:

$$\tau^2 = 1 \text{ and } \tau \boldsymbol{p}_Z(g_1, g_2)\tau = \boldsymbol{p}_Z(g_2, g_1). \tag{63}$$

Then the representations $I_v(s)$ and $\boldsymbol{I}_v(s)$ are related by the following observation.

**Proposition 6.4.2.** *The map*

$$M_v : \boldsymbol{I}_v(s) \to I_v(s) \oplus I_v(s)$$

*defined by*

$$M_v(\varphi)(g) = (\varphi(\boldsymbol{p}_Z(g,1)), \varphi(\tau \boldsymbol{p}_Z(1,g)))$$

*is a linear isomorphism and an intertwining map of* $\mathrm{GL}_2(F_v)\times\mathrm{GL}_2(F_v)$ *representations, if* $\mathrm{GL}_2(F_v)\times\mathrm{GL}_2(F_v)$ *acts on the left through the quotient* $\mathrm{GL}_2(F_v) \times \mathrm{GL}_2(F_v) \twoheadrightarrow \mathrm{GSO}(V)(F_v)$ *and on the right through the first (resp. second) projection on the first (resp. second) factor.*

Then by the well-known theory of principal series for $\mathrm{GL}_2$, we deduce:

**Corollary 6.4.3.** *For all places $v$, the representation $\boldsymbol{I}_v(1/2)$ has a unique irreducible subrepresentation, and the corresponding quotient is the direct sum of the trivial character and the sign character of* $\mathrm{GO}(V)(F_v)$.

6.4.4. Let $\boldsymbol{I}_v^0(1/2)$ be the kernel of the projection from $\boldsymbol{I}_v(1/2)$ to the sign character.

Consider the map

$$[\cdot] : S_{F_v}(\langle e_2\rangle \otimes V) \to \boldsymbol{I}_v(1/2)$$

defined by

$$[\phi](g) = \omega(h_{\nu(g)}, g)\widehat{\phi}(0).$$

A standard calculation shows that $[\cdot]$ is equivariant for the action of $R_0'(F_v) \subset \mathrm{GL}_2(F_v) \times \mathrm{GO}(V)(F_v)$ on both sides, where $R_0'(F_v)$ acts on $\boldsymbol{I}_v(1/2)$ through the projection $R_0' \twoheadrightarrow \mathrm{GO}(V)$. We may then extend $[\phi]$ to a holomorphic section $[\phi](s) \in \boldsymbol{I}_v(s)$ by requiring the restriction of $[\phi](s)$ to the maximal compact subgroup $K_0 \subset \mathrm{GO}(V)(F_v)$ to be independent of $s$.

**Lemma 6.4.5.** *For any place $v$ of $F$:*

1. *The image of $\phi \mapsto [\phi]$ is $\boldsymbol{I}_v^0(1/2)$.*
2. *We have* $\dim \mathrm{Hom}_{\mathrm{SL}_2(F_v)\times\mathrm{SO}(F_v)}(S_{F_v}(\langle e_2\rangle \otimes V), \mathbb{C}) = \dim \mathrm{Hom}_{R_0'(F_v)}(S_{F_v}(\langle e_2\rangle \otimes V), \mathbb{C}) = 1$.

*Proof.* Suppose first that $v$ is nonarchimedean. Then (1) follows from comparing [7, Proposition 5.2(iii)] with Corollary 6.4.3. By [29, Theorem II.1.1], $[\cdot]$ realizes its image as the maximal quotient of $S_{F_v}(\langle e_2\rangle \otimes V)$ on which $\mathrm{SL}_2(F_v)$ acts trivially, so (2) follows from (1).

Now suppose $v$ is archimedean, and let $S_{F_v}(\langle e_2\rangle \otimes V)_{\mathrm{SL}_2(F_v)}$ be the maximal quotient on which $\mathrm{SL}_2(F_v)$ acts trivially. By [13, Theorem 1A], $S_{F_v}(\langle e_2\rangle \otimes V)_{\mathrm{SL}_2(F_v)}$ has a unique irreducible quotient $\rho$; moreover, the proof of this theorem in §4 of *op. cit.* implies that $\rho$ contains a spherical vector for the maximal compact subgroup of $\mathrm{O}(V)(F_v)$. (For the latter claim, see also [13, §7(b)] and the explicit description of the $K$-type correspondence in [10, Proposition 4.2.1].) Since $S_{F_v}(\langle e_2\rangle \otimes V)_{\mathrm{SL}_2(F_v)}$ surjects onto the image of $[\cdot]$ by definition, we conclude that the image of $[\cdot]$ is contained in $\boldsymbol{I}_v^0(1/2)$ and contains a spherical vector, hence (1) holds. Finally, (2) is immediate from (1), Corollary 6.4.3, and the fact that $S_{F_v}(\langle e_2\rangle \otimes V)_{\mathrm{SL}_2(F_v)}$ has a unique irreducible quotient. ∎

6.4.6. *Eisenstein series on GSO(V).* Let $\boldsymbol{I}(s) = \mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GO}(V)(\mathbb{A}_F)} \delta_P^s$ be the global parabolic induction, and for holomorphic sections $\varphi(s) \in \boldsymbol{I}(s)$ consider the Eisenstein series:

$$\boldsymbol{E}(g, s; \varphi) = \sum_{\gamma \in P(F) \backslash \mathrm{GO}(V)(F)} \varphi(s)(\gamma g), \quad g \in \mathrm{GSO}(V)(\mathbb{A}_F), \tag{64}$$

which converges for $\Re(s) \gg 0$. We also consider $I(s) = \mathrm{Ind}_{B(\mathbb{A}_F)}^{GL_2(\mathbb{A}_F)} \delta_B^s$ and, for holomorphic sections $\varphi(s) \in I(s)$, the corresponding family of Eisenstein series:

$$E(g, s; \varphi) = \sum_{\gamma \in B(F) \backslash \mathrm{GL}_2(F)} \varphi(s)(\gamma g), \quad g \in \mathrm{GL}_2(\mathbb{A}_F). \tag{65}$$

Let

$$M = (M_1, M_2) : \boldsymbol{I}(s) \to I(s) \oplus I(s)$$

be the intertwining map given by

$$\begin{aligned} M_1(\varphi)(g) &= \varphi(\boldsymbol{p}_Z(g, 1)) \\ M_2(\varphi)(g) &= \varphi(\tau \boldsymbol{p}_Z(1, g)), \end{aligned} \tag{66}$$

where $\tau \in O(V)(F)$ satisfies (63). This is a restricted tensor product of local maps $M_v = (M_{1,v}, M_{2,v})$.

**Proposition 6.4.7.** *We have*

$$\boldsymbol{E}(\boldsymbol{p}_Z(g_1, g_2); s, \varphi) = E(g_1; s, M_1(\varphi)) + E(g_2; s, M_2(\varphi))$$

*as functions on $\mathbb{C} \times GL_2(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$ for $\Re(s) \gg 0$ and holomorphic sections $\varphi \in \boldsymbol{I}(s)$.* □

By Proposition 6.4.7 and the well-known theory of Eisenstein series for $\mathrm{GL}_2$, $\boldsymbol{E}(g, s; \varphi)$ has a meromorphic continuation to $s \in \mathbb{C}$, with at most a simple pole at $s = \frac{1}{2}$. Let

$$[\cdot] : \mathcal{S}_{\mathbb{A}_F}(\langle e_2 \rangle \otimes V) \to \boldsymbol{I}(1/2) \tag{67}$$

be the tensor product of the local maps, and similarly for $[\cdot](s)$. For each $\phi \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2 \rangle \otimes V)$, we consider the Laurent series expansion:

$$\boldsymbol{E}(g, s; [\phi]) = \frac{A_{-1}(g; \phi)}{s - \frac{1}{2}} + A_0(g; \phi) + \cdots, \quad g \in \mathrm{GSO}(V)(\mathbb{A}_F). \tag{68}$$

**Lemma 6.4.8.** *For each $\phi$, $A_{-1}(g; \phi)$ is a constant function of $g$. Moreover, the linear map*

$$A_0 : \mathcal{S}_{\mathbb{A}_F}(\langle e_2 \rangle \otimes V) \to \mathcal{A}(\mathrm{GSO}(V)(\mathbb{A}_F))$$

*is an $R_0(\mathbb{A}_F)$-intertwining operator modulo constant functions.*

*Proof.* The first claim is immediate from Proposition 6.4.7. For the second, the proof of [7, Proposition 6.4] applies almost verbatim, taking into account Lemma 6.4.5(1). □

6.4.9. *The spherical Eisenstein series.* Let $\varphi^0(s) \in I(s)$ be the unique $\mathrm{GL}_2(\widehat{\mathcal{O}}_F) \cdot \mathrm{SO}(2)$-spherical section such that $\varphi^0(s)(1) = 1$, and let

$$E_0(g, s) := E(g, s; \varphi^0) \tag{69}$$

be the resulting Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_F)$. We record the following:

**Proposition 6.4.10.** *The residue of $E_0(h, s)$ at $s = \frac{1}{2}$ is given by:*

$$\kappa = \frac{\pi^d \operatorname{Res}_{s=1} \zeta_F(s)}{2|D_F|^{\frac{1}{2}} \zeta_F(2)}.$$

*Proof.* Although this is standard, we give a sketch for the reader's convenience. In the Fourier expansion of $E_0(h, s)$, the non-constant Fourier coefficients are holomorphic. We therefore wish to calculate

$$\operatorname{Res}_{s=\frac{1}{2}} \frac{1}{\mathrm{Vol}([N])} \int_{[N]} E_0(n, s)\, dn,$$

where $\mathrm{d}n$ is the Haar measure on $N(\mathbb{A}_F)$ induced by the identification $N(\mathbb{A}_F) \simeq \mathbb{A}_F$ and (2.1.1). Unfolding, we obtain (using the Bruhat decomposition of $\mathrm{GL}_2$):

$$\begin{aligned}\frac{1}{\mathrm{Vol}([N])}\int_{[N]} E_0(n,s)\,\mathrm{d}n &= \frac{1}{\mathrm{Vol}([N])}\int_{[N]}\sum_{\gamma\in B(F)\backslash \mathrm{GL}_2(F)} \varphi^0(s)(\gamma n)\,\mathrm{d}n \\ &= \frac{1}{\mathrm{Vol}([N])}\int_{[N]}\varphi^0(s)(n)\,\mathrm{d}n + \frac{1}{\mathrm{Vol}([N])}\int_{[N]}\sum_{a\in N(\mathbb{Q})}\varphi^0(s)(w_0an)\,\mathrm{d}n \\ &= \frac{1}{\mathrm{Vol}([N])}\int_{[N]}\varphi^0(s)(n)\,\mathrm{d}n + \frac{1}{\mathrm{Vol}([N])}\int_{N(\mathbb{A}_F)}\varphi^0(s)(w_0n)\,\mathrm{d}n,\end{aligned}$$

where

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the Weyl element. The first term is holomorphic in $s$, so we may discard it and compute:

$$\frac{1}{\mathrm{Vol}([N])}\prod_v \int_{N(F_v)} \varphi^0(s)(w_0n_v)\,\mathrm{d}n_v,$$

where $\mathrm{d}n_v$ is the standard Haar measure assigning volume one to $\mathcal{O}_v$. By the Gindikin-Karpelevich formula (e.g. [4, Chapter 7]), this product is

$$\frac{1}{\mathrm{Vol}([N])}\left(\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\right)^d \prod_{v\nmid\infty}\frac{1-q_v^{-2s-1}}{1-q_v^{-2s}} = \frac{1}{\mathrm{Vol}([N])}\left(\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(s+1/2)}\right)^d \frac{\zeta_F(2s)}{\zeta_F(2s+1)}.$$

Taking residue at $s=\frac{1}{2}$, we obtain

$$\kappa = \frac{\pi^d \operatorname{Res}_{s=1}\zeta_F(s)}{2\,\mathrm{Vol}([N])\zeta_F(2)}.$$

Finally, we may calculate

$$\mathrm{Vol}([N]) = \mathrm{Vol}(F\backslash\mathbb{A}_F/\widehat{O}_F) = \mathrm{Vol}(\mathbb{R}^d/\mathcal{O}_F) = |D_F|^{\frac{1}{2}}$$

by strong approximation. □

6.4.11. *Regularized theta integrals.* We now recall the regularization, due to Kudla and Rallis [20], of the (non-convergent) theta integral

$$g \mapsto \int_{[\mathrm{SL}_2]} \theta(h_1h_{\nu(g)}, g;\phi)\,\mathrm{d}h_1, \quad g\in \mathrm{GSO}(V)(\mathbb{A}_F),$$

where $\phi \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$. The first step of the regularization is to define a certain central element $z$ of the universal enveloping algebra of $\mathfrak{sl}_2$; for the precise definition, see [20, §5.1]. Kudla-Rallis' regularized theta integral (adapted to the similitude case) is then:

(70)
$$I(g,s;\phi) \coloneqq \frac{1}{\kappa\cdot(4s^2-1)}\int_{[\mathrm{SL}_2]}\theta(g,h_1h_{\nu(g)};\omega(z)\phi)E_0(h_1,s)\,\mathrm{d}h_1, \qquad g\in\mathrm{GSO}(V)(\mathbb{A}_F).$$

(The factor of $4s^2-1$ is designed to cancel the effect of $\omega(z)$, cf. [20, §5.5]. Our normalization of $s$ differs from *loc. cit.* by a factor of two.) The regularized integral $I(g,s;\phi)$ is a meromorphic function of $s$ whose poles coincide with the poles of $E_0(h_1,s)$. The Laurent expansion about $s=\frac{1}{2}$ has the form:

(71)
$$I(g,s;\phi) = \frac{B_{-2}(g,\phi)}{\left(s-\frac{1}{2}\right)^2} + \frac{B_{-1}(g,\phi)}{s-\frac{1}{2}} + B_0(g,\phi) + \cdots$$

By definition, the linear maps

(72)
$$B_d : \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle\otimes V) \to \mathcal{A}(\mathrm{GSO}(V)(\mathbb{A}_F))$$

are $\mathrm{GSO}(V)(\mathbb{A}_F)$-equivariant, where $g\in\mathrm{GSO}(V)(\mathbb{A}_F)$ acts on the left by $\phi\mapsto\omega(h_{\nu(g)},g)\phi$.

