KOLYVAGIN'S CONJECTURE, BIPARTITE EULER SYSTEMS, AND HIGHER CONGRUENCES OF MODULAR FORMS

NAOMI SWEETING

ABSTRACT. Let $E/\mathbb{Q}$ be an elliptic curve and let K be an imaginary quadratic field. Under a certain Heegner hypothesis, Kolyvagin constructed cohomology classes for E using K-CM points and conjectured they did not all vanish. Conditional on this conjecture, he described the Selmer rank of E using his system of classes. We extend work of Wei Zhang to prove new cases of Kolyvagin's conjecture by considering congruences of modular forms modulo large powers of p. Additionally, we prove an analogous result, and give a description of the Selmer rank, in a complementary "definite" case (using certain modified L-values rather than CM points). Similar methods are also used to improve known results on the Heegner point main conjecture of Perrin-Riou. One consequence of our results is a new converse theorem, that p-Selmer rank one implies analytic rank one, when the residual representation has dihedral image.

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1. Introduction

Let f be a weight 2 cuspidal eigenform, new of level $\Gamma_0(N)$, without complex multiplication. The Birch and Swinnerton-Dyer Conjecture predicts:

$$r(A_f/\mathbb{Q}) = [E_f : \mathbb{Q}]\operatorname{ord}_{s=1} L(f,s), \tag{1}$$

where A_f is a representative of the isogeny class of GL_2-type abelian varieties associated to f, r denotes the Mordell-Weil rank, and E_f is the coefficient field of f. In pioneering works on this problem, Perrin-Riou [57] and Kolyvagin [45, 46] studied ranks of elliptic curves over an auxiliary imaginary quadratic field K through the theory of Heegner points on modular curves. We prove, in new cases, conjectures made by both authors.

Fix a quadratic imaginary field K, and a prime $\wp$ of E_f of residue characteristic p, with $\mathcal{O} = \mathcal{O}_{f,\wp}$ the completion at $\wp$ of the ring of integers $\mathcal{O}_f \subset E_f$. Assume the following generalized Heegner hypothesis:

(Heeg) $N = N^+N^-$, where all $\ell | N^+$ are split in K, all $\ell | N^-$ are inert in K, and N^- is squarefree,

as well as:

(unr) $$p \nmid 2N\operatorname{disc}(K).$$

The rational p-adic Tate module V_pA_f of A_f is equipped with an action of E_f; write $V_f \coloneqq V_pA_f \otimes_{E_f} E_{f,\wp}$ for the $\wp$-adic Galois representation attached to f, and let $T_f \subset V_f$ be a Galois-stable $\mathcal{O}$-lattice. We shall assume that $\overline{T}_f \coloneqq T_f/\wp T_f$ is absolutely irreducible as a representation of the Galois group $G_\mathbb{Q} \coloneqq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For purposes of exposition in this introduction, we sometimes assume:

(sclr) The image of the $G_\mathbb{Q}$ action on $\overline{T}_f$ contains a nontrivial scalar.

By [18, Lemma 6.1], (sclr) holds when $p > 5$.

To formulate Kolyvagin's conjecture, we use the hypothesis

(disc) $$\operatorname{disc}(K) \neq -3, -4,$$

and that the number of prime factors $\nu(N^-)$ is even. If m is a squarefree product of primes inert in K, one can use Heegner points of conductor m on the Shimura curve X_{N^+,N^-} to construct classes

$$c(m) \in H^1(K, T_f/I_m),$$

where I_m is the ideal of $\mathcal{O} = \mathcal{O}_{f,\wp}$ generated by $\ell+1$ and the ℓth Hecke eigenvalue $a_\ell(f)$ for all $\ell|m$. (See (4.3.1) for the definition of X_{N^+,N^-}, and (5.3.4), (8.2.7) for the definition of $c(m)$.) These classes are a mild generalization of the ones constructed by Kolyvagin [46]. We are able to prove the following result towards Kolyvagin's conjecture that the system $\{c(m)\}$ is nontrivial:

Theorem A (Corollary 8.2.8). *Assume (Heeg), (unr), and (disc) hold for $f, \wp$, and K, and $\nu(N^-)$ is even. Suppose the following conditions hold:*

$$(\diamondsuit)\quad \begin{cases} \bullet\ \text{The modulo } \wp \text{ representation } \overline{T}_f \text{ associated to } f \text{ is absolutely irreducible; if } p=3, \text{ then } \overline{T}_f \text{ is not induced from a character of } G_{\mathbb{Q}(\sqrt{-3})}. \\ \bullet\ \text{If } p \text{ is inert in } K, \text{ then there exists some prime } \ell_0 || N. \\ \bullet\ \text{If } a_p \text{ is not a } \wp\text{-adic unit, then there exist primes } \ell_i || N \text{ for } i=1,2 \\ \text{(possibly with } \ell_1=\ell_2\text{) such that } \overline{T}_f|_{G_{\mathbb{Q}_{\ell_i}}} \text{ is ramified for } i=1,2 \text{ and} \\ \overline{T}_f^{G_{\mathbb{Q}_{\ell_1}}} = (\overline{T}_{f\otimes\chi_K})^{G_{\mathbb{Q}_{\ell_2}}} = 0, \text{ where } f\otimes\chi_K \text{ is the quadratic twist.} \end{cases}$$

Then there exists a nonzero Kolyvagin class

$$0 \neq c(m) \in H^1(K, T_f/I_m).$$

As Kolyvagin observed, Theorem A can be used to give a description of the Selmer ranks $r^\pm = \mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}(K,T_f)^\pm$, where superscripts refer to the action of complex conjugation. Indeed, define the vanishing order of the system $\{c(m)\}$ as

$$\nu := \min\{\nu(m) \,:\, c(m)\neq 0\} \tag{2}$$

where as before ν denotes the number of prime factors. Then we have:

Corollary B (Corollary 8.2.8). *Under (sclr) and the assumptions of Theorem A,*

$$\max\left\{r^+, r^-\right\} = \nu+1.$$

Moreover $r^+ + r^-$ is odd, and the larger eigenspace for complex conjugation has eigenvalue $(-1)^{\nu+1}\epsilon_f$, where ϵ_f is the global root number of f.

The latter two assertions can alternatively be deduced from the parity conjecture proven by Nekovář [52].

Since $c(1) \in \mathrm{Sel}(K,T_f)$ is the Kummer image of the classical Heegner point $y_K \in A_f(K)$, the Gross-Zagier formula implies that $L'(f/K,1)\neq 0$ if and only if $c(1)\neq 0$, or equivalently if and only if y_K is non-torsion. Hence Corollary B yields a so-called p-converse theorem (in fact, under slightly weaker hypotheses):

Corollary C (Corollary 8.1.3). *Assume that (Heeg), (unr), and Condition $\diamondsuit$ hold for f, $\wp$, and K, and $\nu(N^-)$ is even. Then $L'(f/K,1)\neq 0$ if and only if $\mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}(K,T_f)=1$, in which case A_f has Mordell-Weil rank $[E_f:\mathbb{Q}]$ over K.*

Now suppose instead that $\nu(N^-)$ is odd; it turns out that Kolyvagin's construction, suitably modified, may still be used to relate Selmer ranks and CM points. The Jacquet-Langlands correspondence associates to f a quaternionic modular form

$$\phi_f : X_{N^+,N^-} \to \mathcal{O}_f, \tag{3}$$

where X_{N^+,N^-} is a double coset space for a definite quaternion algebra, usually called a Shimura set. (See (4.4.2) for the definition.) If m is a squarefree product of primes inert in K, there exist analogues of CM points of conductor m on X_{N^+,N^-}. Using the values of ϕ_f at these points, we construct certain special elements (well-defined up to units)

$$\lambda(m) \in \mathcal{O}/I_m. \tag{4}$$

Here the ideal $I_m \subset \mathcal{O}$ is as before; see (5.3.4) and (8.2.7) for the definition of $\lambda(m)$. The analogues of the elements $\lambda(m)$ for p-power conductor have long been used in anticyclotomic Iwasawa theory, e.g. [3]. However, for squarefree m, a novel observation of this work is that the elements $\lambda(m)$ encode the same information about the Selmer ranks of A_f/K as Kolyvagin's classes $c(m)$.

Theorem D (Corollary 8.2.8). *Suppose that (Heeg), (unr), (sclr), (disc), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is odd. Then the vanishing order*

$$\nu := \min\{\nu(m) : \lambda(m) \neq 0\}$$

is finite and

$$\nu = \max\{r^+, r^-\}.$$

Moreover $(-1)^\nu = \epsilon_f$ and $r^+ + r^-$ is even.

As before, the final statement is a consequence of the parity conjecture; we include it only to emphasize that it follows from the non-vanishing of some $\lambda(m)$, in analogy to the indefinite case.

1.1. **Comparison to previous results.** In the indefinite case, the first results towards Kolyvagin's conjecture were obtained by Zhang [81], under a number of additional assumptions: that $p \geq 5$, that the Galois representation associated to $\overline{T}_f$ is surjective, and additional hypotheses on the residual ramification. In particular, under the hypotheses of [81], there exists a class $c(m)$ whose reduction in $H^1(K, \overline{T}_f)$ is nonzero; this is not the case in general. In the definite case, the classes $\lambda(m)$ are a novel feature of this work and were not considered in [81]. Since the results of this paper were first announced, alternative approaches to Kolyvagin's conjecture have been introduced by Burungale, Castella, Grossi, and Skinner [7] in the split ordinary case, and by Kim [43] in the ordinary case with surjective residual Galois representation and some ramification conditions, based on various forms of the Iwasawa main conjectures for f. In the context of multiplicative reduction, Kolyvagin's conjecture has also been studied by Skinner and Zhang [69].

The converse theorem we obtain (Corollary C) is new in several cases, most notably when the image of the Galois action on $\overline{T}_f$ is dihedral, or when $p = 3$. A number of authors have also obtained p-converse theorems by purely Iwasawa-theoretic methods, without first proving Kolyvagin's conjecture; see for instance the work of Skinner [70] and Wan [78] in the split ordinary case, Kim [42] in the ordinary case, Castella-Wan [17] and Burungale-Büyükboduk-Lei [6] in the non-ordinary case, Burungale-Tian [11] and Burungale-Castella-Skinner-Tian [10] in the CM case, Castella-Grossi-Lee-Skinner [15] in the residually reducible case, and Venerucci [77] in the case of multiplicative reduction.

1.2. **Iwasawa theory.** Now suppose again that $\nu(N^-)$ is even, and assume as well that $\wp$ is a prime of ordinary reduction for f, with p split in K. While the Kolyvagin classes are constructed by varying the conductor of CM points on X_{N^+,N^-} over squarefree integers, one may instead p-adically interpolate CM points of p-power conductor to obtain a class:

$$\boldsymbol{\kappa}_\infty \in H^1(K, T_f \otimes \Lambda(\Psi)), \tag{5}$$

where $\Lambda = \mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$ is the anticyclotomic Iwasawa algebra, given $\mathrm{Gal}(\overline{K}/K)$-action by the tautological character Ψ. (Note that the specialization of $\boldsymbol{\kappa}_\infty$ at the trivial character is a multiple of $c(1)$.) The methods used to prove Theorem A also yield a result towards Perrin-Riou's Heegner point main conjecture. To state it, let W_f be the p-divisible $G_{\mathbb{Q}}$-module V_f/T_f.

Theorem E (Corollary 7.3.2). *Suppose that (Heeg), (unr), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is even. Suppose further that a_p is a $\wp$-adic unit and p splits in K. Then there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}(K_\infty, W_f)^\vee \approx \Lambda \oplus M \oplus M$$

for some torsion Λ-module M, and

$$\mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}(K, T_f \otimes \Lambda)}{\Lambda \cdot \boldsymbol{\kappa}_\infty}\right) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If (sclr) holds, then the equality is true in Λ.

For precise definitions of the above Selmer groups and of $\boldsymbol{\kappa}_\infty$, which is denoted $\boldsymbol{\kappa}(1)$ in the text, see §5.2.

Finally, we have the following result on the anticyclotomic main conjecture for f when $\nu(N^-)$ is odd. Evaluating the quaternionic modular form ϕ_f on CM points of p-power conductor on the Shimura set X_{N^+,N^-}, one constructs the algebraic p-adic L-function

$$\boldsymbol{\lambda}_\infty \in \Lambda, \tag{6}$$

denoted $\boldsymbol{\lambda}(1)$ in the text. The square of $\boldsymbol{\lambda}_\infty$ has an interpolation property for twisted L-values of f.

Theorem F (Corollary 7.3.3)**.** *Suppose that (Heeg), (unr), and Condition $\diamondsuit$ hold for $f, \wp$, and K, and that $\nu(N^-)$ is odd. Suppose further that a_p is a $\wp$-adic unit and p splits in K. Then there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}(K_\infty, W_f)^\vee \approx M \oplus M$$

for some torsion Λ-module M, and

$$(\boldsymbol{\lambda}_\infty) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. If additionally (sclr) holds, then the equality is true in Λ.

One direction of this equality is due to Skinner-Urban's work on the Iwasawa main conjecture [71]; indeed, along with [29] in the non-ordinary case, this is an essential ingredient in all of our results, as explained below.

1.3. **Comparison to previous results.** The hypotheses in Zhang's proof of Kolyvagin's conjecture were carried over to Burungale, Castella, and Kim's proof [8] of the lower bound on the Selmer group in the Heegner point main conjecture, where it was also assumed that p is not anomalous. While the methods used in this paper build on those of [8], Castella and Wan [16] have used a different method to prove a three-variable main conjecture when $\nu(N^-)$ is even. Their result also requires some hypotheses on residual ramification avoided here, and that N be squarefree. (Also see the references cited in §1.1 for several results on the Heegner point main conjecture in other contexts.)

For upper bounds on the Selmer group in Theorem E and Theorem F, various technical assumptions on the residual representation and on the image of the Galois action were used in prior works by Bertolini and Darmon [3] and Howard [37, 38], and in Chida-Hsieh's higher-weight generalization [18].

Since the results of this paper were first announced, Burungale, Castella, and Skinner [9] have also given an independent proof of Theorem E when $N^- = 1$ and $p > 3$ using base change.

1.4. **Overview of the proofs.** To prove Theorems A and D, we extend Kolyvagin's construction to a larger system of classes

$$c(m, Q_1) \in H^1(K, T_f/\wp^M), \quad \lambda(m, Q_2) \in \mathcal{O}/\wp^M, \tag{7}$$

where M is a fixed integer, and m, Q_1, Q_2 are squarefree products of auxiliary primes satisfying certain congruence conditions, such that $\nu(N^-Q_1)$ is even and $\nu(N^-Q_2)$ is odd. The classes (7) form a bipartite Euler system in the sense of Howard [38] for each fixed m and a Kolyvagin system for each fixed Q_1. If $\nu(N^-)$ itself is even, then the classes $c(m, 1) = c(m)$ agree with Kolyvagin's original construction. The Euler system relations are of the form:

$$\mathrm{loc}_q\, c(m, Q_1) \sim \lambda(m, Q_1 q) \sim \partial_{q'} c(m, Q_1 q q'), \tag{8}$$

where q, q' are two additional auxiliary primes not dividing Q_1; and

$$\mathrm{loc}_\ell^\pm\, c(m, Q_1) \sim \partial_\ell^\mp c(m\ell, Q_1), \tag{9}$$

where ℓ is an additional auxiliary prime not dividing m. Here loc_q, $\partial_{q'}$, $\mathrm{loc}_\ell^\pm$, $\partial_\ell^\pm$ are certain localization maps landing in subspaces of the local cohomology free of rank one over $\mathcal{O}/\wp^M$. The classes $c(m, Q_1)$ were introduced by Zhang, although the $\lambda(m, Q_2)$ are only implicit in [81].

If $c(m, Q_1) \neq 0$, then one can use (8) and (9) to find an auxiliary ℓ — either prime or equal to 1 — such that $\partial_q c(m\ell, Q_1) \neq 0$ for some $q|Q_1$. By (8), this implies $\lambda(m\ell, Q_1/q) \neq 0$. On the other hand, if $\lambda(m, Q_2) \neq 0$ and $q|Q_2$, then by (8) $c(m, Q_2/q) \neq 0$. Combining these two observations, we reduce the non-vanishing of some class $c(m, 1)$ or $\lambda(m, 1)$ — depending on the parity of $\nu(N^-)$ — to exhibiting a single Q_2 such that $\lambda(1, Q_2) \neq 0$.

Now, if there exists a newform g of level NQ_2 with a congruence to f modulo $\wp^M$, then $\lambda(1, Q_2)$ is essentially the reduction of the algebraic part of the L-value $L^{\mathrm{alg}}(g/K, 1)$ modulo $\wp^M$, which is related to the length of the Selmer group of g by the Iwasawa main conjecture [71, 29]. To complete the proof, it therefore suffices to choose a suitable Q_2 and construct such a g with a small Selmer group. We remark that our results can only be obtained by working modulo $\wp^M$ for a large M, since in general it will not be possible to choose g such that $L^{\mathrm{alg}}(g/K, 1)$ is a $\wp$-adic unit; in [81], $M = 1$ is fixed throughout, and the need to show that the L-value is a unit is responsible for most of the additional hypotheses.

To construct g, we use deformation-theoretic techniques developed by Ramakrishna [59], and extended by Fakhruddin-Khare-Patrikis [28]. Standard level-raising methods work by producing a modulo $\wp$ eigenform of the desired level, and then using that all modulo $\wp$ eigenforms lift to characteristic zero, but this is not the case modulo $\wp^M$. Instead, we deform the representation $T_f/\wp^M$ to a $\wp$-adic Galois representation of a suitable auxiliary level, and then apply modularity lifting to ensure the resulting representation is modular. The auxiliary level Q_2 must be chosen to control two Selmer groups: the adjoint Selmer group governing the deformation problem, and the Selmer group $\mathrm{Sel}(K, W_g)$ that is related to the L-value. (Here, W_g is the p-divisible Galois module constructed analogously to $W_f = V_f/T_f$.)

We now make some remarks on the construction of the Euler system. The elements $c(m, Q_1)$ (resp. $\lambda(m, Q_2)$) are constructed from CM points of conductor m on the Shimura curve X_{N^+,N^-Q_1} (resp. Shimura set X_{N^+,N^-Q_2}), associated to the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant N^-Q_1 (resp. the definite quaternion algebra with discriminant N^-Q_2). Similar Euler system constructions have been made by many authors, e.g. in [18, 3] as well as in [81], but all have relied on certain hypotheses ensuring an integral multiplicity one property for the space of algebraic modular forms on X_{N^+,N^-Q_i}, which we do not impose here. Instead, we obtain a control on the failure of multiplicity one, using the work of Helm [35] on maps between Jacobians of modular curves and Shimura curves. The construction of the Euler system is intimately related to level-raising, and so our method also improves results on level-raising of f to algebraic eigenforms modulo $\wp^M$ new at multiple auxiliary primes, which had previously been restricted to the multiplicity one case. A precise statement is given in Theorem 4.6.7.

The proof of Theorem E is similar to that of Theorem A: the p-adically interpolated Heegner class $\boldsymbol{\kappa}_\infty$ is viewed as the bottom layer of an Euler system $\{\boldsymbol{\kappa}(Q_1), \boldsymbol{\lambda}(Q_2)\}$. (The squarefree conductor m no longer plays a role.) If g, as above, is a newform of level NQ_2 with a congruence to f, then $\boldsymbol{\lambda}(Q_2)$ is congruent to Bertolini and Darmon's anticyclotomic p-adic L-function of g [3]. Using this and an Euler system argument, we reduce the lower bound on the Selmer group in the Heegner point main conjecture to the lower bound on the Selmer group in the anticyclotomic main conjecture for g, which was proven in [71]. Finally, the upper bound on the Selmer group in Theorems E and F follows by standard arguments from the construction of the Euler system.

In the text, the arguments described above are phrased in the language of ultrapatching, which amounts to a formalism for letting M tend to infinity; this also forces each prime factor of m, Q_1, Q_2 to tend to infinity in order to satisfy the congruence conditions. (The number of prime factors of m, Q_1, and Q_2 remains bounded.) This method was inspired by [67], where ultrapatching was applied to the Taylor-Wiles construction. Our setting is different in that we patch Galois cohomology groups and Selmer groups rather than geometric étale cohomology groups. The benefit of ultrapatching is that it allows us to consider the Euler system classes as characteristic zero objects in patched Selmer groups, significantly streamlining the Euler system arguments. For instance, with patching, we are able to make precise the heuristic that the non-vanishing of each Euler system class $c(m, Q_1)$ or $\lambda(m, Q_2)$ is equivalent to the (m, Q_i)-transverse Selmer group being rank one or zero, respectively, cf. Lemma 8.2.4.

Structure of the paper. In §2, we review basic properties of ultrafilters and introduce patched cohomology and Selmer groups. In §3, we present a simplified version of the theory of bipartite Euler systems that appeared in [38], using patched cohomology. In §4, we establish the geometric inputs that will be used to construct bipartite Euler systems: the work of Helm on maps between modular curves and Shimura curves, the modulo $\wp^M$ level-raising result, and the behavior of Heegner points on Shimura curves under reduction and specialization. In §5, we present a general framework for constructing bipartite Euler systems out of CM points, which we then specialize for our applications. In §6, we give the deformation-theoretic input to construct the newform g (in fact a sequence g_n satisfying increasingly deep congruence conditions). We then prove Theorems E and F in §7. Theorems A and D are proven in §8. Finally, in the appendix we generalize some results on degrees of modular parametrizations, proved in special cases by Ribet-Takahashi [64] and Khare [41, §3.2]. These results are needed to compare different normalizations of periods in §8.

1.5. Notational conventions.

- If N is a squarefree positive integer, then $\nu(N)$ denotes its number of prime factors.
- If L is an algebraic extension of $\mathbb{Q}$, we write $G_L = \mathrm{Gal}(\overline{L}/L)$ for its absolute Galois group.
- If L is a number field, we write $\mathbb{A}_L$ (resp. $\mathbb{A}_{f,L}$) for its ring of adèles (resp. finite adèles). If v is a place of L, then we write G_{L_v} for the absolute Galois group of the completion L_v, and $I_v \subset G_{L_v}$ for the inertia subgroup if v is nonarchimedean. We write $\mathcal{O}_L$ for the ring of integers of L and $\mathcal{O}_{L,\wp}$ (resp. $\mathcal{O}_{L,(\wp)}$) for the completion (resp. localization) at a prime ideal $\wp \subset \mathcal{O}_L$. If Σ is a finite set of places of L, then L^Σ denotes the largest algebraic extension of L unramified outside Σ. If Σ is a

finite set of places of $\mathbb{Q}$ and Σ_L is the set of places of L lying over places in Σ, then we abbreviate $L^\Sigma := L^{\Sigma_L}$.