**Theorem 6.4.12** (Gan-Qiu-Takeda)**.** *For all $\phi \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$ and all $g \in \mathrm{GSO}(V)(\mathbb{A}_F)$, we have:*

$$B_{-2}(g,\phi) = \mathrm{Vol}([\mathrm{SL}_2])A_{-1}(g,\phi)$$
$$B_{-1}(g,\phi) = \mathrm{Vol}([\mathrm{SL}_2])A_0(g,\phi) + C(\nu(g),\phi),$$

*where the volume of $[\mathrm{SL}_2]$ is taken with respect to $\mathrm{d}h_1$, and $C(\nu(g),\phi) \in \mathbb{C}$ is a constant.*

*Proof.* Fix $\phi$; it follows immediately from [7, Theorem 1.2] that the identities hold for all $g \in \mathrm{SO}(V)(\mathbb{A}_F)$, and for some constant $C(1,\phi)$. By Lemma 6.4.8, $A_{-1}(\cdot,\phi)$ is constant, so the map $\phi \mapsto B_{-2}(1,\phi)$ defines an $\mathrm{SL}_2(\mathbb{A}_F) \times \mathrm{SO}(V)(\mathbb{A}_F)$-invariant linear functional on $\mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$. In particular, $\phi \mapsto B_{-2}(1,\phi)$ is invariant for $R_0(\mathbb{A}_F)$ by Lemma 6.4.5(2), and we conclude that $B_{-2}(g,\phi)$ is a constant function of $g$, so the first identity holds.

For the second identity, for all $a \in \mathbb{A}_F^\times$, fix $g_a \in \mathrm{GSO}(V)(\mathbb{A}_F)$ with $\nu(g_a) = a$. Then by Lemma 6.4.8, for all $g \in \mathrm{GSO}(V)(\mathbb{A}_F)$ with $\nu(g) = a$ we have

$$A_0(g,\phi) = A_0(gg_a^{-1}, \omega(h_a, g_a)\phi) + C_1(a,\phi) \tag{73}$$

for some constant $C_1(a,\phi)$; similarly, because $B_{-2}$ is constant, $B_{-1}$ is an intertwining operator modulo constants, and in particular we have

$$B_{-1}(g,\phi) = B_{-1}(gg_a^{-1}, \omega(h_a, g_a)\phi) + C_2(a,\phi) \tag{74}$$

for some constant $C_2(a,\phi)$. Combining (73) and (74) with the identity for isometry groups gives

$$B_{-1}(g,\phi) = \mathrm{Vol}([\mathrm{SL}_2])\left(A_0(g,\phi) - C_1(a,\phi)\right) + C(1,\omega(h_a,g_a)\phi) + C_2(a,\phi)$$

for all $g$ with $\nu(g) = a$, which proves the theorem. □

## 6.5. **Calculation of the period: nontrivial case.**

6.5.1. We now assume that $S = \emptyset$, so that $\Pi = \Pi_\emptyset(\pi_1,\pi_2)$ is generic, and compute $\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}$.

**Theorem 6.5.2.** *(1) Choose vectors $\phi_1 \in \mathcal{S}_{\mathbb{A}_F}(\langle e_2\rangle \otimes V)$, $\phi_2 \in \mathcal{S}_{\mathbb{A}_F}(\langle e_4\rangle \otimes V)$, $\alpha \in \pi_1 \boxtimes \pi_2$, and $\beta \in \pi$. Then:*

$$\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}(\theta_{\phi_1\otimes\phi_2}(\alpha),\beta) = \mathrm{Val}_{s=\frac{1}{2}} \int_{[\mathrm{PGSO}(V)]} \boldsymbol{E}(g,s;[\phi_2])\alpha(g)\theta_{\phi_1}(\beta)(g)\,\mathrm{d}g,$$

*where $\mathrm{PGSO}(V)(\mathbb{A}_F) = \mathrm{PGL}_2(\mathbb{A}_F) \times \mathrm{PGL}_2(\mathbb{A}_F)$ is given the product Haar measure.*

*(2) $\mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}$ is identically zero unless $\pi$ is isomorphic to either $\pi_1^\vee$ or $\pi_2^\vee$.*

*(3) Suppose we are given factorizations:*

$$\phi_1 = \otimes_v \phi_{1,v} \in \mathcal{S}_{\mathbb{A}_F}(V), \;\; \phi_2 = \otimes_v \phi_{2,v} \in \mathcal{S}_{\mathbb{A}_F}(V),$$
$$\alpha = \otimes_v \alpha_v \in \pi_1 \boxtimes \pi_2, \quad \beta = \otimes_v \beta_v \in \pi_2^\vee,$$

*along with decompositions of the global Whittaker functions:*

$$\alpha_{N\times N,\psi\times\psi}(g) = \prod_v W_{\alpha,v}(g_v), \;\; g = (g_v) \in \mathrm{GSO}(V)(\mathbb{A}_F),$$

$$\beta_{N,\psi}(h) = \prod_v W_{\beta,v}(h_v), \;\; h = (h_v) \in \mathrm{GL}_2(\mathbb{A}).$$

*Then for a sufficiently large finite set of primes $S$, we have:*

$$\mathcal{P}_\emptyset(\theta_{\phi_1\otimes\phi_2}(\alpha),\beta) = 2|D_F|^{\frac{1}{2}} \cdot \pi^{-d} \frac{L^S(1,\pi_1 \times \pi_2^\vee)L^S(1,\mathrm{Ad}\,\pi_2)}{\zeta_F^S(2)} \prod_{v\in S} \frac{\mathcal{Z}_v(\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)}{1-q_v^{-1}}$$

*where $\mathcal{Z}_v(\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)$ is the local zeta integral:*

$$\int_{(N\times N\backslash \mathrm{PGSO}(V))(F_v)} \int_{SL_2(F_v)} W_{\alpha,v}(g)W_{\beta,v}(h_1h_c)\omega(h_1h_c,g)\widehat{\phi}_{1,v}(1,0,0,-1)\varphi^0(g_2)M_{1,v}[\phi_{2,v}](g_1)\,\mathrm{d}h_1\,\mathrm{d}g \tag{75}$$
$$c = \det(g_1g_2), \;\; g = \boldsymbol{p}_Z(g_1,g_2).$$

*Here $\varphi^0(g_2)$ is the standard spherical section of $I(1/2)$.*

*(4) The L-values $L^S(1,\pi_1\times\pi_2^\vee)$ and $L^S(1,\operatorname{Ad}\pi_2)$ are nonzero. Moreover, for each place $v$, there exist choices of $\phi_{i,v},\alpha_v$, and $\beta_v$ such that*

$$\mathcal{Z}_v(\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)\neq 0.$$

*Proof.* First, fix the Haar measure $\mathrm{d}g$ on $\mathrm{SO}(V)(\mathbb{A}_F)$ such that, under the surjective natural map $[\mathrm{SO}(V)]\times\mathcal{C}\to[\mathrm{PGSO}(V)]=[\mathrm{PGL}_2]\times[\mathrm{PGL}_2]$, the Haar measure on $[\mathrm{PGL}_2]\times[\mathrm{PGL}_2]$ induced from (2.2.2) pulls back to $\mathrm{d}g\,\mathrm{d}c$. We expand:

$$(*)\qquad \mathcal{P}_{\emptyset,\pi_1,\pi_2,\pi}\left(\theta_{\phi_1\otimes\phi_2}(\alpha),\beta\right)=\int_{[Z_H\backslash H]}\theta_{\phi_1\otimes\phi_2}(\alpha)(h,h')\beta(h)\,\mathrm{d}h,$$

which by definition is:

$$\begin{aligned}(*)&=\int_{[Z_H\backslash H]}\int_{[\mathrm{GSO}(V)^{\nu(h)}]}\theta(h,g;\phi_1)\theta(h',g;\phi_2)\alpha(g)\beta(h)\,\mathrm{d}g\,\mathrm{d}(h,h')\\&=\int_{\mathcal{C}}\int_{[\mathrm{SL}_2]}\int_{[\mathrm{SO}(V)]}\int_{[\mathrm{SL}_2]}\theta(hh_c,gg_c;\phi_1)\theta(h'h_c,gg_c;\phi_2)\alpha(gg_c)\beta(h)\,\mathrm{d}h\,\mathrm{d}g\,\mathrm{d}h'\,\mathrm{d}c\\&=\int_{\mathcal{C}}\int_{[\mathrm{SL}_2]}\int_{[\mathrm{SO}(V)]}\theta(h'h_c,gg_c;\phi_2)\theta_{\phi_1}(\beta)(gg_c)\alpha(gg_c)\,\mathrm{d}g\,\mathrm{d}h'\,\mathrm{d}c.\end{aligned}$$

Now, by the reasoning of [20, §5.5], the latter integral is equal to the residue at $s=\frac{1}{2}$ of:

$$\frac{1}{\kappa\cdot(4s^2-1)}\int_{\mathcal{C}}\int_{[\mathrm{SL}_2]}\int_{[\mathrm{SO}(V)]}\theta(hh_c,gg_c;\omega(z)\phi_2)E_0(h,s)\alpha(g)\theta_{\phi_1}(\beta)(gg_c)\,\mathrm{d}g\,\mathrm{d}h\,\mathrm{d}c,$$

which is meromorphic for $\Re(s)\gg 0$. Here $\kappa$ is as in Proposition 6.4.10. Now, by the principle of meromorphic continuation, we may interchange the integrals over $\mathrm{SL}_2$ and $\mathrm{SO}(V)$, and obtain:

$$\begin{aligned}(*)&=\int_{[\mathrm{PGSO}(V)]}B_{-1}(g,\phi_2)\alpha(g)\theta_{\phi_1}(\beta)(g)\,\mathrm{d}g\\&=\mathrm{Val}_{s=\frac{1}{2}}\int_{[\mathrm{PGSO}(V)]}\boldsymbol{E}(g,s;[\phi_2])\alpha(g)\theta_{\phi_1}(\beta)(g)\,\mathrm{d}g,\end{aligned}$$

by Theorem 6.4.12 and the cuspidality of $\alpha$. This is (1). For (2), since $\theta_{\phi_1}(\beta)(g)$ lies in the automorphic representation $\pi'\boxtimes\pi'$ of $\mathrm{GSO}(V)(\mathbb{A}_F)$, it is a linear combination of functions of the form

$$\boldsymbol{p}_Z(g_1,g_2)\mapsto f_1(g_1)f_2(g_2),\quad f_i\in\pi'.$$

Combining this observation with Proposition 6.4.7, it follows that $(*)$ is a linear combination of integrals of the form

$$(76)\qquad\begin{aligned}&\mathrm{Val}_{s=\frac{1}{2}}\int_{[\mathrm{PGL}_2\times\mathrm{PGL}_2]}E(g_1,s;M_1[\phi_2])\alpha(\boldsymbol{p}_Z(g_1,g_2))f_1(g_1)f_2(g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2,\\&\mathrm{Val}_{s=\frac{1}{2}}\int_{[\mathrm{PGL}_2\times\mathrm{PGL}_2]}E(g_2,s;M_2[\phi_2])\alpha(\boldsymbol{p}_Z(g_1,g_2))f_1(g_1)f_2(g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2.\end{aligned}$$

These clearly both vanish unless $f_2$ lies in either $\pi_1^\vee$ or $\pi_2^\vee$, which proves (2). In order to prove (3), suppose $\pi=\pi_2^\vee$. We replace $(*)$ with an equivalent integral that can be unfolded:

$$\begin{aligned}(*)&=\frac{1}{\kappa}\mathrm{Res}_{s=\frac{1}{2}}\int_{[\mathrm{PGSO}(V)]}E(g_1,s;M_1[\phi_2])E_0(g_2,s)\alpha(g)\theta_{\varphi_1}(\beta)(g)\,\mathrm{d}g,\quad g=\boldsymbol{p}_Z(g_1,g_2)\\&=\frac{1}{\kappa}\mathrm{Res}_{s=\frac{1}{2}}\int_{N(\mathbb{A})\times N(\mathbb{A})\backslash\mathrm{PGSO}(V)(\mathbb{A})}M_1[\phi_2](s)(g_1)\varphi^0(s)(g_2)\alpha_{N\times N,\psi\times\psi}(g)\theta_{\varphi_1}(\beta)_{N\times N,\psi^{-1}\times\psi^{-1}}(g)\,\mathrm{d}g.\end{aligned}$$

This factors into an Euler product

$$(*)=\frac{1}{\kappa}\mathrm{Res}_{s=\frac{1}{2}}\prod_v\mathcal{Z}_v(s,\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v),$$

where the local zeta integrals are (applying Lemma 6.3.3):

$$\int_{(N\times N\backslash \mathrm{PGSO}(V))(F_v)}\int_{SL_2(F_v)} W_{\alpha,v}(g)W_{\beta,v}(h_1h_c)\omega(h_1h_c,g)\widehat{\phi}_1(1,0,0,-1)\varphi^0(g_2)M_1[\phi_2](g_1)\,\mathrm{d}h_1\,\mathrm{d}g \tag{77}$$
$$c=\det(g_1g_2),\quad g=\boldsymbol{p}_Z(g_1,g_2).$$

At an unramified place $v$ such that $W_{\alpha,v}$, $W_{\beta,v}$, $\phi_{i,v}$ are all the standard spherical vectors, the inner integral

$$\int_{\mathrm{SL}_2(F_v)} W_{\beta,v}(h_1h_c)\omega(h_1h_c,g)\widehat{\phi}_1(1,0,0,-1)\,\mathrm{d}h_1$$

is exactly the standard spherical Whittaker function for $\pi_2^\vee\boxtimes\pi_2^\vee$, by the unramified theta correspondence. Then, the standard Rankin-Selberg calculations (see e.g. [15, Proposition 2.3]) show that we have the Euler factor

$$\mathcal{Z}_v(s,\phi_{1,v},\phi_{2,v},\alpha_v,\beta_v)=\frac{L_v(s+\frac{1}{2},\pi_1\boxtimes\pi_2^\vee)L_v(s+\frac{1}{2},\pi_2\boxtimes\pi_2^\vee)}{1-q_v^{-2}}.$$

Now the formula (3) follows by comparing with Proposition 6.4.10. The non-vanishing of the $L$-values in (4) is well-known; see for instance [33] and [12]. The non-vanishing of the local zeta integrals at ramified places also follows from the non-vanishing for Rankin-Selberg local zeta integrals, cf. [16]. ☐

## 7. Proof of main result: special cycles in the generic case

In this section, we apply the results of §6 to the cohomology of Shimura varieties. Since the Schwartz functions at the Archimedean places must be chosen rather carefully to obtain automorphic forms that contribute to cohomology, we begin with several local calculations.