- The symbol Frob_v always denotes an arithmetic Frobenius element.
- If L is a number field and A is a G_L-module, then $L(A)$ is the smallest algebraic extension of L such that $G_{L(A)}$ acts trivially on A.
- We fix, for each place v of $\mathbb{Q}$, an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$, which induces an embedding $G_{\mathbb{Q}_v} \hookrightarrow G_{\mathbb{Q}}$.
- For most of the paper, we will fix a quadratic imaginary field $K \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. In this case we also fix embeddings $\overline{K} \hookrightarrow \overline{K}_w$ for all places w of K, which induce embeddings $G_{K_w} \hookrightarrow G_K$. If w is induced by a place v of $\mathbb{Q}$ and the embedding $K \subset \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$ chosen above, then we make these choices compatibly with the ones for $\mathbb{Q}$, so that G_{K_w} is a subgroup of $G_{\mathbb{Q}_v}$.
- The p-adic cyclotomic character is denoted $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$.
- Starting in §3, we shall fix a cuspidal eigenform f of weight 2 and trivial character, new of level N, and without complex multiplication. For all primes ℓ, let $a_\ell(f)$ be the ℓth coefficient of the normalized q-expansion of f, and let $\epsilon_f = \pm 1$ be the global root number. We denote by $\mathcal{O}_f \subset \mathbb{C}$ the ring of integers of the number field generated over $\mathbb{Q}$ by the Fourier coefficients $a_\ell(f)$. We denote by $\wp$ a prime of $\mathcal{O}_f$ lying over an odd prime p; except in §6, we also assume $p \nmid N$. We write $\mathcal{O}$ for the completion of $\mathcal{O}_f$ at $\wp$, and let $E = \mathrm{Frac}(\mathcal{O})$. We let $\pi \in \mathcal{O}$ be a uniformizer.
- For f as above, we denote by V_f the $\wp$-adic Galois representation associated to f, defined in the beginning of this introduction. Fix a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice $T_f \subset V_f$. We will always assume that $\overline{T}_f := T_f/\pi$ is absolutely irreducible, in which case T_f is determined up to rescaling. We also write $W_f = T_f \otimes_{\mathcal{O}} E/\mathcal{O} \cong V_f/T_f$.

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2. Ultrafilters and patching

2.1. **Ultraproducts.** The following discussion is inspired by the unpublished notes of Manning [48, §I.1].

2.1.1. A (non-principal) ultrafilter $\mathfrak{F}$ for the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ is a collection of subsets of $\mathbb{N}$ satisfying the following properties:

(1) Every set $S \in \mathfrak{F}$ is infinite.
(2) For every $S \subset \mathbb{N}$, either $S \in \mathfrak{F}$ or $\mathbb{N} - S \in \mathfrak{F}$.
(3) If $S_1 \subset S_2 \subset \mathbb{N}$ and $S_1 \in \mathfrak{F}$, then $S_2 \in \mathfrak{F}$.
(4) If $S_1, S_2 \in \mathfrak{F}$, then $S_1 \cap S_2 \in \mathfrak{F}$.

Throughout this paper, we fix once and for all a non-principal ultrafilter $\mathfrak{F}$ on $\mathbb{N}$, which is possible assuming the axiom of choice. We will say that a statement P holds for $\mathfrak{F}$-many $n \in \mathbb{N}$ if the set S of n for which P holds lies in $\mathfrak{F}$.

Proposition 2.1.2. *Suppose that $\mathcal{C}$ is a finite set and $S \subset \mathbb{N}$ lies in $\mathfrak{F}$. Then for any function $t : S \to \mathcal{C}$, there is a unique $c \in \mathcal{C}$ such that $t(n) = c$ for $\mathfrak{F}$-many n.*

Proof. The function t defines a finite partition of $\mathbb{N}$:

$$\mathbb{N} = (\mathbb{N} - S) \sqcup \bigsqcup_{c \in \mathcal{C}} t^{-1}(c).$$

An easy induction argument shows that, for any partition of $\mathbb{N}$ into a finite number sets, exactly one of the sets lies in $\mathfrak{F}$. Since $\mathbb{N} - S \notin \mathfrak{F}$, the result follows. ☐

2.1.3. If $\mathcal{M} = \{M_n\}_{n\in\mathbb{N}}$ is a sequence of sets indexed by $\mathbb{N}$, then $\mathfrak{F}$ defines an equivalence relation $\sim$ on $\prod M_n$:

$$(m_n)_{n\in\mathbb{N}} \sim (m'_n)_{n\in\mathbb{N}} \iff \{n : m_n = m'_n\} \in \mathfrak{F}.$$

The quotient $\prod M_n/\sim$ is called the **ultraproduct** of the sequence $\mathcal{M}$ and is denoted $\mathcal{U}(\mathcal{M})$. The ultraproduct is functorial: let $\mathcal{M}' = \{M'_n\}$ be another sequence of sets and suppose given, for $\mathfrak{F}$-many n, maps $\varphi_n : M_n \to M'_n$. Then there is a natural map $\varphi^{\mathcal{U}} : \mathcal{U}(\mathcal{M}) \to \mathcal{U}(\mathcal{M}')$. In particular, if each M_n is endowed with the structure of an abelian group (resp. R-module for a fixed ring R), then $\mathcal{U}(\mathcal{M})$ is naturally an abelian group (resp. R-module).

Proposition 2.1.4. *(1) Let $\mathcal{M} = \{M_n\}_{n\in\mathbb{N}}$ and $\mathcal{M}' = \{M'_n\}_{n\in\mathbb{N}}$ be sequences of nonempty sets, and suppose given maps $\varphi_n : M_n \to M'_n$ for $\mathfrak{F}$-many n. If φ_n is injective (resp. surjective, bijective) for $\mathfrak{F}$-many n, then $\varphi^{\mathcal{U}}$ is injective (resp. surjective, bijective).*

(2) Let M be a finite set and suppose $\mathcal{M} = \{M\}_{n\in\mathbb{N}}$ is the constant sequence. Then the diagonal map $M \to \mathcal{U}(\mathcal{M})$ is an isomorphism.

(3) Suppose $\mathcal{M} = \{M_n\}_{n\in\mathbb{N}}$, where each M_n is a nonempty finite set such that $\#M_n < C$ for $\mathfrak{F}$-many n. Then $\mathcal{U}(\mathcal{M})$ is finite and $\#\mathcal{U}(\mathcal{M}) = \#M_n$ for $\mathfrak{F}$-many n.

Proof. (1) Suppose φ_n is injective for $\mathfrak{F}$-many n and let $m, m' \in \mathcal{U}(\mathcal{M})$ be the equivalence classes of sequences $(m_n)_{n\in\mathbb{N}}$ and $(m'_n)_{n\in\mathbb{N}}$. If $\varphi^{\mathcal{U}}(m) = \varphi^{\mathcal{U}}(m')$, then for $\mathfrak{F}$-many n, $\varphi_n(m_n) = \varphi_n(m'_n)$. Hence for $\mathfrak{F}$-many n, $m_n = m'_n$, so $m = m'$ in $\mathcal{U}(\mathcal{M})$. Therefore $\varphi^{\mathcal{U}}$ is injective.

Now suppose φ_n is surjective for $\mathfrak{F}$-many n, and let $m' \in \mathcal{U}(\mathcal{M}')$ be an element represented by $(m'_n)_{n\in\mathbb{N}}$. We will show that m' lies in the image of $\varphi^{\mathcal{U}}$. Let $S \in \mathfrak{F}$ be such that φ_n is surjective for $n \in S$. Define a new sequence $(m_n)_{n\in\mathbb{N}}$ by choosing $m_n \in M_n$ arbitrarily for $n \notin S$, and choosing $m_n \in M_n$ such that $\varphi_n(m_n) = m'_n$ for $n \in S$. Then the equivalence class m of this sequence satisfies $\varphi^{\mathcal{U}}(m) = m'$. Hence, $\varphi^{\mathcal{U}}$ is surjective.

(2) The diagonal map is clearly injective, and it is surjective by Proposition 2.1.2.

(3) By Proposition 2.1.2, there exists some $c < C$ such that $\#M_n = c$ for $\mathfrak{F}$-many n. Let $[c] = \{0, \ldots, c-1\}$ and choose isomorphisms of sets

$$\varphi_n : M_n \xrightarrow{\sim} [c]$$

for $\mathfrak{F}$-many n. By (1), $\varphi^{\mathcal{U}}$ induces an isomorphism from $\mathcal{U}(\mathcal{M})$ to the ultraproduct $\mathcal{C}$ of the constant sequence $\{[c]\}_{n\in\mathbb{N}}$. However, $\mathcal{C}$ is canonically isomorphic to $[c]$ by (2).

□

Proposition 2.1.5. *Let $\mathcal{S}$ be the category of sequences of abelian groups indexed by $\mathbb{N}$. Then $\mathcal{U}$ is exact as a functor from $\mathcal{S}$ to the category of abelian groups.*

Proof. Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$, $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$, and $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ be three sequences of abelian groups, and suppose given exact sequences

$$0 \to A_n \xrightarrow{\varphi_n} B_n \xrightarrow{\psi_n} C_n \to 0$$

for all $n \in \mathbb{N}$. We wish to show that

$$0 \to \mathcal{U}(\mathcal{A}) \xrightarrow{\varphi^{\mathcal{U}}} \mathcal{U}(\mathcal{B}) \xrightarrow{\psi^{\mathcal{U}}} \mathcal{U}(\mathcal{C}) \to 0$$

is exact. By Proposition 2.1.4(1), it suffices to show that the kernel of $\psi^{\mathcal{U}}$ is the image of $\varphi^{\mathcal{U}}$. Suppose $(b_n)_{n\in\mathbb{N}}$ represents an element $b \in \ker \psi^{\mathcal{U}}$. Then, by definition, $\psi_n(b_n) = 0$ for $\mathfrak{F}$-many n, so for $\mathfrak{F}$-many n there exists $a_n \in A_n$ with $\varphi_n(a_n) = b_n$. Hence, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ representing an element $a \in \mathcal{U}(\mathcal{A})$ with $\varphi^{\mathcal{U}}(a) = b$. We have shown that $\ker \psi^{\mathcal{U}} \subset \operatorname{im} \varphi^{\mathcal{U}}$. Since the opposite inclusion is clear, this completes the proof. □

2.2. Ultraprimes.

2.2.1. Fix a number field L with algebraic closure $\overline{L}$ and let M_L be its set of places. For each $s \in M_L$, we assume fixed an embedding $\overline{L} \hookrightarrow \overline{L_s}$, which identifies G_{L_s} with a subgroup of G_L.[1] If $\mathcal{M}_L$ is the constant sequence of sets $\{M_L\}_{n\in\mathbb{N}}$, then we define the set of **ultraprimes** of L as

$$\mathsf{M}_L = \mathcal{U}(\mathcal{M}_L).$$

By definition, an ultraprime $\mathsf{s} \in \mathsf{M}_L$ is an equivalence class of sequences $(s_n)_{n\in\mathbb{N}}$, where each s_n is a place of L.

The map $s \mapsto (s, s, \ldots)$ induces an embedding $M_L \hookrightarrow \mathsf{M}_L$, written $s \mapsto \underline{s}$, and we say an ultraprime is constant if it lies in the image of this embedding.

We will say $\mathsf{s} \in \mathsf{M}_L$ is archimedean if it is a constant ultraprime $\mathsf{s} = \underline{s}$ for an archimedean place s of L; s is non-archimedean otherwise.

Proposition 2.2.2. *Let s be a non-constant ultraprime. Then there exists a unique Frobenius element $\mathrm{Frob}_\mathsf{s} \in G_L$ with the following property: for each finite Galois extension $L \subset E \subset \overline{L}$, and for any representative $(s_n)_{n\in\mathbb{N}}$ of s, there are $\mathfrak{F}$-many n such that s_n is unramified in E/L and the Frobenius of s_n in $\mathrm{Gal}(E/L)$ is the natural image of Frob_s.*

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a representative of s, and fix for the time being a finite Galois extension E/L inside $\overline{L}$. If s_n is archimedean or ramified in E for $\mathfrak{F}$-many n, then Proposition 2.1.2 implies that s is constant. Thus for $\mathfrak{F}$-many n, the Frobenius of s_n is a well-defined element of $\mathrm{Gal}(E/L)$ (determined exactly, and not only up to conjugacy, by the fixed embeddings $E \hookrightarrow \overline{L} \hookrightarrow \overline{L_{s_n}}$). By Proposition 2.1.2, the map $n \mapsto \mathrm{Frob}_{s_n} \in \mathrm{Gal}(E/L)$ sends $\mathfrak{F}$-many n to a (unique) common value $g_E \in \mathrm{Gal}(E/L)$. Note that g_E does not depend on the representative $(s_n)_{n\in\mathbb{N}}$. By the uniqueness of g_E, the association $E \mapsto g_E$ is compatible with restriction to subextensions $E' \subset E$, hence defines an element of the absolute Galois group. □

2.2.3. Let s be an ultraprime. We define its abstract Galois group G_s as G_{L_s} if $\mathsf{s} = \underline{s}$ is constant, and as the semi direct product

$$\widehat{\mathbb{Z}}(1) \rtimes \langle \mathrm{Frob}_\mathsf{s} \rangle$$

otherwise. Here, $\langle \mathrm{Frob}_\mathsf{s} \rangle$ denotes the free profinite group on one generator, where the generator acts on $\widehat{\mathbb{Z}}(1)$ by $\mathrm{Frob}_\mathsf{s} \in G_L$. We define the inertia group $I_\mathsf{s} \subset G_\mathsf{s}$ of s to be the usual inertia group if s is constant, and the normal subgroup $\widehat{\mathbb{Z}}(1) \subset G_\mathsf{s}$ otherwise.

2.3. Local cohomology.

2.3.1. For any (topological) G_L-module A and for any $\mathsf{s} \in \mathsf{M}_L$, there is a natural action of G_s on A (factoring through the quotient $G_\mathsf{s} \to \langle \mathrm{Frob}_\mathsf{s} \rangle$ if s is nonconstant). We define local cohomology groups by:

$$\mathsf{H}^i(L_\mathsf{s}, A) := H^i(G_\mathsf{s}, A),$$
$$\mathsf{H}^i(I_\mathsf{s}, A) := H^i(I_\mathsf{s}, A), \quad i \geq 0,$$

where on the right-hand side we take continuous cochain cohomology. (In particular, for all $s \in M_L$, $\mathsf{H}^i(L_{\underline{s}}, A) = H^i(L_s, A)$.)

Proposition 2.3.2. *Let $\mathsf{s} \in \mathsf{M}_L$ be an ultraprime represented by a sequence $(s_n)_{n\in\mathbb{N}}$. If A is a finite, discrete G_L-module unramified outside finitely many primes, then for $\mathfrak{F}$-many n there are canonical isomorphisms (functorial in A, compatible with cup products, and compatible with the natural restriction maps):*

$$H^i(L_{s_n}, A) \simeq \mathsf{H}^i(L_\mathsf{s}, A),$$
$$H^i(I_{s_n}, A) \simeq \mathsf{H}^i(I_\mathsf{s}, A), \quad i \geq 0.$$

Proof. If s is the constant ultraprime $\underline{s}$, then $s_n = s$ for $\mathfrak{F}$-many n, and the desired isomorphisms are given by Proposition 2.1.4(2); so suppose s is nonconstant. For $\mathfrak{F}$-many n, the action of the decomposition group G_{s_n} at s_n on A is unramified and the Frobenius of s_n acts by Frob_s. Let ℓ_n be the prime of $\mathbb{Q}$ lying under s_n; since $L/\mathbb{Q}$ is a finite extension and A is finite, for $\mathfrak{F}$-many n we have $\ell_n \nmid |A|$. Restricting to these n, the inflation map induces isomorphisms:

$$H^i(G^t_{s_n}, A) \simeq H^i(L_{s_n}, A), \quad H^i(I^t_{s_n}, A) \simeq H^i(I_{s_n}, A),$$

[1]We will usually apply this formalism with $L = \mathbb{Q}$ or $L = K$, in which case the choices of embeddings $\overline{L} \hookrightarrow \overline{L_s}$ are fixed according to the conventions of §1.5.

where $G^t_{s_n}$ and $I^t_{s_n}$ denote the tame quotients and we again take continuous cochain cohomology. The tame Galois group $G^t_{s_n}$ is identified with the semidirect product:

$$I^t_{s_n} \rtimes \langle \mathrm{Frob}_{s_n} \rangle \simeq \widehat{\mathbb{Z}}^{(\ell_n)}(1) \rtimes \langle \mathrm{Frob}_{s_n} \rangle.$$

In particular, both $G^t_{s_n}$ and G_{s} have cohomological dimension two; and both $I^t_{s_n}$ and I_{s} have cohomological dimension one. So it suffices to prove the first isomorphism of the proposition when $i \leq 2$ and the second when $i \leq 1$. Now note that, on any finite quotient of $\widehat{\mathbb{Z}}(1)$, the actions of Frob_{s_n} and $\mathrm{Frob}_{\mathsf{s}}$ agree for $\mathfrak{F}$-many n. The proposition therefore follows from the following easy lemma in group cohomology (applied to both $G^t_{s_n}$ and G_{s}). □

Lemma 2.3.3. *Let $G = I \rtimes \langle F \rangle$ be a group, where I is abelian and profinite of cohomological dimension at most one, and $\langle F \rangle$ denotes the free profinite group on one generator, acting on I by an automorphism. If A is a finite $\mathbb{Z}[F]$-module, viewed as a G-module via $G \twoheadrightarrow \langle F \rangle$, then there are canonical isomorphisms for the continuous cochain cohomology:*

$$\begin{aligned} H^i(I, A) &= H^i(I/|A|, A), && i = 0, 1, \\ H^i(G, A) &= H^i(I/|A| \rtimes \langle F \rangle, A), && i = 0, 1, \\ H^2(G, A) &= H^1(\langle F \rangle, \mathrm{Hom}(I/|A|, A)), && \end{aligned}$$

Proof. The only identities that are not immediate are for $H^i(G, A)$, with $i = 1, 2$. For $i = 1$, we claim that the inflation map induces an isomorphism

$$H^1(I/|A| \rtimes \langle F \rangle, A) \xrightarrow{\sim} H^1(G, A).$$

Equivalently, if $H := |A| I \subset G$, we wish to show that the restriction map $H^1(G, A) \to H^1(H, A)$ is trivial. Indeed, the restriction map factors through $H^1(I, A) \to H^1(H, A)$, which is the zero map since I acts trivially on A.

For $i = 2$, the Hochschild-Serre spectral sequence gives a canonical isomorphism

$$H^2(G, A) = H^1(\langle F \rangle, H^1(I, A)) = H^1(\langle F \rangle, \mathrm{Hom}(I/|A|, A)),$$

since both $\langle F \rangle$ and I have cohomological dimension at most 1.

□

2.4. **Patched cohomology.**

2.4.1. Let $\mathsf{S} \subset \mathsf{M}_L$ be a finite set of ultraprimes $\left\{\mathsf{s}^{(1)}, \mathsf{s}^{(2)}, \ldots, \mathsf{s}^{(r)}\right\}$. A **representative** of S is a sequence $(S_n)_{n\in\mathbb{N}}$, with $S_n \subset M_L$, such that $S_n = \left\{s_n^{(1)}, \cdots, s_n^{(r)}\right\}$ for some sequences $(s_n^{(i)})_{n\in\mathbb{N}}$ representing $\mathsf{s}^{(i)}$. If $L = \mathbb{Q}$ and S contains no archimedean ultraprimes, we will also refer to S being represented by the sequence of squarefree integers $\prod_{\ell_n \in S_n} \ell_n$. If A is a G_L-module, we say A is unramified outside $\mathsf{S} \subset \mathsf{M}_L$ if it is unramified outside $\mathsf{S} \cap M_L$.

Definition 2.4.2. Let A be a finite G_L-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$, represented by a sequence $(S_n)_{n\in\mathbb{N}}$ with $S_n \subset M_L$. Then we define the ith unramified-outside-S patched cohomology, for all $i \geq 0$, by:

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \mathcal{U}\left(\left\{H^i(L^{S_n}/L, A)\right\}_{n\in\mathbb{N}}\right).$$

Remark 2.4.3. When $i = 0$, the patched cohomology $\mathsf{H}^0(L^{\mathsf{S}}, A)$ is canonically isomorphic to $H^0(L, A)$.

Proposition 2.4.4. *Let A be a finite G_L-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$. Then:*

1. *The patched cohomology $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ is independent of the choice of representative $(S_n)_{n\in\mathbb{N}}$ of S.*
2. *The maps $A \mapsto \mathsf{H}^i(L^{\mathsf{S}}/L, A)$ are functorial in A.*
3. *Given a finite set $\mathsf{S}' \subset \mathsf{M}_L$ containing S, there is a natural map $\mathsf{H}^i(L^{\mathsf{S}}/L, A) \to \mathsf{H}^i(L^{\mathsf{S}'}/L, A)$, compatible with the functoriality of (2).*

Proof. Given two sequences $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}} \subset M_L$ representing S, we have $S_n = T_n$ for $\mathfrak{F}$-many n; for these n we have

$$H^i(L^{S_n}/L, A) = H^i(L^{T_n}/L, A).$$

Hence Proposition 2.1.4(1) shows that $\mathcal{U}\left(\left\{H^i(L^{S_n}/L, A)\right\}_{n\in\mathbb{N}}\right)$ and $\mathcal{U}\left(\left\{H^i(L^{T_n}/L, A)\right\}_{n\in\mathbb{N}}\right)$ are canonically isomorphic. This shows (1); (2) and (3) are immediate from the functoriality of the ultraproduct. □

Lemma 2.4.5. *Let A be a finite G_L-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$. Then $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ is finite for all $i \geq 0$.*