### 7.1. Archimedean calculations.

7.1.1. We first establish some general conventions for the local Weil representation for the pair $(V,W_{2n})$ over $\mathbb{R}$, where $V=V_{M_2}$. Fix coordinates on $W_{2n}\otimes V$ by:

$$\begin{aligned}(\underline{x}_1,\cdots,\underline{x}_{2n})&\longleftrightarrow\sum e_i\otimes\underline{x}_i,\\ \underline{x}_i=(x_i,y_i,z_i,w_i)&\longleftrightarrow\begin{pmatrix}x_i & y_i\\ z_i & w_i\end{pmatrix}.\end{aligned}\tag{78}$$

Let $K_n\subset G=\mathrm{GSp}_{2n,\mathbb{R}}$ be as in (2.3.2), let $H=\mathrm{GSO}(V)$, let $R_0$ be as in (4.1.1), and let $L=Z_H\cdot\boldsymbol{p}_Z(\mathrm{SO}(2)\times\mathrm{SO}(2))\subset H(\mathbb{R})$. Also let $L_1\subset L$ be the kernel of $\nu_H$ restricted to $L$, so that $L_1=\boldsymbol{p}_Z(\mathrm{SO}(2)\times\mathrm{SO}(2))$. For any integers $m_1,m_2$ with $m_1\equiv m_2 \pmod 2$, let $\chi_{m_1,m_2}$ be the character of $L$ which is given by $\omega_{m_1}^{-1}=\omega_{m_2}^{-1}$ on $Z_H$ and by $\chi_{m_1}\boxtimes\chi_{m_2}$ on $L_1$. Finally let $(K\times L)_0=(K\times L)\cap R_0$.

7.1.2. Let $S^0(n)\subset\mathcal{S}_{F_v}(\langle e_2,\cdots,e_{2n}\rangle\otimes V)$ be the subspace of Schwartz functions of the form

$$\phi(\underline{x}_2,\cdots,\underline{x}_{2n})=p(\underline{x}_2,\cdots,\underline{x}_{2n})\exp(-\pi(|\underline{x}_2|^2+\cdots+|\underline{x}_{2n}|^2),$$

where $p$ is a polynomial, and let $S^0_d(n)\subset S^0(n)$ be the subset such that $p$ is homogeneous of degree $d$. As a $(\mathfrak{r}_0,(K_n\times L)_0)$-module, $S^0(n)$ is isomorphic to the Fock space $\mathcal{F}(n)$ of complex polynomials in $4n$ variables, see [13, §2]; the isomorphism does not preserve degrees, but it does carry $S^0_{\le d}(n)=\oplus_{i\le d}S^0_i(n)$ isomorphically onto $\mathcal{F}_{\le d}(n)$, the subspace of polynomials of degree less than or equal to $d$. The following proposition is the key fact we will need about the structure of the $(K_n\times L)_0$-module $S^0_{\le d}(n)$.

**Proposition 7.1.3.** *(1) Suppose the $U(n)$-representation of highest weight $(a_1,\cdots,a_n)$ appears in $S^0_{\le d}(n)$. Then $|a_1|+\cdots+|a_n|\le d$.*

*(2) If $m_1\equiv m_2 \pmod 2$ are integers such that $m_1=\pm m_2$ if $n=1$, define*

$$a=\frac{|m_1+m_2|}{2},\quad b=\frac{|m_1-m_2|}{2},$$

*and let $\tau$ be the unique representation of $K_n$ whose restriction to $\mathbb{R}^\times$ is $\omega_m^{-1}$ and which has weight $(a,0,\cdots,0,-b)$ when restricted to $U(n)$. Then*

$$\dim\left(S^0_{\le a+b}(n)\otimes\tau\otimes\chi^\vee_{m_1,m_2}\right)^{(K_n\times L)_0}=1.$$

*Proof.* This follows from [10, Proposition 4.2.1]; see Remark 3.2.2 of *op. cit.* to translate the $O(2) \times O(2)$ parameters into $\boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2))$-parameters. $\square$

In our calculations below, we will need two explicit Schwartz functions, described by the next two propositions.

**Proposition 7.1.4.** *Suppose $n = 1$ and $m \geq 0$, and choose $\epsilon \in \{\pm 1\}$. Then:*

(1) *A generator for the one-dimensional space*

$$(S^0_{\leq m}(1) \otimes \chi_{\epsilon m} \otimes \chi^\vee_{m,\epsilon m})^{(K_1 \times L)_0}$$

*is given by*

$$\phi^\epsilon_m(x, y, z, w) := (x + \epsilon iy + iz - \epsilon w)^m \exp(-\pi|\underline{x}|^2).$$

(2) *If $m \geq 2$ and $\pi_m$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $m \geq 2$, then the local theta correspondence yields a map*

$$\mathcal{S}(\langle e_2 \rangle \otimes V) \twoheadrightarrow (\pi_m \boxtimes \pi_m)^\vee \boxtimes \pi_m$$

*of $R_0(\mathbb{R})$-representations, under which $\phi^\epsilon_m$ has nontrivial image.*

*Proof.* For (1), it suffices to show that for all

$$(k, \boldsymbol{p}_Z(k_1, k_2)) \in U(1) \times \boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2)) \subset \mathrm{SL}_2(F_v) \times \mathrm{SO}(V)(F_v),$$

we have:

$$\omega(k, \boldsymbol{p}_Z(k_1, k_2))\phi^\epsilon_m = \chi_{-\epsilon m}(k)\chi_m(k_1)\chi_{\epsilon m}(k_2)\phi^\epsilon_m.$$

The action of $\boldsymbol{p}_Z(\mathrm{SO}(2) \times \mathrm{SO}(2))$ can be checked directly.

For the action of $U(1) \subset \mathrm{SL}_2$, we calculate on the Lie algebra level using the following formulas for differential $\mathrm{d}\omega$ of the Weil representation:

$$\mathrm{d}\omega\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right) = \frac{1}{2\pi i}\left(\frac{\partial^2}{\partial z \partial y} - \frac{\partial^2}{\partial w \partial x}\right),$$
$$\mathrm{d}\omega\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right) = 2\pi i(xw - yz).$$

Since

$$\mathrm{d}\omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right)\phi^\epsilon_m = -im\epsilon\phi^\epsilon_m,$$

(1) follows.

For (2), the local theta correspondence between $\mathrm{GSp}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GSO}(V)$ is well-known, see e.g. [10, Proposition 4.4.2], and the non-vanishing of the image of $\phi^\epsilon_m$ follows from the discussion in [13, p. 545]. $\square$

**Proposition 7.1.5.** *Let $m \geq 0$ be an even integer. Then the Schwartz function*

$$\phi^0_m := \left((x + iz)^2 + (y + iw)^2\right)^{\frac{m}{2}} \exp(-\pi|\underline{x}|^2) \in S^0_m(1)$$

*satisfies*

$$\omega(k, \boldsymbol{p}_Z(k_1, k_2))\phi^0_m = \chi_m(k_1)\phi^0_m.$$

*Proof.* This is a direct calculation, similar to Proposition 7.1.4. $\square$

7.1.6. For the remainder of this subsection, fix $n = 2$. We now define the vector-valued Schwartz functions adapted to constructing cohomology classes on Shimura varieties as in §3, and compute a related local zeta integral. As in (5.3.1), let $m_1 \geq m_2 \geq 2$ be integers such that $m_1 \equiv m_2 \pmod 2$, and let

$$\ell_1 = \frac{m_1 + m_2}{2},$$
$$\ell_2 = \frac{m_1 - m_2 + 4}{2}.$$

Fix a choice of sign $\epsilon \in \{\pm\}$, and let $\tau^\epsilon_{\ell_1,\ell_2}$ be the representation of $K_2$ defined in (5.3.1). Let $\varphi^\epsilon_{m_1,m_2}$ be a generator of the one-dimensional space

$$\left(S^0_{\leq m_1}(2) \otimes \tau^\epsilon_{\ell_1,\ell_2} \otimes \chi^\vee_{m_1,-\epsilon m_2}\right)^{(K_2 \times L)_0}.$$

We denote by $\overline{\varphi}^{\epsilon}_{m_1,m_2} \in S^0_{\leq m_1}(2)$ the projection to the component of $\tau^{\epsilon}_{\ell_1,\ell_2}$ of weight $(-\epsilon m_2, 0)$ for the maximal torus of $U(2)$. Both $\varphi^{\epsilon}_{m_1,m_2}$ and $\overline{\varphi}^{\epsilon}_{m_1,m_2}$ are well-defined only up to scalar.

**Remark 7.1.7.** In practice, it would suffice to make a single choice of $\epsilon$ at this point; we have included both for maximum clarity and for the convenience of the reader.

Calculating $\overline{\varphi}^{\epsilon}_{m_1,m_2}$ explicitly would be highly tedious; a convenient shortcut is given by the following proposition.

**Proposition 7.1.8.** *Let $\pi_{m_i}$ denote the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $m_i$ and central character $\omega_{m_i}$. Then under the canonical projection*

$$S^0_{\leq m_1}(2) \to (\pi_{m_1} \boxtimes \pi_{m_2})^{\vee} \boxtimes \Pi^+(\pi_{m_1}, \pi_{m_2})$$

*arising from the archimedean local theta correspondence, the image of $\phi^{-\epsilon}_{m_2} \otimes \phi^0_{m_1-m_2}$ lies in the linear span of the image of $\overline{\varphi}^{\epsilon}_{m_1,m_2}$.*

In fact, both images are nontrivial; see Remark 7.1.10.

*Proof.* By Propositions 7.1.4 and 7.1.5, $\phi^{-\epsilon}_{m_2} \otimes \phi^0_{m_1-m_2}$ is a vector of weight $(m_1, -\epsilon m_2)$ for $\boldsymbol{p}_Z(\mathrm{SO}(2)\times\mathrm{SO}(2))$ and of weight $(\epsilon m_2, 0)$ for the maximal torus of $U(2)$. Since the $\boldsymbol{p}_Z(\mathrm{SO}(2)\times\mathrm{SO}(2))$-type $(m_1, -\epsilon m_2)$ appears with multiplicity one in $\pi_{m_1}\boxtimes\pi_{m_2}$, and since the dual of $\tau^{\epsilon}_{\ell_1,\ell_2}$ appears with multiplicity one in $\Pi^+(\pi_{m_1}, \pi_{m_2})$, it suffices to show that the only $U(2)$-type appearing in both

$$U(2) \cdot \left(\phi^{-\epsilon}_{m_2} \otimes \phi^0_{m_1-m_2}\right) \subset S^0_{\leq m_1}(2)$$

and $\Pi^+(\pi_{m_1}, \pi_{m_2})$ is $\tau^{\epsilon}_{\ell_1,\ell_2}|_{U(2)}$. Indeed, if a $U(2)$-type of highest weight $(a, b)$ appears in $U(2)\cdot\left(\phi^{-\epsilon}_{m_2} \otimes \phi^0_{m_1-m_2}\right)$, then we have $|a| + |b| \leq m_1$ (Proposition 7.1.3), $a + b = \epsilon m_2$, and $a \geq \epsilon m_2 \geq b$. Hence the possible $(a, b)$ are:

$$(a,b) = \begin{cases} (m_2, 0),\ (m_2+1, -1), \ldots, (\ell_1, 2-\ell_2), & \epsilon = +, \\ (0, -m_2),\ (1, -m_2-1), \ldots, (\ell_2-2, -\ell_1), & \epsilon = -. \end{cases} \tag{79}$$

On the other hand, recall that $\Pi^+(\pi_{m_1}, \pi_{m_2})|_{\mathrm{Sp}_4(\mathbb{R})}$ is a direct sum of two discrete series representations, with Harish-Chandra parameters $\lambda + \rho = (\ell_{1,v} - 1, 2 - \ell_{2,v})$ and $(\ell_{2,v} - 2, 1 - \ell_{1,v})$. By the Blattner formula [11], it follows that the only $U(2)$-types in (79) that can appear in $\Pi^+(\pi_{m_1}, \pi_{m_2})$ are $(\ell_1, 2-\ell_2)$ and $(\ell_2-2, -\ell_1)$, precisely the duals of $\tau^{\epsilon}_{\ell_1,\ell_2}$. ☐

Finally, for our later applications, we now calculate an archimedean local zeta integral related to $\phi^{-\epsilon}_{m_2} \otimes \phi^0_{m_1-m_2}$. For each integer $n \geq 2$ and pair of signs $\epsilon, \delta \in \{\pm\}$, let $W^{\epsilon}_{n,\psi^{\delta}}$ be the normalized weight $\epsilon n$ vector in the $\psi^{\delta}$-Whitaker model of the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $n$; thus

$$W^{\epsilon}_{n,\psi^{\delta}} \begin{pmatrix} \epsilon\delta t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} = t^{n/2} e^{-2\pi t}, \quad W^{\epsilon}_{n,\psi^{\delta}} \begin{pmatrix} -\epsilon\delta t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} = 0, \quad \forall t > 0, \tag{80}$$

see [27, Proposition 3.2.1].