Proof. By Proposition 2.1.4(3), it suffices to show that the cardinality of $H^i(L^S/L, A)$ remains bounded as S ranges over finite sets of M_L of cardinality $|\mathsf{S}|$ such that A is unramified outside S. This is clear when $i = 0$, and the case $i \geq 3$ is handled by [51, Chapter 1, Theorem 4.10(c)]. For $i = 1$ and 2, let S_0 be the set of primes at which A is ramified, or with residue characteristic dividing $|A|$. Now the map

$$H^i(L^S/L, A) \to \prod_{v\in S\cup S_0} H^i(L_v, A)$$

has kernel contained in $Ш^i_{S_0}(A)$, which is finite by part (a) of *loc. cit.* Moreover S_0 is finite and $H^i(L_v, A)$ is finite for all $v \in S_0$. For $v \in S \setminus S_0$, we have $|H^i(L_v, A)| \leq |A|^2$ by the local Euler characteristic formula and local Poitou-Tate duality; hence

$$|H^i(L^S/L, A)| \leq |Ш^i_{S_0}(A)| \cdot \prod_{v\in S_0} |H^i(L_v, A)| \cdot |A|^{2|\mathsf{S}|},$$

which gives the desired uniform bound. □

2.4.6. Suppose A is a topological G_L-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$. If A is profinite, then its unramified-outside-S patched cohomology is defined as:

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \varprojlim_{A\twoheadrightarrow A'} \mathsf{H}^i(L^{\mathsf{S}}/L, A')$$

where the inverse limit runs over finite quotients and the transition maps are induced by Proposition 2.4.4(2). Similarly, if A is ind-finite, then its unramified-outside-S patched cohomology is defined as

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \varinjlim_{A'\subset A} \mathsf{H}^i(L^{\mathsf{S}}/L, A'),$$

where the direct limit runs over finite submodules. If A is finite, then both these definitions recover Definition 2.4.2. If A is either profinite or ind-finite, then the totally patched cohomology is defined as

$$\mathsf{H}^i(L, A) = \varinjlim_{\mathsf{S}\subset\mathsf{T}\subset\mathsf{M}_L} \mathsf{H}^i(L^{\mathsf{T}}/L, A),$$

where the direct limit is over finite subsets such that A is unramified outside S and the transition maps are induced by Proposition 2.4.4(3).

By construction, the maps $A \mapsto \mathsf{H}^i(L^{\mathsf{S}}/L, A)$ and $A \mapsto \mathsf{H}^i(L, A)$ are functorial in topological G_L-modules which are either profinite or ind-finite.

Definition 2.4.7. A profinite topological G_L-module A is said to be countably profinite if it admits only countably many finite quotients, or equivalently, if it admits a presentation as a countable inverse limit of finite discrete G_L-modules.

Lemma 2.4.8. *Suppose A is countably profinite. Then for all $i \geq 0$, the natural map induces an isomorphism*

$$\mathsf{H}^i(L_{\mathsf{s}}, A) \simeq \varprojlim_{A\twoheadrightarrow A'} \mathsf{H}^i(L_{\mathsf{s}}, A')$$

where the inverse limit runs over finite quotients of A. Similarly, if A is ind-finite, then for all $i \geq 0$, the natural map induces an isomorphism

$$\mathsf{H}^i(L_{\mathsf{s}}, A) \simeq \varinjlim_{A'\subset A} \mathsf{H}^i(L_{\mathsf{s}}, A')$$

where the inverse limit runs over finite sub-G_L-modules of A.

Proof. In the countably profinite case, this follows from [55, Corollary 2.7.6] applied to G_{s}. In the ind-finite case, it is clear by the exactness of direct limits. $\square$

2.4.9. Suppose A is a topological G_L-module which is ind-finite or countably profinite, and unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$; let $\mathsf{s} \in \mathsf{M}_L$ be any ultraprime. Using Lemma 2.4.8, we can define localization maps

$$\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}} : \mathsf{H}^i(L^{\mathsf{S}}/L, A) \to \mathsf{H}^i(L_{\mathsf{s}}, A)$$
$$\mathrm{Res}_{\mathsf{s}} : \mathsf{H}^i(L, A) \to \mathsf{H}^i(L_{\mathsf{s}}, A)$$

as follows.

- If A is finite, define $\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}}$ as the composite
$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \mathcal{U}\left(\left\{H^i(L^{S_n}/L, A)\right\}_{n\in\mathbb{N}}\right) \to \mathcal{U}\left(\left\{H^i(L_{s_n}, A)\right\}_{n\in\mathbb{N}}\right) \simeq \mathsf{H}^i(L_{\mathsf{s}}, A),$$
where $(s_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ are sequences representing s and S and the last isomorphism is by Proposition 2.3.2.
- If A is countably profinite, define $\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}}$ as the composite
$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \varprojlim_{A\twoheadrightarrow A'} \mathsf{H}^i(L^{\mathsf{S}}/L, A') \xrightarrow{\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}}} \varprojlim_{A\twoheadrightarrow A'} \mathsf{H}^i(L_{\mathsf{s}}, A') \simeq \mathsf{H}^i(L_{\mathsf{s}}, A),$$
where the inverse limit runs over finite quotients of A and the last isomorphism is by Lemma 2.4.8.
- If A is ind-finite, similarly define $\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}}$ as the composite
$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) = \varinjlim_{A'\subset A} \mathsf{H}^i(L^{\mathsf{S}}/L, A') \xrightarrow{\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}}} \varinjlim_{A'\subset A} \mathsf{H}^i(L_{\mathsf{s}}, A') \simeq \mathsf{H}^i(L_{\mathsf{s}}, A),$$
where the direct limit runs over finite submodules of A.
- Suppose $\mathsf{S}' \supset \mathsf{S}$ is a finite subset of M_L. By construction $\mathrm{Res}_{\mathsf{s}}^{\mathsf{S}}$ coincides with the composite
$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \to \mathsf{H}^i(L^{\mathsf{S}'}/L, A) \to \mathsf{H}^i(L_{\mathsf{s}}, A);$$
hence we obtain a well-defined map
$$\mathrm{Res}_{\mathsf{s}} : \mathsf{H}^i(L, A) \to \mathsf{H}^i(L_{\mathsf{s}}, A).$$

We now observe that patched cohomology recovers the usual Galois cohomology when S contains only constant ultraprimes.

Proposition 2.4.10. *Suppose A is a countably profinite or ind-finite topological G_L-module unramified outside a finite set $S \subset M_L$; let $\mathsf{S} \subset \mathsf{M}_L$ be the corresponding set of constant ultraprimes. Then:*

(1) $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ is canonically isomorphic to $H^i(L^S/L, A)$.

(2) If $s \in M_L$ is any place and $\mathsf{s} = \underline{s}$ is the corresponding constant ultraprime, then the restriction map
$$\mathrm{Res}_{\mathsf{s}} : \mathsf{H}^i(L^{\mathsf{S}}/L, A) \to \mathsf{H}^i(L_{\mathsf{s}}, A) = H^i(L_s, A)$$
coincides with the usual one under the identification of (1).

Proof. If A is finite, (1) is immediate from the definition; the general case of (1) follows by taking limits, using [55, Corollary 2.7.6] in the countably profinite case. (The finiteness condition there is satisfied by Lemma 2.4.5.) Given (1), (2) is clear from the definition. $\square$

Proposition 2.4.11. *If A is either countably profinite or ind-finite, then, for all i, the natural map $\mathsf{H}^i(L^{\mathsf{S}}/L, A) \to \mathsf{H}^i(L, A)$ identifies $\mathsf{H}^i(L^{\mathsf{S}}/L, A)$ with the kernel of the composition*

$$\mathsf{H}^i(L, A) \xrightarrow{\mathrm{Res}_{\mathsf{s}}} \prod_{\mathsf{s}\in\mathsf{M}_L-\mathsf{S}} \mathsf{H}^i(L_{\mathsf{s}}, A) \to \mathsf{H}^i(I_{\mathsf{s}}, A).$$

Proof. It suffices to show that, for all finite sets $\mathsf{T} \subset \mathsf{M}_L - \mathsf{S}$,

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \simeq \ker\left(\mathsf{H}^i(L^{\mathsf{S}\cup\mathsf{T}}, A) \to \prod_{\mathsf{s}\in\mathsf{T}} \mathsf{H}^i(I_{\mathsf{s}}, A)\right).$$

This holds when A is finite by Propositions 2.1.5 and 2.3.2; the general case follows by taking limits. $\square$

Lemma 2.4.12. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of either countably profinite or ind-finite G_L-modules unramified outside S. Then there is an induced long exact sequence beginning:

$$\begin{aligned} 0 \to \mathsf{H}^0(L^{\mathsf{S}}/L, A) \to \mathsf{H}^0(L^{\mathsf{S}}/L, B) \to \mathsf{H}^0(L^{\mathsf{S}}/L, C) \to \\ \to \mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \mathsf{H}^1(L^{\mathsf{S}}/L, B) \to \cdots \end{aligned}$$

Proof. If A, B, and C are all finite, then this follows from Proposition 2.1.5.

Now suppose that A, B, and C are all profinite. Let I, J, and K be directed sets indexing the finite quotients $A \twoheadrightarrow A_i$, $B \twoheadrightarrow B_j$, and $C \twoheadrightarrow C_k$, respectively. We define morphisms of directed sets $t : J \to I$ and $s : J \to K$ by

$$A_{t(j)} = \mathrm{im}(A \to B_j), \;\; C_{s(j)} = B_j/A_{t(j)}.$$

Because the subgroup and quotient topologies on A and C agree with the profinite topologies, the images of t and s are cofinal in I and K, respectively. We therefore have:

$$\mathsf{H}^*(L^{\mathsf{S}}/L, A) = \varprojlim_{j \in J} \mathsf{H}^*(L^{\mathsf{S}}/L, A_{t(j)}), \;\; \mathsf{H}^*(L^{\mathsf{S}}/L, C) = \varprojlim_{j \in J} \mathsf{H}^*(L^{\mathsf{S}}/L, C_{s(j)}).$$

For each j, the finite case of the lemma yields a long exact sequence in patched cohomology associated to the short exact sequence of finite G_L-modules

$$0 \to A_{t(j)} \to B_j \to C_{s(j)} \to 0;$$

by Lemma 2.4.5, each term in the long exact sequence is finite. Since countable inverse limits of finite abelian groups are exact, taking limits completes the proof. The ind-finite case is analogous. □

The following lemma will be needed in the proof of Corollary 8.2.8 below.

Lemma 2.4.13. *Let A be a countably profinite or ind-finite G_L-module unramified outside a finite set S. If $\mathsf{s} \notin \mathsf{S}$ has $\mathrm{Frob}_{\mathsf{s}} = 1 \in G_L$, then $\mathrm{Res}_{\mathsf{s}} \mathsf{H}^1(L^{\mathsf{S}}/L, A) = 0$.*

One can easily show using the Chebotarev density theorem that there are infinitely many $\mathsf{s} \in \mathsf{M}_L$ with $\mathrm{Frob}_{\mathsf{s}} = 1$, so the lemma is not vacuous.

Proof. Without loss of generality, we may assume A is finite. Let S and s be represented by sequences $(S_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$, respectively, with $s_n \notin S_n$. Because A is finite, a class $c \in \mathsf{H}^1(L^{\mathsf{S}}/L, A)$ is represented by a sequence of cocycles $c_n \in H^1(L^{S_n}/L, A)$. The restriction of c_n to $G_{L(A)}$ is a homomorphism $G_{L(A)} \to A$; let $L(c_n)$ be the fixed field of its kernel, which is a finite extension of L. Then for $\mathfrak{F}$-many n, s_n is unramified in $L(c_n)/L$ with $\mathrm{Frob}_{s_n} = 1 \in \mathrm{Gal}(L(c_n)/L)$. For these n, $\mathrm{Res}_{s_n} c_n \in H^1(L_{s_n}, A)$ is the trivial unramified cocycle $\mathrm{Frob}_{s_n} \mapsto 0$. Hence $\mathrm{Res}_{s_n} c_n = 0$ for $\mathfrak{F}$-many n, which shows $\mathrm{Res}_{\mathsf{s}} c = 0$. □

2.5. Selmer structures and patched Selmer groups.

2.5.1. For any topological G_L-module A and any ultraprime $\mathsf{s} \in \mathsf{M}_L$, define

$$\mathsf{H}^1_{\mathrm{unr}}(L_{\mathsf{s}}, A) := \ker\left(\mathsf{H}^1(L_{\mathsf{s}}, A) \to \mathsf{H}^1(I_{\mathsf{s}}, A)\right).$$

Definition 2.5.2. Let A be a countably profinite or ind-finite $\mathbb{Z}_p[G_L]$-module. A **(generalized) Selmer structure** $(\mathcal{F}, \mathsf{S})$ for A consists of:

- a finite set $\mathsf{S} \subset \mathsf{M}_L$, containing all constant ultraprimes $\mathsf{s} = \underline{s}$ with s lying over p or ∞, such that A is unramified outside S;
- for each $\mathsf{s} \in \mathsf{M}_L$, a closed $\mathbb{Z}_p$-submodule (the **local condition**)

$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) \subset \mathsf{H}^1(L_{\mathsf{s}}, A)$$

  such that $\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) = \mathsf{H}^1_{\mathrm{unr}}(L_{\mathsf{s}}, A)$ for all $\mathsf{s} \notin \mathsf{S}$.

If A is an R-module for some ring R and G_L acts on A by R-module automorphisms, then the patched local and global cohomology groups inherit an R-module structure; a Selmer structure for A **over** R is a Selmer structure such that every local condition is an R-submodule.

2.5.3. If $B \subset A$ is any closed Galois-stable submodule, then a Selmer structure $(\mathcal{F}, \mathsf{S})$ for A induces Selmer structures on B and A/B defined in the usual way:

$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, B) = \ker\left(\mathsf{H}^1(L_{\mathsf{s}}, B) \to \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}\right),$$
$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A/B) = \operatorname{im}\left(\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) \to \mathsf{H}^1(L_{\mathsf{s}}, A/B)\right).$$

Note that these Selmer structures are well-defined because, if $\mathsf{s} \notin \mathsf{S}$, then the unramified local condition for A at s induces the unramified local condition for B and A/B; the proof is the same as for [49, Lemma 1.1.9].

2.5.4. To a generalized Selmer structure we associate the **patched Selmer group**, defined by the exact sequence:

$$0 \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}. \tag{10}$$

By Proposition 2.4.11, $\mathrm{Sel}_{\mathcal{F}}(A)$ does not depend on S but only on the local conditions, so we sometimes omit S from the notation. By Proposition 2.4.10, if S consists only of constant ultraprimes, then this definition recovers the usual notion of Selmer groups.

2.5.5. If $B \subset A$ is Galois-stable, and $B, A/B$ are equipped with the induced Selmer structures, then by definition there are natural maps:

$$\mathrm{Sel}_{\mathcal{F}}(B) \to \mathrm{Sel}_{\mathcal{F}}(A) \to \mathrm{Sel}_{\mathcal{F}}(A/B).$$

Proposition 2.5.6. *Let $(\mathcal{F}, \mathsf{S})$ be a generalized Selmer structure for A. If A is countably profinite and each continuous finite quotient $A \twoheadrightarrow A'$ is equipped with the Selmer structure induced by $\mathcal{F}$, then:*

$$\varprojlim \mathrm{Sel}_{\mathcal{F}}(A') \simeq \mathrm{Sel}_{\mathcal{F}}(A).$$

If instead A is ind-finite and each finite submodule $A' \subset A$ is given its induced Selmer structure, then:

$$\varinjlim \mathrm{Sel}_{\mathcal{F}}(A') \simeq \mathrm{Sel}_{\mathcal{F}}(A).$$

Proof. We show the countably profinite case; the ind-finite case is similar. By definition, $\mathrm{Sel}_{\mathcal{F}}(A)$ is the kernel of

$$\varprojlim \mathsf{H}^1(L^{\mathsf{S}}/L, A') \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)},$$

whereas

$$\begin{aligned}\varprojlim \mathrm{Sel}_{\mathcal{F}}(A') &= \varprojlim \ker\left(\mathsf{H}^1(L^{\mathsf{S}}/L, A') \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1(L_{\mathsf{s}}, A')}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A')}\right)\\ &= \ker\left(\mathsf{H}^1(L^{\mathsf{S}}/L, A) \to \varprojlim \frac{\mathsf{H}^1(L_{\mathsf{s}}, A')}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A')}\right).\end{aligned}$$

Since $\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)$ is a closed subgroup of $\mathsf{H}^1(L_{\mathsf{s}}, A)$, it is isomorphic to the inverse limit:

$$\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) = \varprojlim \operatorname{im}\left(\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) \to \mathsf{H}^1(L_{\mathsf{s}}, A')\right) = \varprojlim \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A').$$

This implies the result. □

Remark 2.5.7. The following remark will be needed in §3.3. Suppose $L/\mathbb{Q}$ is Galois, and let A be a countably profinite or ind-finite topological G_L-module unramified outside a finite set $\mathsf{S} \subset \mathsf{M}_L$. For an element $\gamma \in G_{\mathbb{Q}}$, consider the twist A_γ of A, where the G_L action is precomposed with the conjugation map

$$\varphi_\gamma : G_L \to G_L.$$

If $\mathsf{s} \mapsto \mathsf{s}^\gamma$ denotes the natural action of $G_{\mathbb{Q}}$ on M_L (factoring through $\mathrm{Gal}(L/\mathbb{Q})$), then there are canonical isomorphisms

$$\mathsf{H}^i(L^{\mathsf{S}}/L, A) \cong \mathsf{H}^i(L^{\mathsf{S}^\gamma}/L, A_\gamma), \tag{11}$$

$$\mathsf{H}^i(L_{\mathsf{s}}, A) \cong \mathsf{H}^i(L_{\mathsf{s}^\gamma}, A_\gamma), \quad \forall \mathsf{s} \in \mathsf{M}_L. \tag{12}$$

Indeed, when A is finite, these are induced functorially from the map induced by $\varphi_\gamma : G_L \to G_L$ on usual Galois cohomology, using Proposition 2.3.2 in the local case; in general, they are induced by taking limits, applying Lemma 2.4.8 in the local case. In particular, a generalized Selmer structure $(\mathcal{F}, \mathsf{S})$ for A induces a generalized Selmer structure $(\mathcal{F}^\gamma, \mathsf{S}^\gamma)$ for A_γ by setting $\mathsf{H}^i_{\mathcal{F}^\gamma}(L_\mathsf{s}, A_\gamma)$ to be the image of $\mathsf{H}^i_{\mathcal{F}}(L_{\mathsf{s}^{\gamma^{-1}}}, A)$ under (12) for all $\mathsf{s} \in \mathsf{M}_L$. Note that (11) then induces a canonical isomorphism

$$\mathrm{Sel}_{\mathcal{F}}(A) \cong \mathrm{Sel}_{\mathcal{F}^\gamma}(A_\gamma). \tag{13}$$

2.6. **Selmer groups and duality over discrete valuation rings.**

2.6.1. Let $\mathcal{O}$ be a discrete valuation ring, with uniformizer π, which is a finite flat extension of $\mathbb{Z}_p$; write $E = \mathcal{O}[1/\pi]$.

Proposition 2.6.2. *Let $\mathsf{s} \in \mathsf{M}_L$ be any ultraprime and let $R = \mathcal{O}$, $\mathcal{O}/\pi^j$, or $E/\mathcal{O}$. Then there is a canonical isomorphism*

$$\mathsf{H}^2(L_\mathsf{s}, R(1)) \cong \begin{cases} R, & \mathsf{s} \text{ non-archimedean}; \\ R[2], & \mathsf{s} = \underline{s},\ s \text{ archimedean},\ L_s = \mathbb{R}; \\ 0, & \mathsf{s} = \underline{s},\ s \text{ archimedean},\ L_s = \mathbb{C}. \end{cases}$$

Proof. For $R = \mathcal{O}/\pi^j$, this follows from Proposition 2.3.2. The cases $R = \mathcal{O}$ and $R = E/\mathcal{O}$ follow from the case $\mathcal{O}/\pi^j$ by Lemma 2.4.8. □

2.6.3. For a topological $\mathcal{O}[G_L]$-module A, let

$$A^* = \mathrm{Hom}_{\mathcal{O}}(A, E/\mathcal{O}(1))$$

be the Cartier dual. By Proposition 2.6.2, the cup product induces a pairing

$$\langle \cdot, \cdot \rangle_\mathsf{s} : \mathsf{H}^i(L_\mathsf{s}, A) \times \mathsf{H}^{2-i}(L_\mathsf{s}, A^*) \to E/\mathcal{O}, \quad i = 0, 1, 2, \tag{14}$$

for all $\mathsf{s} \in \mathsf{M}_L$.

Proposition 2.6.4. *Suppose A is a countably profinite $\mathcal{O}[G_L]$-module unramified outside a finite set S containing all $\mathsf{s} = \underline{s}$ with $s | p\infty$. Then:*

1. *(1) The pairing $\langle \cdot, \cdot \rangle_\mathsf{s}$ is perfect if $\mathsf{s} \in \mathsf{M}_L$ is non-archimedean, or if $i = 1$.*
2. *(2) If $\mathsf{s} \in \mathsf{M}_L$ is not equal to $\underline{s}$ for a prime $s|p$, then $\mathsf{H}^1_{\mathrm{unr}}(L_\mathsf{s}, A)$ and $\mathsf{H}^1_{\mathrm{unr}}(L_\mathsf{s}, A^*)$ are exact annihilators under $\langle \cdot, \cdot \rangle_\mathsf{s}$.*
3. *(3) For $i = 0, 1, 2$, the induced pairing*

$$\Sigma_{\mathsf{s} \in \mathsf{S}} \langle \cdot, \cdot \rangle_\mathsf{s} : \mathsf{H}^i(L^\mathsf{S}/L, A) \times \mathsf{H}^{2-i}(L^\mathsf{S}/L, A^*) \to E/\mathcal{O}$$

is identically zero.