**Proposition 7.1.9.** *With notation as above, let $v|\infty$ be a place of $F$ and identify $F_v \simeq \mathbb{R}$. Then*

$$\mathcal{Z}_v\left(\phi^{-\epsilon}_{m_2}, \phi^0_{m_1-m_2}, W^-_{m_1,\psi} \otimes W^{\epsilon}_{m_2,\psi}, W^{-\epsilon}_{m_2,\psi}\right) \neq 0.$$

*Proof.* First, we consider the inner integral for $\mathcal{Z}_v\left(\phi^{-\epsilon}_{m_2}, \phi^0_{m_1-m_2}, W^-_{m_1,\psi} \otimes W^{\epsilon}_{m_2,\psi}, W^{-\epsilon}_{m_2,\psi}\right)$:

$$\begin{aligned} I(g_1, g_2) = \int_{\mathrm{SL}_2(F_v)} \omega(h_1 h_c, \boldsymbol{p}_Z(g_1, g_2)) \widehat{\phi}^{-\epsilon}_{m_2}(1, 0, 0, -1) W^{-\epsilon}_{m_2,\psi}(h_1 h_c)\, \mathrm{d}h_1, \\ c = \det(g_1)\det(g_2). \end{aligned} \tag{81}$$

Because the proof of Lemma 6.3.3 identifies $I(g_1, g_2)$ with a local Whittaker function in the discrete series representation $\pi_{m_2} \boxtimes \pi_{m_2}$, by Proposition 7.1.4 we have

$$I(g_1, g_2) = \lambda W^+_{m_2,\psi^{-1}}(g_1) W^{-\epsilon}_{m_2,\psi^{-1}}(g_2) \tag{82}$$

for a scalar $\lambda \neq 0$.

By the equivariance properties of $\phi^0_{m_1-m_2}$,

$$M_{1,v}[\phi^0_{m_1-m_2}] \in \mathrm{Ind}_{\overline{B}(\mathbb{R})}^{\mathrm{PGL}_2(\mathbb{R})} |\cdot|^{1/2}$$

is a section of weight $m_1 - m_2$ for SO(2). Thus it is determined by:

$$\mu := M_{1,v}[\phi^0_{m_1-m_2}](1) = \int \phi^0_{m_1-m_2}(x,y,0,0)\,\mathrm{d}x\,\mathrm{d}y = \frac{\left(\frac{m_1-m_2}{2}\right)!}{\pi^{\frac{m_1-m_2}{2}}} \neq 0. \tag{83}$$

Now, our local zeta integral is given by

$$\lambda\left(\int_{N\backslash \mathrm{PGL}_2(F_v)} W^-_{m_1,\psi}(g_1) M_{1,v}[\phi^0_{m_1-m_2}](g_1) W^+_{m_2,\psi^{-1}}(g_1)\,\mathrm{d}g_1\right) \cdot \left(\int_{N\backslash \mathrm{PGL}_2(F_v)} W^\epsilon_{m_2,\psi}(g_2) W^{-\epsilon}_{m_2,\psi^{-1}}(g_2)\varphi^0(g_2)\,\mathrm{d}g_2\right). \tag{84}$$

Since both integrands are right SO(2)-invariant, and since the Haar measure on $\mathrm{PGL}_2(\mathbb{R})$ is given by

$$\mathrm{d}g = \frac{\mathrm{d}a\,\mathrm{d}t\,\mathrm{d}\theta}{\pi t^2}, \quad g = \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}, \quad t \in \mathbb{R}^\times, \theta \in [0,\pi),$$

we obtain

$$\begin{aligned}\lambda\mu\left(\int_0^\infty t^{\frac{m_1+m_2}{2}-1} e^{-4\pi t}\,\mathrm{d}t\right)\cdot\left(\int_0^\infty t^{m_2-1}e^{-4\pi t}\,\mathrm{d}t\right)\\ = \frac{\lambda\mu\left(\frac{m_1+m_2}{2}-1\right)!(m_2-1)!}{(4\pi)^{\frac{m_1+3m_2}{2}}} \neq 0,\end{aligned} \tag{85}$$

as claimed.

∎

**Remark 7.1.10.** Proposition 7.1.9 implies that $\phi^{-\epsilon}_{m_2} \otimes \phi^0_{m_1-m_2}$ has nonzero image under the local theta correspondence map of Proposition 7.1.8; otherwise, the local zeta integral in Proposition 7.1.9 would have to vanish by Theorem 6.5.2, the local-global compatibility of the theta correspondence, and an easy globalization argument.

## 7.2. **Cohomological span of special cycle.**

7.2.1. Let $H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_4$, viewed as an algebraic group over $F$. Then $\boldsymbol{H}$ possesses a Shimura datum, and we have a natural embedding of pro-algebraic varieties

$$\iota : S(\boldsymbol{H}) \hookrightarrow S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2),$$

induced from the map on the level of groups: $(h_1, h_2) \mapsto ((h_1,h_2),h_1)$. For all weights $\boldsymbol{m}_1, \boldsymbol{m}_2$ as in (5.3.1) above, abbreviate by $\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2}$ the Betti local system $\mathcal{V}^\vee_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)} \boxtimes \mathcal{V}_{\boldsymbol{m}_2-2}$ on $S(\mathbf{GSp}_4)_\mathbb{C} \times S(\mathbf{GL}_2)_\mathbb{C}$. Note that, by [17, Theorem 2.5], the constant local system $\underline{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)}$ on $S(\boldsymbol{H})$ is a direct factor with multiplicity one of the pullback $\iota^*(\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})$, and in particular, we have a composite map (well-defined up to a scalar):

$$H^{4d}_c(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2), \mathcal{W}^\vee_{\boldsymbol{m}_1,\boldsymbol{m}_2}) \to H^{4d}_c(S(\boldsymbol{H}), \iota^*(\mathcal{W}^\vee_{\boldsymbol{m}_1,\boldsymbol{m}_2})) \to H^{4d}_c(S(\boldsymbol{H}), \mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)). \tag{86}$$

**Definition 7.2.2.** The cycle class $[\mathcal{Z}] \in H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})(2d)$ is the image of the fundamental class of $S(\boldsymbol{H})$ under the map

$$H^0(S(\boldsymbol{H}), \mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)) \to H^{2d}(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})(2d)$$

induced by the dual of (86). We write

$$[\mathcal{Z}]_* : H^{3d}_c(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)})(d) \to H^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2})$$

for the induced map.

7.2.3. Let $\pi$ be an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\boldsymbol{m}_2$ whose central character agrees with that of $\pi_1$ and $\pi_2$. If $\pi$ is defined over $E$, recall that the trace map induces a perfect pairing:

$$(87)\quad \langle\cdot,\cdot\rangle \; : \; H^d_c(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2,E})[\pi_f] \times H^d(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}_2-2,E})[\pi_f^\vee] \;\to\; H^{2d}_c(S(\mathbf{GL}_2), E) \;\to\; E(d).$$

**Proposition 7.2.4.** *Let $\pi$ be as above, and let $\pi_1, \pi_2$ be as in (5.3.1), with $\Pi = \Pi_S(\pi_1, \pi_2)$, for some $S = S_f \sqcup S_\infty$ such that $|S|$ is even.*

*(1) For choices of signs $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$, let $s_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'} : \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty} \to \mathbb{C}$ be the projection onto the weight $(-\boldsymbol{\epsilon}'\boldsymbol{m}_2, 0)$-component (hence $\sigma$ is trivial unless $S_\infty = \emptyset$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$). Then the following diagram commutes up to a nonzero scalar:*

$$\begin{array}{ccc}
\left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty}\right)^{\boldsymbol{K}_2} \otimes \left(\pi^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1} & \xrightarrow{s_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'}\otimes \mathrm{id}} & \Pi \otimes \pi^\vee \\
\downarrow \mathrm{cl}^{\boldsymbol{\epsilon}}_S \otimes \mathrm{cl}'_{\boldsymbol{\epsilon}'} & & \\
H^{3d}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}] \otimes H^d_{(2)}(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}_2-2,\mathbb{C}})[\pi_f^\vee] & & \\
\downarrow \wr & & \downarrow \mathcal{P}_S \\
H^{3d}_c(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_{S_f}] \otimes H^d_c(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}_2-2,\mathbb{C}})[\pi_f^\vee] & & \\
\downarrow [\mathcal{Z}]_* \otimes \mathrm{id} & & \\
H^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2,\mathbb{C}}) \otimes H^d_c(S(\mathbf{GL}_2), \mathcal{V}^\vee_{\boldsymbol{m}_2-2,\mathbb{C}}) & & \\
\downarrow \langle\cdot,\cdot\rangle & & \\
\mathbb{C} & = & \mathbb{C}
\end{array}$$

*(2) Suppose $S_\infty = \emptyset$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$. After fixing isomorphisms*

$$\Pi_f \simeq \left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\right)^{\boldsymbol{K}_2}, \;\; \pi_f^\vee \simeq \left(\pi^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1},$$

*the composites with the map from (1):*

$$\Pi_f \otimes \pi_f^\vee \xrightarrow{\sim} \left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\right)^{\boldsymbol{K}_2} \otimes \left(\pi^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1} \longrightarrow \mathbb{C}$$

*are independent of $\boldsymbol{\epsilon}$ up to a nonzero scalar.*

*Proof.* Let $\ell : V_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)} \otimes V^\vee_{\boldsymbol{m}_2-2} \to \mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)$ be an $H(F)$-invariant projection.

The composite map:

$$(88)\quad \begin{aligned}\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty} \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}_2} &\to \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}\mathfrak{p}^*_{\mathrm{GSp}_4} \otimes \wedge^{1-\boldsymbol{p}(\boldsymbol{\epsilon}'),1-\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}^*_{\mathrm{GL}_2} \otimes V_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)} \otimes V^\vee_{\boldsymbol{m}_2-2} \\ &\xrightarrow{\iota^*\otimes\ell} \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty)+1-\boldsymbol{p}(\boldsymbol{\epsilon}'),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)+1-\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}_H \xrightarrow{\simeq} \mathbb{C}\end{aligned}$$

is a map of $U(1)^d \times U(1)^d$-modules, where the action on $\boldsymbol{\chi}_{-\boldsymbol{\epsilon}'\boldsymbol{m}_2}$, $\wedge^{\boldsymbol{p}(\boldsymbol{\epsilon}'),\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}_{\mathrm{GL}_2}$, and $V_{\boldsymbol{m}_2-2}$ is through projection to the first factor. In particular, (88) is trivial unless $S_\infty = \emptyset$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$, in which case it is proportional to the projection onto the weight $(-\boldsymbol{\epsilon}\boldsymbol{m}_2, 0)$-component of $\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}$; moreover, a direct calculation shows it is nonzero. In particular, it follows formally that (up to a nonzero constant depending on the normalizations):

$$\langle [\mathcal{Z}]_* \,\mathrm{cl}_{\boldsymbol{\epsilon}}(\alpha), \mathrm{cl}'_{\boldsymbol{\epsilon}'}(\beta)\rangle = \int_{[H]} \boldsymbol{s}_{\boldsymbol{\epsilon},\boldsymbol{\epsilon}'}(\alpha)(\iota(h_1,h_2)) \cdot \beta(h_1)\,\mathrm{d}(h_1,h_2)$$

for all

$$\alpha \in \left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty}\right)^{\boldsymbol{K}_2}, \; \beta \in \left(\pi^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1},$$

which is (1).

For (2), let

$$g_{\boldsymbol{\epsilon}} = \begin{pmatrix} \epsilon_v & & & \\ & \epsilon_v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}_{v|\infty} \in \mathrm{GSp}_4(F\otimes\mathbb{R}) \subset \mathrm{GSp}_4(\mathbb{A}_F), g'_{\boldsymbol{\epsilon}} = \begin{pmatrix} \epsilon_v & \\ & 1 \end{pmatrix}_{v|\infty} \in \mathrm{GL}_2(F\otimes\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{A}_F).$$

We have an obvious commutative diagram

$$\begin{array}{ccc} \left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\right)^{\boldsymbol{K}_2} \otimes \left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1} & \xrightarrow{s_{\boldsymbol{\epsilon},-\boldsymbol{\epsilon}}\otimes\mathrm{id}} & \Pi\otimes\pi^\vee \\ \downarrow & & \downarrow \\ \left(\Pi\otimes\boldsymbol{\tau}^{+}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\right)^{\boldsymbol{K}_2} \otimes \left(\pi^\vee\otimes\boldsymbol{\chi}^\vee_{-\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1} & \xrightarrow{s_{+,-}\otimes\mathrm{id}} & \Pi\otimes\pi^\vee \end{array}$$

in which the vertical arrows are translation by $(g_{\boldsymbol{\epsilon}}, g'_{\boldsymbol{\epsilon}})$ and $\pm$ stands for the constant sign $(\pm)_v|\infty$. However, since $(g_{\boldsymbol{\epsilon}}, g'_{\boldsymbol{\epsilon}})$ lies in $H(\mathbb{A}_F) \subset \mathrm{GSp}_4(\mathbb{A}_F)\times\mathrm{GL}_2(\mathbb{A}_F)$, this translation has no effect on the period integral $\mathcal{P}_S$, and (2) follows. □

**Theorem 7.2.5.** *Let $\pi_1, \pi_2, \pi$ be cuspidal automorphic representations of $G'(\mathbb{A})$ of weights $\boldsymbol{m}_1 = (m_{1,v})_{v|\infty}$, $\boldsymbol{m}_2 = (m_{2,v})_{v|\infty}$, and $\boldsymbol{m}_2$, respectively, where $m_{1,v} \geq m_{2,v}+2 \geq 4$ for all $v|\infty$ and all $m_{i,v}$ have the same parity. Define $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ as in (52). Assume that the central characters of $\pi_1$, $\pi_2$, and $\pi$ agree, and have infinity type $\omega_{\boldsymbol{m}_i}$. Let $\Pi_{S_f}$ be as in (5.3.1) for a set $S_f$ of finite places of $F$. Then, for any coefficient field $E \supset \mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)$ such that $\Pi, \pi_i$, and $\pi$ are defined over $E$, the induced map*

$$[\mathcal{Z}]_* : H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)})(d)[\Pi_{S_f}] \to H^d_!(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2})[\pi_f]$$