Proof. Let d be the valuation of the different of $E/\mathbb{Q}_p$, i.e.

$$\{x \in E \,:\, \mathrm{tr}(x\mathcal{O}) \subset \mathbb{Z}_p\} = \pi^{-d} \cdot \mathcal{O},$$

and let $\mathrm{tr}' : E/\mathcal{O} \to \mathbb{Q}_p/\mathbb{Z}_p$ be the composite

$$E/\mathcal{O} \xrightarrow{\pi^{-d}} E/\pi^{-d}\mathcal{O} \xrightarrow{\mathrm{tr}} \mathbb{Q}_p/\mathbb{Z}_p.$$

If we set $A' \coloneqq \mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Q}_p/\mathbb{Z}_p(1))$, then the map $\varphi \mapsto \mathrm{tr}' \circ \varphi$ defines an isomorphism $A^* \xrightarrow{\sim} A'$. This isomorphism fits into a commutative diagram

$$\begin{array}{ccc} \mathsf{H}^i(L_\mathsf{s}, A) \times \mathsf{H}^{2-i}(L_\mathsf{s}, A^*) & \xrightarrow{\langle \cdot, \cdot \rangle_\mathsf{s}} & E/\mathcal{O} \\ \downarrow \sim & & \downarrow \mathrm{tr}' \\ \mathsf{H}^i(L_\mathsf{s}, A) \times \mathsf{H}^{2-i}(L_\mathsf{s}, A') & \xrightarrow{\langle \cdot, \cdot \rangle_\mathsf{s}} & \mathbb{Q}_p/\mathbb{Z}_p \end{array}$$

for $i = 0, 1, 2$. It is then not difficult to check that each of (1), (2), (3) follows from its analog for A'. So without loss of generality we may suppose $\mathcal{O} = \mathbb{Z}_p$. We may also assume without loss of generality that A is finite, since the general case of each statement follows by taking limits using Lemma 2.4.8. Then (1) and (2) follow from Proposition 2.3.2 and the usual local Tate duality. For (3), S is represented by a sequence $(S_n)_{n\in\mathbb{N}}$ such that A is unramified outside S_n and each S_n contains all $s|p\infty$. Since A is finite, the pairing map $\Sigma_{\mathsf{s}\in\mathsf{S}}\langle\cdot,\cdot\rangle_{\mathsf{s}}$ is the ultraproduct of the maps

$$\Sigma_{s_n\in S_n}\langle\cdot,\cdot\rangle_{s_n} : H^i(L^{S_n}/L, A) \times H^{2-i}(L^{S_n}/L, A^*) \to \mathbb{Q}_p/\mathbb{Z}_p,$$

which are all identically zero by global Tate duality. □

2.6.5. Suppose A is an $\mathcal{O}[G_L]$-module such that either A or A^* is countably profinite. If $(\mathcal{F}, \mathsf{S})$ is a generalized Selmer structure for A over $\mathcal{O}$, then we define the dual Selmer structure $(\mathcal{F}^*, \mathsf{S})$ for A^* by:

$$\mathsf{H}^1_{\mathcal{F}^*}(L_{\mathsf{s}}, A^*) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)^{\perp}.$$

Here $\perp$ denotes the orthogonal complement under the pairing of (14); the local conditions outside S are unramified by Proposition 2.6.4(2). Moreover, the proof of that proposition shows that the dual local conditions do not change if we instead view A as a $\mathbb{Z}_p[G_L]$ module. We observe that the dual Selmer structure to $(\mathcal{F}^*, \mathsf{S})$ is again $(\mathcal{F}, \mathsf{S})$. When A is finite, the dual Selmer groups are related by the Greenberg-Wiles formula:

Proposition 2.6.6. *Let A be a finite $\mathcal{O}[G_L]$-module, and let $(\mathcal{F}, \mathsf{S})$ be a Selmer structure for A over $\mathcal{O}$. We have:*

$$\frac{\#\operatorname{Sel}_{\mathcal{F}}(A)}{\#\operatorname{Sel}_{\mathcal{F}^*}(A^*)} = \frac{\#\mathsf{H}^0(L^{\mathsf{S}}/L, A)}{\#\mathsf{H}^0(L^{\mathsf{S}}/L, A^*)}\prod_{\mathsf{s}\in\mathsf{S}}\frac{\#\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)}{\#\mathsf{H}^0(L_{\mathsf{s}}, A)}.$$

Proof. Suppose $\mathsf{s} \in \mathsf{S}$ is represented by the sequence $(s_n)_{n\in\mathbb{N}}$. Then Proposition 2.3.2 implies that, for $\mathfrak{F}$-many n, we have isomorphisms

$$H^i(L_{s_n}, A) \cong \mathsf{H}^i(L_{\mathsf{s}}, A)$$

and

$$H^i(L_{s_n}, A^*) \cong \mathsf{H}^i(L_{\mathsf{s}}, A^*)$$

for $i = 0, 1, 2$, compatible with the duality of Proposition 2.6.4(1). Let $H^1_{\mathcal{F}_n}(L_{s_n}, A) \subset H^1(L_{s_n}, A)$ be the image of $\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)$ under the first isomorphism; then its orthogonal complement $H^1_{\mathcal{F}_n^*}(L_{s_n}, A^*) \subset H^1(L_{s_n}, A^*)$ is the image of $\mathsf{H}^1_{\mathcal{F}^*}(L_{\mathsf{s}}, A^*)$. This defines dual Selmer structures $(\mathcal{F}_n, S_n)$ and $(\mathcal{F}_n^*, S_n)$ on A for $\mathfrak{F}$-many n, where $(S_n)_{n\in\mathbb{N}}$ represents S. By Propositions 2.1.5 and 2.3.2,

$$\operatorname{Sel}_{\mathcal{F}}(A) = \mathcal{U}\left(\{\operatorname{Sel}_{\mathcal{F}_n}(A)\}_{n\in\mathbb{N}}\right), \ \operatorname{Sel}_{\mathcal{F}^*}(A^*) = \mathcal{U}\left(\{\operatorname{Sel}_{\mathcal{F}_n^*}(A^*)\}_{n\in\mathbb{N}}\right),$$

so by [24, Theorem 2.19] and Proposition 2.1.4(3), we have

$$\frac{\#\operatorname{Sel}_{\mathcal{F}}(A)}{\#\operatorname{Sel}_{\mathcal{F}^*}(A^*)} = \frac{\#H^0(L, A)}{\#H^0(L, A^*)} \cdot \prod_{s_n\in S_n}\frac{\#H^1_{\mathcal{F}_n}(L_{s_n}, A)}{\#H^0(L_{s_n}, A)}$$

for $\mathfrak{F}$-many n; since

$$\#H^1_{\mathcal{F}_n}(L_{s_n}, A) = \#\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A)$$

for $\mathfrak{F}$-many n by definition, and likewise for H^0, this implies the proposition. □

2.6.7. Given two Selmer structures $(\mathcal{F}, \mathsf{S})$ and $(\mathcal{G}, \mathsf{T})$ for A, we may define Selmer structures $(\mathcal{F}+\mathcal{G}, \mathsf{S}\cup\mathsf{T})$ and $(\mathcal{F}\cap\mathcal{G}, \mathsf{S}\cup\mathsf{T})$ by the local conditions:

$$\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_{\mathsf{s}}, A) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) + \mathsf{H}^1_{\mathcal{G}}(L_{\mathsf{s}}, A),$$
$$\mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_{\mathsf{s}}, A) = \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, A) \cap \mathsf{H}^1_{\mathcal{G}}(L_{\mathsf{s}}, A).$$

Corollary 2.6.8. *Let A be an $\mathcal{O}[G_L]$-module such that either A or A^* is countably profinite, and let $(\mathcal{F}, \mathsf{S})$ and $(\mathcal{G}, \mathsf{T})$ be generalized Selmer structures for A over $\mathcal{O}$. Then*

$$\frac{\operatorname{Sel}_{\mathcal{F}+\mathcal{G}}(A)}{\operatorname{Sel}_{\mathcal{F}\cap\mathcal{G}}(A)} \subset \prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}}\frac{\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_{\mathsf{s}}, A)}{\mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_{\mathsf{s}}, A)}$$

and

$$\frac{\mathrm{Sel}_{\mathcal{F}^*+\mathcal{G}^*}(A^*)}{\mathrm{Sel}_{\mathcal{F}^*\cap\mathcal{G}^*}(A^*)} \subset \prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\mathsf{H}^1_{\mathcal{F}^*+\mathcal{G}^*}(L_\mathsf{s},A^*)}{\mathsf{H}^1_{\mathcal{F}^*\cap\mathcal{G}^*}(L_\mathsf{s},A^*)}$$

are exact annihilators under the perfect pairing

$$\Sigma_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}}\langle\cdot,\cdot\rangle_\mathsf{s} : \prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_\mathsf{s},A)}{\mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_\mathsf{s},A)} \times \prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\mathsf{H}^1_{\mathcal{F}^*+\mathcal{G}^*}(L_\mathsf{s},A^*)}{\mathsf{H}^1_{\mathcal{F}^*\cap\mathcal{G}^*}(L_\mathsf{s},A^*)} \to E/\mathcal{O}$$

induced by (14).

Proof. By Proposition 2.5.6 and Lemma 2.4.8, we may assume without loss of generality that A is finite. Proposition 2.6.4(3) implies that the spaces

$$\frac{\mathrm{Sel}_{\mathcal{F}+\mathcal{G}}(A)}{\mathrm{Sel}_{\mathcal{F}\cap\mathcal{G}}(A)}, \ \frac{\mathrm{Sel}_{\mathcal{F}^*+\mathcal{G}^*}(A^*)}{\mathrm{Sel}_{\mathcal{F}^*\cap\mathcal{G}^*}(A^*)}$$

annihilate each other under $\Sigma_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}}\langle\cdot,\cdot\rangle_\mathsf{s}$, so it suffices to show

$$\frac{\#\,\mathrm{Sel}_{\mathcal{F}+\mathcal{G}}(A)}{\#\,\mathrm{Sel}_{\mathcal{F}\cap\mathcal{G}}(A)} \cdot \frac{\#\,\mathrm{Sel}_{\mathcal{F}^*+\mathcal{G}^*}(A)}{\#\,\mathrm{Sel}_{\mathcal{F}^*\cap\mathcal{G}^*}(A)} = \prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\#\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_\mathsf{s},A)}{\#\mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_\mathsf{s},A)}. \tag{15}$$

By Proposition 2.6.6, the left-hand side of (15) is

$$\prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\#\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_\mathsf{s},A)}{\#\mathsf{H}^0(L_\mathsf{s},A)} \cdot \frac{\#\mathsf{H}^1_{\mathcal{F}^*+\mathcal{G}^*}(L_\mathsf{s},A^*)}{\#\mathsf{H}^0(L_\mathsf{s},A^*)},$$

or equivalently

$$\prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\#\mathsf{H}^1_{\mathcal{F}+\mathcal{G}}(L_\mathsf{s},A)}{\#\mathsf{H}^0(L_\mathsf{s},A)} \cdot \frac{\#\mathsf{H}^1(L_\mathsf{s},A)}{\#\mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_\mathsf{s},A)\#\mathsf{H}^0(L_\mathsf{s},A^*)}$$

because $\mathsf{H}^1_{\mathcal{F}\cap\mathcal{G}}(L_\mathsf{s},A)$ is the exact annihilator of $\mathsf{H}^1_{\mathcal{F}^*+\mathcal{G}^*}(L_\mathsf{s},A^*)$ under $\langle\cdot,\cdot\rangle_\mathsf{s}$. So it suffices to show

$$\prod_{\mathsf{s}\in\mathsf{S}\cup\mathsf{T}} \frac{\#\mathsf{H}^1(L_\mathsf{s},A)}{\#\mathsf{H}^0(L_\mathsf{s},A)\cdot\#\mathsf{H}^0(L_\mathsf{s},A^*)} = 1. \tag{16}$$