*is trivial unless $\pi = \pi_2$ and $S_f = \emptyset$. In the latter case, $[\mathcal{Z}]_*$ takes the form:*

$$\Pi_\emptyset \otimes H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3)})_{\Pi_\emptyset}(d) \xrightarrow{\ell\otimes s} \pi_{2,f}\otimes H^d_!(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2})_{\pi_{2,f}},$$

*where $s$ is an surjection and $\ell$ is a nontrivial $E$-linear map.*

*Proof.* Without loss of generality, suppose $E = \mathbb{C}$. By Proposition 7.2.4, Theorem 6.2.2, and Theorem 6.5.2, we immediately reduce to the case $S_f = \emptyset$ and $\pi = \pi_2$. In this case, write $\Pi = \Pi_\emptyset(\pi_1,\pi_2)$. Under the decomposition

$$H^{3d}_!(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_f] = \bigoplus_{S_\infty}\bigoplus_{\boldsymbol{\epsilon}} H^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_f]$$

provided by Propositions 5.3.3 and 5.3.5, Proposition 7.2.4(1) implies that $[\mathcal{Z}]_*$ is trivial on components with $S_\infty \neq \emptyset$, and maps $H^{\boldsymbol{p}(\boldsymbol{\epsilon},\emptyset),\boldsymbol{q}(\boldsymbol{\epsilon},\emptyset)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})(d)[\Pi_f]$ to $H^{\boldsymbol{p}(\boldsymbol{\epsilon}),\boldsymbol{q}(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2,\mathbb{C}})[\pi_{2,f}]$. Moreover, by Proposition 7.2.4(2) and Proposition 3.7.3, $[\mathcal{Z}]_*$ is a pure tensor $\ell\otimes s$, and $s$ is surjective provided it is nontrivial. Thus, for any single choice of $\boldsymbol{\epsilon}$, it suffices to show that

$$(89)\quad \begin{aligned} H^{\boldsymbol{p}(\boldsymbol{\epsilon},\emptyset),\boldsymbol{q}(\boldsymbol{\epsilon},\emptyset)}_{(2)}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}})[\Pi_f] \otimes H^{\boldsymbol{1}-p(\boldsymbol{\epsilon}),\boldsymbol{1}-q(\boldsymbol{\epsilon})}_{(2)}(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2,\mathbb{C}})[\pi^\vee_{2,f}] \\ \xrightarrow{\langle[\mathcal{Z}]_*\cdot,\cdot\rangle} \mathbb{C} \end{aligned}$$

is nontrivial.

Indeed, let

$$\boldsymbol{\varphi}^{\boldsymbol{\epsilon}}_\infty = \otimes_{v|\infty}\varphi^{\epsilon_v}_{m_{1,v},m_{2,v}} \in \mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2,e_4\rangle\otimes V)\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\otimes\boldsymbol{\chi}^\vee_{\boldsymbol{m}_1,-\boldsymbol{\epsilon}\boldsymbol{m}_2},$$

where $\varphi^{\epsilon_v}_{m_{1,v},m_{2,v}}$ is the vector-valued Schwartz function of (7.1.6). Also let

$$(90)\quad \theta_{\boldsymbol{\epsilon}} : \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2,e_4\rangle\otimes V) \twoheadrightarrow \left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\right)^{\boldsymbol{K}_2}$$

be the $\mathbb{C}$-linear map

$$\phi_f \mapsto \theta_{\phi_f\otimes\varphi^{\boldsymbol{\epsilon}}_\infty}(f_1\otimes f_2),$$

where $f_1\in\pi_1$ and $f_2\in\pi_2$ are nonzero newforms of weights $-\boldsymbol{m}_1$ and $\boldsymbol{\epsilon}\boldsymbol{m}_2$, respectively.

Now Proposition 7.2.4 implies that the composite map

$$\mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2,e_4\rangle\otimes V)\otimes(\pi_2^\vee\otimes\boldsymbol{\chi}_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}^\vee)^{\boldsymbol{K}_1}\xrightarrow{\theta_{\boldsymbol{\epsilon}}\otimes\mathrm{id}}\left(\Pi\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\emptyset}\right)^{\boldsymbol{K}_2}\otimes(\pi_2^\vee\otimes\boldsymbol{\chi}_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}^\vee)^{\boldsymbol{K}_1} \xrightarrow{\langle[\mathcal{Z}]_*\circ\mathrm{cl}^{\boldsymbol{\epsilon}}_\emptyset,\mathrm{cl}'_{\boldsymbol{\epsilon}}\rangle}\mathbb{C} \tag{91}$$

is given by

$$\phi_f\otimes\beta\mapsto\mathcal{P}_\emptyset(\theta_{\phi_f\otimes\overline{\varphi}^{\boldsymbol{\epsilon}}_\infty}(f_1\otimes f_2),\beta) \tag{92}$$

up to a nonzero scalar, where

$$\overline{\varphi}^{\boldsymbol{\epsilon}}_\infty=\otimes_v\overline{\varphi}^{\epsilon_v}_{m_{1,v},m_{2,v}}$$

for $\overline{\varphi}^{\epsilon_v}_{m_{1,v},m_{2,v}}\in\mathcal{S}_{F_v}(\langle e_2,e_4\rangle\otimes V)$ as in (7.1.6). Now let

$$\phi_\infty=\bigotimes_{v|\infty}(\phi^{-\epsilon_v}_{m_{2,v}}\otimes\phi^0_{m_{1,v}-m_{2,v}})\in\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2,e_4\rangle\otimes V).$$

We wish to show that (92) is nontrivial; by Proposition 7.1.8, it suffices to show instead that the map

$$\phi_f\otimes\beta\mapsto\mathcal{P}_\emptyset(\theta_{\phi_f\otimes\phi_\infty}(f_1\otimes f_2),\beta)$$

is nontrivial. However, this is immediate from Theorem 6.5.2 and Proposition 7.1.9. □

## 8. Non-tempered theta lifts on $\mathrm{GSp}_6$

### 8.1. Theta lift from $\mathrm{GSO}(V_B)$ to $\mathrm{GSp}_6$.

8.1.1. For the remainder of this section, fix a *non-split* quaternion algebra $B$ over $F$, and let $\pi$ be a tempered automorphic representation of $PB^\times$. We consider the representation $\pi\boxtimes\mathbb{1}$ of $\mathrm{GSO}(V_B)(\mathbb{A}_F)\simeq(B^\times\times B^\times/\mathbb{G}_m)(\mathbb{A}_F)$ and its theta lift $\Theta(\pi\boxtimes\mathbb{1})$ to $\mathrm{GSp}_6(\mathbb{A}_F)$; this is well-defined because $V_B$ is anisotropic, and descends to $\mathrm{PGSp}_6(\mathbb{A}_F)$ because the similitude theta lift preserves central characters. We remark that $\Theta(\pi\boxtimes\mathbb{1})$ need not be irreducible (because we are using the connected similitude group $\mathrm{GSO}(V_B)$).

**Proposition 8.1.2.** *The theta lift $\Theta(\pi\boxtimes\mathbb{1})$ lies in the $L^2$ subspace of $\mathcal{A}(\mathrm{PGSp}_6(\mathbb{A}_F))$.*

By the usual criterion for square-integrability [25, Lemma I.4.11], we must check that, for each standard parabolic subgroup $P=MN$ of $\mathrm{GSp}_6$, the characters of $Z(M)$ appearing in the cuspidal component of the normalized Jacquet module

$$\Theta(\pi\boxtimes\mathbb{1})_N\otimes\delta_P^{-1/2}$$

all lie in the interior of the cone spanned by the *negatives* of the characters appearing in the action of $Z(M)$ on $N$. Since $\pi\boxtimes\mathbb{1}$ is not a theta lift from $\mathrm{GSp}_2=\mathrm{GL}_2$, [29, Theorem I.1.1] implies that the Jacquet modules are given by:

$$\Theta(\pi\boxtimes\mathbb{1})_N=\begin{cases}|\cdot|^2\Theta'(\pi\boxtimes\mathbb{1}), & M=\mathrm{GSp}_4\times\mathrm{GL}_1\\ 0, & \text{otherwise.}\end{cases} \tag{93}$$

Here $\Theta'(\pi\boxtimes\mathbb{1})$ denotes the theta lift to $\mathrm{GSp}_4$, and $|\cdot|$ is the norm character of $\mathrm{GL}_1$. On the other hand, the action of $Z(M)$ on $N$ is through positive powers of $|\cdot|$, and $\delta_P=|\cdot|^6$; thus the criterion for square-integrability is satisfied.

**Proposition 8.1.3.** *Suppose $\Pi$ is an irreducible constituent of $\Theta(\pi\boxtimes\mathbb{1})$. Then for all non-archimedean $v$ of odd residue characteristic, if $\pi_v$ is unramified with Satake parameters $\{\alpha,\alpha^{-1}\}$, then $\Pi_v$ is unramified with a Langlands parameter $\phi_v$ such that, under the composite*

$$W_{F_v}\xrightarrow{\phi_v}\widehat{\mathrm{PGSp}_6}=\mathrm{Spin}_7\xrightarrow{r_{spin}}\mathrm{GL}_8,$$

*the eigenvalues of $\mathrm{Frob}_v$ (with multiplicity) are $q^{\pm1/2}\alpha^{\pm1},q^{\pm1},1,$ and $1$.*

*Proof.* Propositions 8.1.2 and 4.2.3 imply that $\Pi_v$ is an irreducible constituent of $\Theta_v(\pi_v\boxtimes\mathbb{1})$ for all $v$. Since $\pi_v$ is tempered at all unramified $v$, $\mathrm{Ind}^{\mathrm{GO}(V)(F_v)}_{\mathrm{GSO}(V)(F_v)}\pi_v\boxtimes\mathbb{1}$ is irreducible. Adopting the notation of (4.3.2) with $G=\mathrm{GSO}(V)$ and $H=\mathrm{GSp}_6$, $\pi_v\boxtimes\mathbb{1}$ is the spherical representation $\pi_\chi$ for the unramified character of $T_G(F_v)$ defined by $(q^{1/2}\alpha,q^{-1/2}\alpha,\alpha^{-1})$. Then Proposition 4.3.3 implies that $\Theta_v(\pi_v\boxtimes\mathbb{1})$ is the irreducible representation $\sigma_\mu$ with $\mu=(q^{1/2}\alpha,q^{-1/2}\alpha,q,\alpha^{-1}q^{-1/2})$. Recall that any $\mu$ determines an unramified Langlands parameter for $H$: the characters $x_i, \lambda \in \mathrm{Hom}_{F_v}(T_G, \mathbb{G}_m)$ correspond to cocharacters in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_m, \widehat{T}_G)$, and any unramified character $\mu = (\beta_1, \beta_2, \beta_3, t)$ may be viewed as the element

$$\lambda(t) \prod_i x_i(\beta_i) \in \widehat{T}_H(\mathbb{C}).$$

Then the Langlands parameter of $\sigma_\mu$ is the conjugacy class of the unramified map

$$\phi_\mu : W_{F_v} \to \widehat{T}_H(\mathbb{C}) \hookrightarrow \widehat{H}(\mathbb{C})$$

such that $\phi(\mathrm{Frob}_v)$ is the element corresponding to $\mu$. Now, the eigenvalues of $r_{\mathrm{spin}} \circ \phi_\mu(\mathrm{Frob}_v)$ on $\mathbb{C}^8$ are given by $t \prod_{i \in S} \beta_i$ as $S$ ranges over subsets of $\{1, 2, 3\}$; the proposition follows.

□

## 8.2. **Contributions to the cohomology of Shimura varieties.**

8.2.1. Consider the Shimura variety for $\mathbf{GSp}_6$ as in §3.6, and let $\boldsymbol{k} = (k_v)_{v|\infty}$ be a tuple of integers with $k_v \geq 3$ for all $v$. Following the notation of (3.4.1), we obtain a local system $\mathcal{V}_{(\boldsymbol{k}-3,\boldsymbol{k}-3,0)}$ of $\mathbb{Q}(\boldsymbol{k})$-vector spaces. Let $\sigma_{k_v}$ be the unique irreducible representation of $K_{3,v}$ with trivial central character and whose restriction to $U(3)$ has highest weight $(k_v, 0, -k_v)$, and let $\boldsymbol{\sigma_k}$ be the representation $\otimes_{v|\infty} \sigma_{k_v}$ of $\boldsymbol{K}_3$. One calculates that

$$\dim \mathrm{Hom}_{\boldsymbol{K}_3} \left( \boldsymbol{\sigma_k}, V_{(\boldsymbol{k}-3,\boldsymbol{k}-3,0),\mathbb{C}} \otimes \wedge^{\mathbf{2},\mathbf{2}} \mathfrak{p}^*_{\mathrm{GSp}_6} \right) = 1, \tag{94}$$

where $(\mathbf{2}, \mathbf{2})$ is the constant plectic Hodge type. Thus we have, from (30), a class map

$$\left( \mathcal{A}_{(2)}(\mathrm{GSp}_6(\mathbb{A}_F)) \otimes \boldsymbol{\sigma_k} \right)^{\boldsymbol{K}_3} \to H^{\mathbf{2},\mathbf{2}}_{(2)}(S(\mathbf{GSp}_6), \mathcal{V}_{(\boldsymbol{k}-3,\boldsymbol{k}-3,0),\mathbb{C}}). \tag{95}$$

8.2.2. We now choose a totally indefinite, non-split quaternion algebra $B$ over $F$. Let $\pi$ be an auxiliary automorphic representation of $PB^\times(\mathbb{A}_F)$ of weight $2\boldsymbol{k} = (2k_v)_{v|\infty}$.

**Lemma 8.2.3.** *Fix a prime $\ell$ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. Let $\widetilde{\Pi}$ be a discrete automorphic representation of $\mathrm{GSp}_6(\mathbb{A}_F)$ which is nearly equivalent to a constituent of the theta lift $\iota^*\Theta(\pi \boxtimes \mathbb{1})$, and let $H$ be any $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F^c) \times \mathrm{GSp}_6(\mathbb{A}_{F,f})]$-stable subquotient of*

$$H^{4d}_{\text{ét}}(S(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{k}-3,\boldsymbol{k}-3,0),\overline{\mathbb{Q}}_\ell})[\widetilde{\Pi}_f]$$

*which is pure of weight $4d$. Then $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$ acts on $H$ via $\chi^{-2d}$, where $\chi$ is the $\ell$-adic cyclotomic character.*

*Proof.* Let $K = \prod K_v \subset \mathrm{GSp}_6(\mathbb{A}_{F,f})$ be a neat compact open subgroup. It suffices to show that $\mathrm{Frob}_p$ acts as $p^{-2d}$ on $H^K$ for almost all $p$ such that $p$ splits completely in $F^c$ and $K_p$ is hyperspecial. Assume without loss of generality that $\widetilde{\Pi}_p$ is unramified with local Langlands parameter

$$\phi_p : W_{\mathbb{Q}_p} \to {}^L\,\mathbf{GSp}_6 = \mathrm{GSpin}_7(\mathbb{C})^d \times W_{\mathbb{Q}_p},$$

and consider the $8d$-dimensional representation defined by the composite:

$$W_{\mathbb{Q}_p} \xrightarrow{\phi_p} \mathrm{GSpin}_7(\mathbb{C})^d \xrightarrow{r_{\mathrm{spin}}^{\otimes d}} \mathrm{GL}_{8d}(\mathbb{C}). \tag{96}$$

In light of our chosen isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, (96) defines an $8d$-dimensional $\ell$-adic unramified local Galois representation $V_p$. By [22, §2], since $p$ splits in $F^c$, the action of the geometric Frobenius $\mathrm{Frob}_p^{-1}$ on $H^{4d}_{\text{ét}}(S_K(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{k}-3,\boldsymbol{k}-3,0),\overline{\mathbb{Q}}_\ell})[\widetilde{\Pi}_f]$ satisfies the characteristic polynomial of $p^3 \,\mathrm{Frob}_p^{-1}$ on $V_p$. Now by Proposition 8.1.3, for almost every such $p$ the representation $V_p$ is given by

$$\bigotimes_{v|p} \left( \overline{\mathbb{Q}}_\ell(-1) \oplus \overline{\mathbb{Q}}_\ell^2 \oplus \overline{\mathbb{Q}}_\ell(1) \oplus \rho_\pi|_{F_v} \oplus \rho_\pi|_{F_v}(1) \right),$$

where $\rho_\pi$ is the 2-dimensional $\ell$-adic Galois representation associated to $\pi$, which we normalize to have weight one. (Recall, e.g. from [1], that $\rho_\pi$ is pure since it appears in the cohomology of a compact Shimura curve; for this we use that the Jacquet-Langlands transfer $\mathrm{JL}(\pi)$ is discrete series at some finite place of $F$, because $B$ is nonsplit.) On the other hand, we have assumed that $H$ is pure of weight $4d$, and $H^K$ is a subquotient of $H^{4d}_{\text{ét}}(S_K(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{k}-3,\boldsymbol{k}-3,0),\overline{\mathbb{Q}}_\ell})[\widetilde{\Pi}_f]$ because the $K$-invariants functor is exact; comparing with the weights in $V_p$, it follows that $\mathrm{Frob}_p$ acts as $p^{-2d}$ on $H^K$. □

## 9. Triple product periods

### 9.1. The vector-valued period problem.

9.1.1. Let $\pi_1$, $\pi_2$, $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, $\boldsymbol{\ell}_1$, $\boldsymbol{\ell}_2$, $\boldsymbol{\epsilon}$, $\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}$, and $\boldsymbol{\sigma}_{\boldsymbol{\ell}_1}$ be as in (5.3.1) and (8.2.1) for $\boldsymbol{k} = \boldsymbol{\ell}_1$, and let $B$ be a *non-split* totally indefinite quaternion algebra over $F$, ramified at a set $S$ of places of $F$ at which $\pi_i$ are both discrete series.

9.1.2. For auxiliary automorphic representations $\pi$ of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{\ell}_1$, we will consider triple product period integrals of $\Theta(\pi \boxtimes \mathbb{1})$ along the subgroup

$$\widetilde{H} \coloneqq (\mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2) \subset \mathrm{GSp}_6\,. \tag{97}$$

The maximal compact-modulo-center subgroup of $\widetilde{H}(F\otimes\mathbb{R})$ is

$$(\boldsymbol{K}_2 \times \boldsymbol{K}_1)_0 \coloneqq (\boldsymbol{K}_2 \times \boldsymbol{K}_1) \cap \widetilde{H}(F\otimes\mathbb{R}). \tag{98}$$

To define the vector-valued period integral, note that (by the classical branching law for unitary groups), the space

$$\mathrm{Hom}_{(\boldsymbol{K}_2\times\boldsymbol{K}_1)_0}(\boldsymbol{\sigma}_{\boldsymbol{\ell}_1} \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}, \mathbb{C}) \tag{99}$$

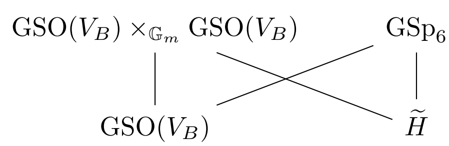
is one-dimensional, say with generator $s_{\boldsymbol{\epsilon}}$.

We then define, for any auxiliary representation $\pi$ of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{\ell}_1$, the triple product period:

$$\begin{gathered}\widetilde{\mathcal{P}}^{\boldsymbol{\epsilon}}_{S,\pi_1,\pi_2,\pi}(\alpha,\beta,\gamma) = \int_{[\widetilde{H}]} s_{\boldsymbol{\epsilon}}(\alpha(g,g')\otimes\beta(g)\otimes\gamma(g'))\,\mathrm{d}(g,g') \neq 0,\\ \alpha \in (\Theta(\pi\boxtimes\mathbb{1})\otimes\boldsymbol{\sigma}_{\boldsymbol{\ell}_1})^{\boldsymbol{K}_3},\ \ \beta \in (\Pi_S(\pi_1\otimes\pi_2)\otimes\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2})^{\boldsymbol{K}_2},\ \ \gamma \in (\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2})^{\boldsymbol{K}_1}.\end{gathered} \tag{100}$$

Since we will not give a precise formula for $\widetilde{\mathcal{P}}^{\boldsymbol{\epsilon}}_{S,\pi_1,\pi_2,\pi}$ and are only interested in its non-vanishing, we ignore the problem of normalization. The non-vanishing of (100), for a good choice of $\pi$, is the key input to the non-vanishing of the Hodge classes we construct in the next section.

9.1.3. The strategy for calculating (100) is to use the seesaw diagram:




There are two main inputs to the non-vanishing of our period integral (for a good choice of $\pi$): the first is a vector-valued version of the usual global seesaw identity, and the second is a non-vanishing result for the vector-valued theta lifts along the "other" diagonal in the seesaw diagram, i.e. from $\mathrm{GSp}_4$ and $\mathrm{GL}_2$ to $\mathrm{GSO}(V_B)$.

### 9.2. Vector-valued seesaw identity.

9.2.1. Continuing with the notation from (7.1.1), let $m_1 \geq m_2 + 2 \geq 4$ be integers such that $m_1 \equiv m_2 \pmod 2$, with $\ell_1$ and $\ell_2$ as in (7.1.6). Let $\sigma_{\ell_1}$ be the unique representation of $K_3$ of trivial central character and whose restriction to $U(3)$ has highest weight $(\ell_1, 0, -\ell_1)$. Let

$$\widetilde{\varphi}_{\ell_1} \in (S^0_{\leq 2\ell_1}(3) \otimes \sigma_{\ell_1} \otimes \chi_{-2\ell_1,0})^{(K_3\times L)_0}$$

be a generator, which makes sense by Proposition 7.1.3. If $(K_2 \times K_1)_0$ is the intersection of $K_3$ with $\mathrm{GSp}_{4,\mathbb{R}} \times_{\mathbb{G}_m} \mathrm{GL}_{2,\mathbb{R}}$ inside $\mathrm{GSp}_{6,\mathbb{R}}$, then we have, for any $\epsilon = \pm 1$,

$$\dim \mathrm{Hom}_{(K_2\times K_1)_0}(\sigma_{\ell_1}, \tau^\epsilon_{\ell_1,\ell_2} \otimes \chi^\vee_{-\epsilon m_2}) = 1; \tag{101}$$

let $s'$ denote a generator. Also let $(K_2 \times K_1 \times L)_0 = (K_2\times K_1)_0 \times L \cap (K_3 \times L)_0$.

**Proposition 9.2.2.** *The Schwartz function*

$$s'(\widetilde{\varphi}_{\ell_1}) \in \big(\mathcal{S}_{\mathbb{R}}(\langle e_2, e_4, e_6\rangle \otimes V) \otimes \tau^\epsilon_{\ell_1,\ell_2} \otimes \chi^\vee_{-\epsilon m_2} \otimes \chi_{-2\ell_1,0}\big)^{(K_2\times K_1\times L)_0}$$

*is a nonzero scalar multiple of the tensor product* $\varphi^\epsilon_{m_1,m_2} \otimes \phi^\epsilon_{m_2}$.

*Proof.* Assume $\epsilon = +$; the other case is similar. Let $\phi \in S^0_{\leq 2\ell_1}(3)$ be the contraction of $\varphi^\epsilon_{m_1,m_2} \otimes \phi^\epsilon_{m_2}$ with any nonzero vector of pure weight. Then $\phi$ generates the irreducible $U(2)\times U(1)$-representation of highest weight $(\ell_2 - 2, -\ell_1, m_2)$. We wish to show that $U(3)\cdot\phi$ is the irreducible $U(3)$-representation of highest weight $(\ell_1, 0, -\ell_1)$.

First, if $U(3)\cdot\phi = V_1 \oplus V_2$ is any nontrivial $U(3)$-stable decomposition, then the irreducible $U(2)\times U(1)$-representation $(U(2)\times U(1))\cdot\phi$ must project nontrivially to both $V_1$ and $V_2$. Hence if the $U(3)$-representation with highest weight $(a,b,c)$ appears in $U(3)\cdot\phi$, it follows (using the branching law for unitary groups) that

$$a \geq \ell_2 - 2 \geq b \geq -\ell_1 \geq c, \qquad a+b+c = 0. \tag{102}$$

On the other hand, considering Proposition 7.1.3(2), we have

$$|a| + |b| + |c| \leq 2\ell_1. \tag{103}$$

The combination of (102) and (103) force $c = -\ell_1$. On the other hand, by [10, Proposition 4.2.1], if $(a,b,c)$ appears in $S^0(3)$ with $c < -1$ then either $b = a = 1$ or $b = 0$. Since $a + b = \ell_1 \geq 3$, we conclude $(a,b,c) = (\ell_1, 0, -\ell_1)$. Since $U(3)\cdot\phi$ is generated by a single vector which has pure weight, it is also multiplicity-free as a representation of $U(3)$, so it is irreducible with highest weight $(\ell_1, 0, -\ell_1)$, as desired. □

9.2.3. We now return to the global situation. Choose an isomorphism $V_B \otimes_F \mathbb{R} \simeq V \otimes_F \mathbb{R}$, which induces an isomorphism $\mathrm{GSO}(V_B)(F\otimes\mathbb{R}) \simeq \mathrm{GSO}(V)(F\otimes\mathbb{R})$. Then let $\boldsymbol{L} = \prod_{v|\infty} L \subset \mathrm{GSO}(V_B)(F\otimes\mathbb{R})$, and similarly for $(\boldsymbol{K}_n \times \boldsymbol{L})_0$, etc. We fix vector-valued Schwartz functions as follows:

$$\begin{aligned}
\boldsymbol{\varphi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_1,\boldsymbol{m}_2} = \otimes_{v|\infty}\varphi^{-\epsilon_v}_{m_{1,v},m_{2,v}} &\in \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2, e_4\rangle \otimes V) \otimes \boldsymbol{\tau}^{-\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m}_1,\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0} \\
&\overset{(7.1.6)}{\simeq} \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_2, e_4\rangle \otimes V_B) \otimes (\boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2})^\vee \otimes \boldsymbol{\chi}_{-\boldsymbol{m}_1,-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0} \\
\boldsymbol{\phi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_2} = \otimes_{v|\infty}\phi^{-\epsilon_v}_{m_{2,v}} &\in \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_6\rangle \otimes V) \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon}\boldsymbol{m}_2} \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m}_2,-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{(\boldsymbol{K}_1\times\boldsymbol{L})_0} \\
&\overset{(\text{Proposition 7.1.4})}{\simeq} \left(\mathcal{S}_{F\otimes\mathbb{R}}(\langle e_6\rangle \otimes V_B) \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon}\boldsymbol{m}_2} \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m}_2,-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{(\boldsymbol{K}_1\times\boldsymbol{L})_0} \\
\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{\ell}_1} = \otimes_{v|\infty}\widetilde{\varphi}_{\ell_1,v} &\in \left(\mathcal{S}_{\mathbb{R}}(\langle e_2, e_4, e_6\rangle \otimes V) \otimes \boldsymbol{\sigma}_{\boldsymbol{\ell}_1} \otimes \boldsymbol{\chi}_{-2\boldsymbol{\ell}_1,0}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0} \\
&\overset{(9.2.1)}{\simeq} \left(\mathcal{S}_{\mathbb{R}}(\langle e_2, e_4, e_6\rangle \otimes V_B) \otimes \boldsymbol{\sigma}_{\boldsymbol{\ell}_1} \otimes \boldsymbol{\chi}_{-2\boldsymbol{\ell}_1,0}\right)^{(\boldsymbol{K}_2\times\boldsymbol{L})_0}
\end{aligned} \tag{104}$$

**Proposition 9.2.4.** *Let $s_{\boldsymbol{\epsilon}}$ be as above. For all*

$$\alpha \in (\mathcal{A}_0(\mathrm{PGSO}(V_B)(\mathbb{A}_F)) \otimes \boldsymbol{\chi}_{2\boldsymbol{\ell}_1,0})^{\boldsymbol{L}}, \quad \beta \in \left(\Pi_S(\pi_1,\pi_2) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}\right)^{\boldsymbol{K}_2},$$

$$\gamma \in \left(\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1}, \quad \phi_{1,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle \otimes V_B), \quad \phi_{2,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_6\rangle \otimes V_B),$$

*and up to a nonzero scalar depending on the normalizations, we have the identity:*

$$\begin{aligned}
&\int_{[Z_{\widetilde{H}}\backslash\widetilde{H}]} s_{\boldsymbol{\epsilon}}\left(\theta_{\phi_{1,f}\otimes\phi_{2,f}\otimes\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{\ell}_1}}(\alpha)(g,g') \otimes \beta(g) \otimes \gamma(g')\right) \mathrm{d}(g,g') = \\
&\qquad \int_{[\mathrm{PGSO}(V_B)]} \alpha(g)\theta_{\phi_{1,f}\otimes\boldsymbol{\varphi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_1,\boldsymbol{m}_2}}(\beta)(g)\theta_{\phi_{2,f}\otimes\boldsymbol{\phi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_2}}(\gamma)(g)\,\mathrm{d}g.
\end{aligned} \tag{105}$$

*Proof.* This is formal from Proposition 9.2.2 and the usual seesaw identity, i.e. exchanging the order of integration. □

## 9.3. **Proof of the non-vanishing result.**

**Proposition 9.3.1.** *Let $\pi_i^B$ be the Jacquet-Langlands transfers of $\pi_i$ to $B(\mathbb{A}_F)^\times$.*

(1) *The map*

$$\theta_{\boldsymbol{\varphi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_1,\boldsymbol{m}_2}} : \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2, e_4\rangle \otimes V_B) \otimes (\Pi_S(\pi_1,\pi_2) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2})^{\boldsymbol{K}_2} \to (\mathcal{A}(\mathrm{GSO}(V_B))(\mathbb{A}_F) \otimes \boldsymbol{\chi}_{-\boldsymbol{m}_1,-\boldsymbol{\epsilon}\boldsymbol{m}_2})^{\boldsymbol{L}},$$

*defined by*

$$(\phi,\alpha) \mapsto \theta_{\phi\otimes\boldsymbol{\varphi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_1,\boldsymbol{m}_2}}(\alpha),$$

*has image containing* $\left((\pi_1^B \boxtimes \pi_2^B) \otimes \boldsymbol{\chi}_{-\boldsymbol{m}_1,-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{L}}$.

(2) *The map*

$$\theta_{\boldsymbol{\phi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_2}} : \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_6\rangle \otimes V_B) \otimes \left(\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1} \to \left(\mathcal{A}(\mathrm{GSO}(V_B))(\mathbb{A}_F) \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m}_2,-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{L}},$$

*defined by*

$$(\phi,\alpha) \mapsto \theta_{\phi\otimes\boldsymbol{\phi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_2}}(\alpha),$$

*has image containing* $\left(((\pi_2^B)^\vee \boxtimes (\pi_2^B)^\vee) \otimes \boldsymbol{\chi}^\vee_{\boldsymbol{m}_2,-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{L}}$.

*Proof.* In the general setup of §4, suppose $\Theta_{V,W}(\pi) = \Pi$ for cuspidal automorphic representations $\pi$ of $G(V)(\mathbb{A}_F)$ and $\Pi$ of $H(W)(\mathbb{A}_F)$. Then by definition we have a surjective composite

$$\mathcal{S}_{\mathbb{A}_F}(W_2 \otimes V) \xrightarrow{\theta} \mathcal{A}(R_0(\mathbb{A}_F)) \twoheadrightarrow \Pi \otimes \pi^\vee. \tag{106}$$

Now, the theta kernel satisfies

$$\theta(\phi)(g,h) = \overline{\theta(\overline{\phi})}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} g \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, h\right)$$

(cf. [30]), so we deduce that $\overline{\Pi\otimes\pi^\vee} = \Pi^\vee \otimes \pi$ also appears in the spectrum of the theta kernel. (Recall that the central characters of $\Pi$ and $\pi$ must agree since the central character of the Weil representation is trivial.) In particular $\Theta_{W,V}(\Pi)$ contains the nonzero irreducible constituent $\pi$. For (1), take $W = W_4$, $V = V_B$, and $\Pi = \Pi_S(\pi_1,\pi_2)$. As in the proof of Proposition 4.2.3, the global theta lift gives rise to a nontrivial map:

$$\mathcal{S}_{\mathbb{A}_F}(\langle e_2,e_4\rangle \otimes V_B) \twoheadrightarrow \Pi_S(\pi_1,\pi_2)^\vee \otimes \Theta_{W_4,V_B}(\Pi_S(\pi_1,\pi_2)) \twoheadrightarrow \Pi_S(\pi_1,\pi_2)^\vee \otimes (\pi_1 \boxtimes \pi_2). \tag{107}$$

(Since $\mathrm{GSO}(V_B)$ is anisotropic, all theta lifts are square-integrable.) The map (107) is a restricted tensor product of local maps. To prove (1), it suffices to show that, for all $v|\infty$ and for some vector $0 \neq s \in \tau^{\epsilon_v}_{\ell_{1,v},\ell_{2,v}}$, the contraction $s(\varphi^{-\epsilon_v}_{m_{1,v},m_{2,v}})$ has nontrivial image under the local component

$$\mathcal{S}_{F_v}(\langle e_2,e_4\rangle \otimes V_B) \twoheadrightarrow \Pi^+(\pi_{1,v},\pi_{2,v})^\vee \otimes \left(\pi^B_{1,v} \boxtimes \pi^B_{2,v}\right) \tag{108}$$

of (107). This follows from Remark 7.1.10. The proof of (2) is analogous, invoking instead Proposition 7.1.4(2). □

Finally we come to the main result of this section:

**Lemma 9.3.2.** *There exists an automorphic representation $\pi$ of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{\ell}_1$, along with vectors*

$$\alpha \in (\Theta(\pi\boxtimes \mathbb{1}) \otimes \boldsymbol{\sigma}_{\boldsymbol{\ell}_1})^{\boldsymbol{K}_3},\ \ \beta \in (\Pi_S(\pi_1\otimes\pi_2) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2})^{\boldsymbol{K}_2},\ \ \gamma \in (\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2})^{\boldsymbol{K}_1},$$

*such that:*

$$\widetilde{\mathcal{P}}^{\boldsymbol{\epsilon}}_{S,\pi_1,\pi_2,\pi}(\alpha,\beta,\gamma) = \int_{[\widetilde{H}]} s_{\boldsymbol{\epsilon}}(\alpha(g,g') \otimes \beta(g) \otimes \gamma(g'))\,\mathrm{d}(g,g') \neq 0.$$

*Proof.* First, fix newforms

$$f_1 \in \pi_1^B,\ \ f_2^{\boldsymbol{\epsilon}} \in \pi_2^B,\ \ f_2^\vee \in (\pi_2^B)^\vee,\ \ (f_2^{\boldsymbol{\epsilon}})^\vee \in (\pi_2^B)^\vee$$

of weights $\boldsymbol{m}_1$, $\boldsymbol{\epsilon}\boldsymbol{m}_2$, $\boldsymbol{m}_2$, and $-\boldsymbol{\epsilon}\boldsymbol{m}_2$, respectively. Then Proposition 9.3.1 implies that we may choose vectors

$$\beta \in \left(\Pi_S(\pi_1,\pi_2) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}\right)^{\boldsymbol{K}_2},\ \ \gamma \in (\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2})^{\boldsymbol{K}_1}$$

and Schwartz functions

$$\phi_{1,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_2,e_4\rangle \otimes V_B),\ \ \phi_{2,f} \in \mathcal{S}_{\mathbb{A}_{F,f}}(\langle e_6\rangle \otimes V_B)$$

such that:

$$\theta_{\phi_{1,f}\otimes\boldsymbol{\varphi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_1,\boldsymbol{m}_2}} = f_1 \otimes f_2^{\boldsymbol{\epsilon}};\quad \theta_{\phi_{2,f}\otimes\boldsymbol{\varphi}^{-\boldsymbol{\epsilon}}_{\boldsymbol{m}_2}}(\gamma) = f_2^\vee \otimes (f_2^{\boldsymbol{\epsilon}})^\vee. \tag{109}$$

Now, the automorphic function $g \mapsto f_1(g) \cdot f_2^\vee(g)$ corresponds to a Hilbert modular form on $B^\times$ of weight $2\boldsymbol{\ell}_1$ and trivial central character. We may therefore choose some automorphic representation $\pi$ of $PB(\mathbb{A}_F)^\times$ of weight $2\boldsymbol{\ell}_1$, with a vector $\alpha_0$ of weight $-2\boldsymbol{\ell}_1$, such that

$$\int_{[PB^\times]} \alpha_0(g) f_1(g) f_2^\vee(g)\,\mathrm{d}g \neq 0.$$

Now, we turn $\alpha_0$ into an automorphic form $\alpha$ on $\mathrm{PGSO}(V_B)(\mathbb{A}_F)$ by setting $\alpha(\boldsymbol{p}_Z(g_1,g_2)) = \alpha_0(g_1)$. It is clear that $\alpha$ is a vector in $((\pi \boxtimes \mathbb{1}) \otimes \boldsymbol{\chi}_{2\boldsymbol{\ell}_1,0})^{\boldsymbol{L}}$. Then Proposition 9.2.4 allows us to compute:

$$\widetilde{\mathcal{P}}_{S,\pi_1,\pi_2,\pi}(\theta_{\phi_{1,f}\otimes\phi_{2,f}\otimes\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{m}_1,\boldsymbol{m}_2}}(\alpha),\beta,\gamma) = \left(\int_{[PB^\times]} \alpha_0(g) f_1(g) f_2^\vee(g)\,\mathrm{d}g\right)\cdot\left(\int_{[PB^\times]} f_2^{\boldsymbol{\epsilon}}(g)(f_2^{\boldsymbol{\epsilon}})^\vee(g)\,\mathrm{d}g\right) \neq 0.$$

□

## 10. Proof of main result: Hodge classes in the non-generic case

### 10.1. **Construction.**

10.1.1. Consider the inclusions of Shimura varieties:

$$S(\mathbf{GSp}_6) \xleftarrow{\iota_1} S(\widetilde{\boldsymbol{H}}) \xrightarrow{\iota_2} S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \tag{110}$$

where

$$\widetilde{H} := \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2 \subset \mathrm{GSp}_6\,.$$

Note that $\iota_1^*\mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_1-3,0)}$ contains $\iota_2^*\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2}$ as a direct factor with multiplicity one by [17, Theorem 2.5]. Since $\iota_2$ is an open and closed embedding at sufficiently small level, one obtains from (110) a map

$$H^i(S(\mathbf{GSp}_6),\mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3,0)}) \to H^i(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2}). \tag{111}$$

It follows from Saito's theory of mixed Hodge modules [32] that this is a map of mixed Hodge strucutres. The Hodge classes we construct will be the images of classes on $S(\mathbf{GSp}_6)$ under the map (111).

10.1.2. Let $\pi_1,\pi_2$, and $\Pi_{S_f}$ be as in (5.3.1), where $|S_f| \geq 2$ is even. We let $B$ be the unique quaternion algebra over $F$ which is ramified at $S_f$ and split at all archimedean places. For any finite set $\Sigma \supset S_f$ of places of $F$, including all infinite ones, we consider the unramified Hecke algebra with $\mathbb{Q}$-coefficients:

$$\widetilde{\mathbb{T}}^\Sigma = \otimes_{v\notin\Sigma}\mathcal{H}(\mathrm{GSp}_6(F_v),\mathrm{GSp}_6(\mathcal{O}_v)). \tag{112}$$

For an auxiliary automorphic representation $\pi$ of $PB^\times$ which is tempered, unramified outside of $\Sigma$, and of weight $2\boldsymbol{\ell}_1$, the Hecke action on $\Theta(\pi\boxtimes\mathbb{1})$ defines a maximal ideal $I^\Sigma \subset \widetilde{\mathbb{T}}^\Sigma$.

**Definition 10.1.3.** Fix $\pi$ and $\Sigma$ as above and a sufficiently small compact open subgroup $K = \prod K_v \subset \mathrm{GSp}_6(\mathbb{A}_{F_f})$ such that $K_v = \mathrm{GSp}_6(\mathcal{O}_v)$ for $v \notin \Sigma$. Then we define

$$\mathrm{Hdg}(\pi,K,\Sigma) \subset H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})_0$$

to be the image of the composite map

$$\begin{aligned} H^{4d}(S_K(\mathbf{GSp}_6),\mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3,0)})(2d)[I^\Sigma] &\xrightarrow{(111)} H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2}) \\ &\longrightarrow H^{4d}(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2),\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})_0. \end{aligned}$$

Here the final map is the projection from Lemma 3.2.3(2).

**Lemma 10.1.4.** *Any $\xi \in \mathrm{Hdg}(\pi,K,\Sigma)$ is a Hodge class of weight $(2d,2d)$. Moreover, for any finite prime $\lambda$ of $\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)$ lying above $\ell$, $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$ acts by $\chi^{-2d}$ on*

$$\mathrm{Hdg}(\pi,K,\Sigma)\otimes_{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)}\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)_\lambda \subset H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)\times S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}},\mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2,\lambda}),$$

*where $\chi$ is the $\ell$-adic cyclotomic character.*

*Proof.* We first compute the Galois action, for which we may extend scalars to $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{S}$ be the set of automorphic representations of $\mathrm{GSp}_6(\mathbb{A}_F)$ such that $\widetilde{\Pi}_f^K \neq 0$ is annihilated by $I^\Sigma$, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ which induces the prime $\lambda$ of $\mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)$. Then it suffices to show that, for all $\widetilde{\Pi} \in \mathcal{S}$ and all automorphic representations $\Pi \boxtimes \tau$ of $\mathrm{GSp}_4(\mathbb{A}_F) \times \mathrm{GL}_2(\mathbb{A}_F)$ with $\Pi_f \boxtimes \tau_f$ non-Eisenstein, $\mathrm{Gal}(\overline{\mathbb{Q}}/F^c)$ acts by $\chi^{-2d}$ on the image of

$$H^{4d}_{\text{ét}}(S_K(\mathbf{GSp}_6)_{\overline{\mathbb{Q}}}, \mathcal{V}_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3,0),\overline{\mathbb{Q}}_\ell})[\widetilde{\Pi}_f] \to H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}} \times S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2,\overline{\mathbb{Q}}_\ell})[\Pi_f \boxtimes \tau_f].$$

Since the calculation in Lemma 8.2.3 is based only on local Langlands parameters at cofinitely many primes, and is insensitive to replacing the local Langlands parameters with $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-conjugates, it also applies to all $\widetilde{\Pi} \in \mathcal{S}$, and the claim follows since Lemma 3.2.3(1) implies that

$$H^{4d}_{\text{ét}}(S(\mathbf{GSp}_4)_{\overline{\mathbb{Q}}} \times S(\mathbf{GL}_2)_{\overline{\mathbb{Q}}}, \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2,\overline{\mathbb{Q}}_\ell})[\Pi_f \boxtimes \tau_f]$$

is pure of weight $4d$.

Now, $H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})_0$ is a pure Hodge structure of weight $4d$ by Lemma 3.2.3, and by construction

$$\mathrm{Hdg}(\pi, K, \Sigma) \subset H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2})_0$$

is a sub-Hodge structure. By Theorem 3.1.3, we have a canonical isomorphism

$$\mathrm{Hdg}(\pi, K, \Sigma) \otimes_{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)} \mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)_\lambda \otimes_{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)_\lambda} \widehat{\overline{\mathbb{Q}}}_\ell \cong \oplus_j \mathrm{gr}^j\, \mathrm{Hdg}(\pi, K, \Sigma) \otimes_{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)} \mathbb{C} \otimes_{\mathbb{C},\iota^{-1}} \widehat{\overline{\mathbb{Q}}}_\ell(-j)$$

compatible with the actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)_\lambda)$ on both sides. In particular,

$$\mathrm{gr}^j(\mathrm{Hdg}(\pi, K, \Sigma) \otimes_{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)} \mathbb{C}) = \begin{cases} \mathrm{Hdg}(\pi, K, \Sigma) \otimes_{\mathbb{Q}(\boldsymbol{m}_1,\boldsymbol{m}_2)} \mathbb{C}, & j = 2d, \\ 0, & \text{else}. \end{cases}$$

Hence the Hodge structure on $\mathrm{Hdg}(\pi, K, \Sigma)$ is trivial and in particular each $\xi \in \mathrm{Hdg}(\pi, K, \Sigma)$ is a Hodge class, as desired. □

10.2. **Nonvanishing.**

10.2.1. To test the non-degeneracy of the subspace $\mathrm{Hdg}(\pi, K, \Sigma)$, we will use the following proposition.

**Proposition 10.2.2.** *Let $\Pi = \Pi_S$, where $S = S_f \sqcup S_\infty$, and choose an auxiliary $\pi$ as above. Suppose given*

$$\alpha \in (\Theta(\pi \boxtimes \mathbb{1}) \otimes \boldsymbol{\sigma}_{\boldsymbol{\ell}_1})^{\boldsymbol{K}_3},\ \beta \in \left(\Pi_S(\pi_1 \otimes \pi_2) \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty}\right)^{\boldsymbol{K}_2},\ \gamma \in \left(\pi_2^\vee \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}\boldsymbol{m}_2}\right)^{\boldsymbol{K}_1}.$$

*(1) Fix choices of signs $\boldsymbol{\epsilon} = \{\epsilon_v\}_{v|\infty}$ and $\boldsymbol{\epsilon}' = \{\epsilon_v\}_{v|\infty}$. Then:*

$$\langle \iota_{2,*} \circ \iota_1^*(\mathrm{cl}(\alpha)), \mathrm{cl}^{\boldsymbol{\epsilon}}_S(\beta) \boxtimes \mathrm{cl}'_{\boldsymbol{\epsilon}'}(\gamma)\rangle = \begin{cases} \widetilde{\mathcal{P}}_{S,\pi_1,\pi_2,\pi}(\alpha, \beta, \gamma), & \textit{if } S_\infty = \emptyset \textit{ and } \boldsymbol{\epsilon} = \boldsymbol{\epsilon}'; \\ 0, & \textit{otherwise.} \end{cases}$$

*(2) After choosing isomorphisms*

$$\Pi_{S_f} \simeq \left(\Pi \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2}\right)^{\boldsymbol{K}_3},\ \ \pi^\vee_{2,f} \simeq (\pi_2^\vee \otimes \boldsymbol{\chi}_{-\boldsymbol{\epsilon}\boldsymbol{m}_2})^{\boldsymbol{K}_1},$$

*the composite maps*

$$\Pi_{S_f} \otimes \pi^\vee_{2,f} \xrightarrow{\mathrm{cl}^{\boldsymbol{\epsilon}}_S \otimes \mathrm{cl}'_{\boldsymbol{\epsilon}}} H^{4d}_{(2)}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1,\boldsymbol{m}_2,\mathbb{C}}) \xrightarrow{\langle \iota_{2,*}\iota_1^*\alpha, \cdot\rangle} \mathbb{C}$$

*are independent of $\boldsymbol{\epsilon}$ up to a scalar.*

*Proof.* The proof is essentially identical to Proposition 7.2.4. The only new ingredient is the calculation of the $(\boldsymbol{K}_2 \times \boldsymbol{K}_1)_0$-equivariant composite:

$$\begin{aligned} \boldsymbol{\sigma}_{\boldsymbol{\ell}_1} \otimes \boldsymbol{\tau}^{\boldsymbol{\epsilon}}_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,S_\infty} \otimes \boldsymbol{\chi}^\vee_{-\boldsymbol{\epsilon}'\boldsymbol{m}_2} &\to \wedge^{\mathbf{2},\mathbf{2}}\mathfrak{p}^*_{\mathrm{GSp}_6} \otimes V_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_1-3,0),\mathbb{C}} \otimes \wedge^{\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty),\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)}\mathfrak{p}^*_{\mathrm{GSp}_4} \\ &\quad \otimes V_{(\boldsymbol{\ell}_1-3,\boldsymbol{\ell}_2-3),\mathbb{C}} \otimes \wedge^{1-\boldsymbol{p}(\boldsymbol{\epsilon}'),1-\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}^*_{\mathrm{GL}_2} \otimes V^\vee_{\boldsymbol{m}_2-2,\mathbb{C}} \\ &\to \wedge^{3+\boldsymbol{p}(\boldsymbol{\epsilon},S_\infty)-\boldsymbol{p}(\boldsymbol{\epsilon}'),3+\boldsymbol{q}(\boldsymbol{\epsilon},S_\infty)-\boldsymbol{q}(\boldsymbol{\epsilon}')}\mathfrak{p}^*_{\widetilde{H}} \xrightarrow{\mathbb{1}_{\widetilde{H}}} \mathbb{C}, \end{aligned} \tag{113}$$

which is automatically trivial unless $S_\infty = \emptyset$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$, in which case one can check that it is not trivial. □

**Corollary 10.2.3.** *There exists a triple $(\pi, K, \Sigma)$ as in Definition 10.1.3 and an element*

$$\xi_{\mathbb{C}} \in \mathrm{Hdg}(\pi, K, \Sigma) \otimes \mathbb{C} \subset H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1, \boldsymbol{m}_2, \mathbb{C}})$$

*such that the induced map*

$$(\xi_{\mathbb{C}})_* : H_c^{3d}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3, \boldsymbol{\ell}_2-3), \mathbb{C}})(d)[\Pi_{S_f}] \to H^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, \mathbb{C}}) \twoheadrightarrow H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, \mathbb{C}})[\pi_{2,f}]$$

*is of the form*

$$\Pi_{S_f} \otimes H_!^{3d}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3, \boldsymbol{\ell}_2-3), \mathbb{C}})_{\Pi_{S_f}}(d) \xrightarrow{\ell \otimes s} \pi_{2,f} \otimes H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, \mathbb{C}})_{\pi_{2,f}},$$

*where $s$ is a surjection and $\ell$ is a nontrivial $\mathbb{C}$-linear map.*

*Proof.* By Proposition 10.2.2 and Lemma 9.3.2, there exists an automorphic representation $\pi$ as in Definition 10.1.3 and a vector

$$\alpha \in (\Theta(\pi \boxtimes \mathbb{1}) \otimes \boldsymbol{\sigma}_{\boldsymbol{\ell}_1})^{\boldsymbol{K}_3}$$

such that

$$\iota_{2,*} \circ \iota_1^*(\mathrm{cl}(\alpha)) \in H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1, \boldsymbol{m}_2, \mathbb{C}})$$

induces, by Poincaré duality, a nontrivial map

$$H_c^{3d}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3, \boldsymbol{\ell}_2-3), \mathbb{C}})(d)[\Pi_{S_f}] \to H^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, \mathbb{C}}) \twoheadrightarrow H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, \mathbb{C}})[\pi_{2,f}].$$

Moreover, this map is of the form claimed in the corollary by Proposition 10.2.2(2) and the same argument as in Theorem 7.2.5. For $K$ sufficiently small and $\Sigma$ sufficiently large,

$$\mathrm{Hdg}(\pi, K, \Sigma) \otimes \mathbb{C} \subset H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1, \boldsymbol{m}_2, \mathbb{C}})_0$$

contains the image of $\iota_{2,*} \circ \iota_1^*(\mathrm{cl}(\alpha))$ under the projection

$$H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1, \boldsymbol{m}_2, \mathbb{C}}) \to H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1, \boldsymbol{m}_2, \mathbb{C}})_0,$$

and this implies the corollary. □

**Theorem 10.2.4.** *Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ of weights $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, respectively, where $\boldsymbol{m}_i = (m_{i,v})_{v|\infty}$ for integers $m_{i,v}$, all of the same parity, such that $m_{1,v} \geq m_{2,v} + 2 \geq 4$. Assume that the central characters of $\pi_1$ and $\pi_2$ agree and have infinity type $\omega_{\boldsymbol{m}_i}$. Let $\Pi_{S_f}$ be as in (5.3.1) for a set $S_f$ of finite places of $F$ such that $|S_f| \geq 2$ is even, and choose a coefficient field $E \supset \mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)$ over which $\pi_i$ and $\Pi_{S_f}$ are defined. Then there exists a triple $(\pi, K, \Sigma)$ as in Definition 10.1.3 and a Hodge class*

$$\xi \in \mathrm{Hdg}(\pi, K, \Sigma)(2d) \subset H^{4d}(S(\mathbf{GSp}_4) \times S(\mathbf{GL}_2), \mathcal{W}_{\boldsymbol{m}_1, \boldsymbol{m}_2})(2d)$$

*such that the induced map*

$$\xi_* : H_c^{3d}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3, \boldsymbol{\ell}_2-3), E})(d)[\Pi_{S_f}] \to H^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, E}) \twoheadrightarrow H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, E})[\pi_{2,f}]$$

*is nontrivial, and moreover the image of $\xi_*$ spans the $E[\mathrm{GL}_2(\mathbb{A}_{F,f})]$-module $H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, E})[\pi_{2,f}]$.*

*Proof.* Let $(\pi, K, \Sigma)$ be as in Corollary 10.2.3, and let $\xi_1, \ldots, \xi_n \in \mathrm{Hdg}(\pi, K, \Sigma)$ be a basis for this finite-dimensional vector space over $\mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)$. Also, let

$$\xi_{\mathbb{C}} = \sum \alpha_i \xi_i, \quad \alpha_i \in \mathbb{C}$$

be a vector satisfying the conclusion of Corollary 10.2.3. We may choose a functional $\lambda \in \mathrm{Hom}(\pi_{2,f}^E, E)$ so that the composite

$$\lambda_{\mathbb{C}} \circ (\xi_{\mathbb{C}})_* : H_c^{3d}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3, \boldsymbol{\ell}_2-3), \mathbb{C}})[\Pi_{S_f}] \otimes \mathbb{C} \to H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, \mathbb{C}})_{\pi_{2,f}}$$

is surjective. We claim that there exist scalars $\beta_1, \cdots, \beta_n \in \mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)$ such that

$$\lambda \circ (\beta_1 \xi_1^* + \cdots + \beta_n \xi_n^*) : H_c^{3d}(S(\mathbf{GSp}_4), \mathcal{V}_{(\boldsymbol{\ell}_1-3, \boldsymbol{\ell}_2-3), E})[\Pi_{S_f}] \to H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, E})_{\pi_{2,f}}$$

is surjective; indeed, this condition corresponds to a Zariski-open subset $U \subset \mathbb{A}^n_{\mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)}$ because

$$H_!^d(S(\mathbf{GL}_2), \mathcal{V}_{\boldsymbol{m}_2-2, E})_{\pi_{2,f}}$$

is finite-dimensional, and it is satisfied by $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, so it is also satisfied by infinitely many tuples $(\beta_1, \ldots, \beta_n) \in \mathbb{Q}(\boldsymbol{m}_1, \boldsymbol{m}_2)^n$. For such a tuple, the Hodge class

$$\xi \coloneqq \sum \beta_i \xi_i \in \mathrm{Hdg}(\pi, K, \Sigma)$$

satisfies the conclusion of the theorem.

□

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