Now note that, by the local Euler characteristic formula, Proposition 2.6.4(1), and Proposition 2.3.2,

$$\frac{\#\mathsf{H}^1(L_\mathsf{s},A)}{\#\mathsf{H}^0(L_\mathsf{s},A)\#\mathsf{H}^0(L_\mathsf{s},A^*)} = \frac{\#\mathsf{H}^1(L_\mathsf{s},A)}{\#\mathsf{H}^0(L_\mathsf{s},A)\#\mathsf{H}^2(L_\mathsf{s},A)} = 1$$

unless $\mathsf{s}=\underline{s}$, where $s|\infty$ or $s|p$; moreover we have

$$\prod_{\substack{\mathsf{s}=\underline{s},\\ s|p}} \frac{\#\mathsf{H}^1(L_\mathsf{s},A)}{\#\mathsf{H}^0(L_\mathsf{s},A)\#\mathsf{H}^0(L_\mathsf{s},A^*)} = \prod_{\substack{\mathsf{s}=\underline{s},\\ s|\infty}} \frac{\#\mathsf{H}^0(L_\mathsf{s},A)\#\mathsf{H}^0(L_\mathsf{s},A^*)}{\#\mathsf{H}^1(L_\mathsf{s},A)} = (\#A)^{[L:\mathbb{Q}]}.$$

This shows (16) and completes the proof. □

2.6.9. Now suppose that $A=T$ is a topological $\mathcal{O}[G_L]$ module, unramified outside a finite set $\mathsf{S}\subset\mathsf{M}_L$, which is free of finite rank over $\mathcal{O}$. In particular, T is countably profinite. Suppose $\mathsf{S}\subset\mathsf{M}_L$ is a finite set containing all archimedean places and all places over p, such that T is unramified outside S. If $T^\dagger=\mathrm{Hom}_\mathcal{O}(T,\mathcal{O}(1))$ is the dual, then the cup product induces a local Tate pairing

$$\langle\cdot,\cdot\rangle_\mathsf{s} : \mathsf{H}^1(L_\mathsf{s},T)\times\mathsf{H}^1(L_\mathsf{s},T^\dagger)\to\mathcal{O}. \tag{17}$$

Proposition 2.6.10. *The kernels on the left and right of (17) are the $\mathcal{O}$-torsion submodules; moreover, the induced pairing*

$$\mathsf{H}^1(L^\mathsf{S}/L,T)\times\mathsf{H}^1(L^\mathsf{S}/L,T^\dagger)\to\prod_{\mathsf{s}\in\mathsf{S}}\mathsf{H}^1(L_\mathsf{s},T)\times\mathsf{H}^1(L_\mathsf{s},T^\dagger)\xrightarrow{\sum\langle\cdot,\cdot\rangle_\mathsf{s}}\mathcal{O}$$

is identically zero.

Proof. For each s, we have the short exact sequence

$$0 \to \mathsf{H}^1(L_\mathsf{s}, T)/\pi^j \to \mathsf{H}^1(L_\mathsf{s}, T/\pi^j) \to \mathsf{H}^2(L_\mathsf{s}, T)[\pi^j] \to 0$$

arising from the long exact sequence in G_s-cohomology associated to $0 \to T \to T \to T/\pi^j$. Hence

$$0 \to \varprojlim_j \mathrm{Hom}_\mathcal{O}\left(\mathsf{H}^2(L_\mathsf{s}, T)[\pi^j], \mathcal{O}/\pi^j\right) \to \varprojlim_j \mathrm{Hom}_\mathcal{O}(\mathsf{H}^1(T/\pi^j), \mathcal{O}/\pi^j) \to \mathrm{Hom}_\mathcal{O}(\mathsf{H}^1(L_\mathsf{s}, T), \mathcal{O}) \to 0$$

is exact. By Proposition 2.6.4(1) and Lemma 2.4.8, this is canonically identified with an exact sequence

$$0 \to \varprojlim_j \mathsf{H}^0(L_\mathsf{s}, T^*)/\pi^j \to \mathsf{H}^1(L_\mathsf{s}, T^\dagger) \to \mathrm{Hom}_\mathcal{O}(\mathsf{H}^1(L_\mathsf{s}, T), \mathcal{O}) \to 0, \tag{18}$$

where the third arrow is induced by (17). The first term in (18) is clearly π-power-torsion, while the last term is torsion-free, and so we can identify $\varprojlim_j \mathsf{H}^0(L_\mathsf{s}, T^*)/\pi^j$ with $\mathsf{H}^1(L_\mathsf{s}, T^\dagger)[\pi^\infty]$. This proves the first claim, and the second follows from Proposition 2.6.4(3) using the commutativity of the diagram

$$\begin{array}{ccc} \mathsf{H}^1(L_\mathsf{s}, T) \times \mathsf{H}^1(L_\mathsf{s}, T^\dagger) & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ \mathsf{H}^1(L_\mathsf{s}, T/\pi^j) \times \mathsf{H}^1(L_\mathsf{s}, T^\dagger/\pi^j) & \longrightarrow & \mathcal{O}/\pi^j \end{array}$$

for all $\mathsf{s} \in \mathsf{S}$ and all $j \geq 1$. □

Given a Selmer structure $(\mathcal{F}, \mathsf{S})$ for T over $\mathcal{O}$, taking the orthogonal complement of each local condition under (17) yields a Selmer structure $(\mathcal{F}^\dagger, \mathsf{S})$ for $T^\dagger$.

Definition 2.6.11. A closed $\mathcal{O}$-submodule $\mathsf{H}^1_\mathcal{F}(L_\mathsf{s}, T) \subset \mathsf{H}^1(L_\mathsf{s}, T)$ is said to be **saturated** if the quotient $\mathsf{H}^1(L_\mathsf{s}, T)/\mathsf{H}^1_\mathcal{F}(L_\mathsf{s}, T)$ is π-torsion free. A Selmer structure $(\mathcal{F}, \mathsf{S})$ for T over $\mathcal{O}$ is saturated if each local condition $\mathsf{H}^1_\mathcal{F}(L_\mathsf{s}, T)$ is saturated.

Note that, if T is unramified outside $\mathsf{S} \subset \mathsf{M}_L$, then $\mathsf{H}^1_{\mathrm{unr}}(L_\mathsf{s}, T)$ is saturated for all $\mathsf{s} \notin \mathsf{S}$ because $\mathsf{H}^1(I_\mathsf{s}, T)$ is π-torsion-free.

Proposition 2.6.12. *If $\mathsf{H}^1_\mathcal{F}(L_\mathsf{s}, T)$ is saturated, then for all $j \geq 1$, we have*

$$\mathsf{H}^1_{\mathcal{F}^*}(L_\mathsf{s}, T^*[\pi^j]) = \mathsf{H}^1_{\mathcal{F}^\dagger}(L_\mathsf{s}, T^\dagger/\pi^j)$$

under the natural identification $T^*[\pi^j] \simeq T^\dagger/\pi^j$. In particular, if $(\mathcal{F}, \mathsf{S})$ is a saturated Selmer structure for T, then

$$\mathrm{Sel}_{\mathcal{F}^*}(T^*[\pi^j]) = \mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger/\pi^j).$$

Proof. For ease of notation, we abbreviate $\mathsf{H}^i(T^\dagger) = \mathsf{H}^i(L_\mathsf{s}, T^\dagger)$, etc. We have an identification $T^\dagger \otimes_\mathcal{O} (E/\mathcal{O}) \simeq T^*$ and an embedding $T^\dagger/\pi^j \hookrightarrow T^*$; let $\mathsf{H}^1_{\mathcal{F}^*}(T^\dagger/\pi^j)$ be the induced local condition from this embedding. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{H}^0(T^*)_{/\,\mathrm{div}} & \longrightarrow & \mathsf{H}^1_{\mathcal{F}^\dagger}(T^\dagger) & \longrightarrow & \mathrm{Hom}_\mathcal{O}\left(\frac{\mathsf{H}^1(T)}{\mathsf{H}^1_\mathcal{F}(T)}, \mathcal{O}\right) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & \mathsf{H}^0(T^*)/\pi^j & \longrightarrow & \mathsf{H}^1_{\mathcal{F}^*}(T^\dagger/\pi^j) & \longrightarrow & \mathrm{Hom}_\mathcal{O}\left(\frac{\mathsf{H}^1(T)}{\mathsf{H}^1_\mathcal{F}(T)}, \mathcal{O}/\pi^j\right) & \longrightarrow & 0. \end{array}$$

Here, the first horizontal map on each row is the Kummer map, the subscript $/\,\mathrm{div}$ refers to the quotient by the maximal divisible submodule, and the exactness of the top row uses (18). By the saturation hypothesis, γ is surjective, and α clearly is as well, so β is surjective by the snake lemma. □

Proposition 2.6.13. *Let $(\mathcal{F}, \mathsf{S})$ be a Selmer structure for T over $\mathcal{O}$. Then:*

$$\mathrm{rk}_\mathcal{O}\, \mathrm{Sel}_\mathcal{F}(T) - \mathrm{rk}_\mathcal{O}\, \mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger) = \mathrm{rk}_\mathcal{O}\, H^0(L, T) - \mathrm{rk}_\mathcal{O}\, H^0(L, T^\dagger) + \sum_{\mathsf{s} \in \mathsf{S}} \left(\mathrm{rk}_\mathcal{O}\, \mathsf{H}^1_\mathcal{F}(L_\mathsf{s}, T) - \mathrm{rk}_\mathcal{O}\, \mathsf{H}^0(L_\mathsf{s}, T)\right).$$

Proof. We first reduce to the case that $(\mathcal{F}, \mathsf{S})$ is saturated. Indeed, if we define

$$\mathsf{H}^1_{\widetilde{\mathcal{F}}}(L_{\mathsf{s}}, T) := \ker\left(\mathsf{H}^1(L_{\mathsf{s}}, T) \to \frac{\mathsf{H}^1(L_{\mathsf{s}}, T)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, T)} \otimes \mathbb{Q}_p\right)$$

for all s, then $(\widetilde{\mathcal{F}}, \mathsf{S})$ is a saturated Selmer structure for T. Now, there is an exact sequence $0 \to \mathrm{Sel}_{\mathcal{F}}(T) \to \mathrm{Sel}_{\widetilde{\mathcal{F}}}(T) \to \prod_{\mathsf{s}\in\mathsf{S}} \frac{\mathsf{H}^1_{\widetilde{\mathcal{F}}}(L_{\mathsf{s}},T)}{\mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}},T)}$. Since the final term is a finitely generated torsion $\mathcal{O}$-module, we have

$$\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}}(T) = \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\widetilde{\mathcal{F}}}(T).$$

So, replacing $\mathcal{F}$ with $\widetilde{\mathcal{F}}$ if necessary, we may assume $(\mathcal{F}, \mathsf{S})$ is saturated. By Propositions 2.6.6 and 2.6.12, we have for each j:

$$\begin{aligned}\lg_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}}(T/\pi^j) - \lg_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger/\pi^j) &= \lg_{\mathcal{O}} \mathsf{H}^0(L, T/\pi^j) - \lg_{\mathcal{O}} \mathsf{H}^0(L, T^\dagger/\pi^j) \\ &\quad + \sum_{\mathsf{s}\in\mathsf{S}} \left(\lg_{\mathcal{O}} \mathsf{H}^1_{\mathcal{F}}(L_{\mathsf{s}}, T/\pi^j) - \lg_{\mathcal{O}} \mathsf{H}^0(L_{\mathsf{s}}, T/\pi^j)\right).\end{aligned}$$

Since $\mathrm{Sel}_{\mathcal{F}}(T)$ is a finitely generated $\mathcal{O}$-module, it follows from [49, Lemma 3.7.1] that

$$\lg_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}}(T/\pi^j) = j \cdot \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}}(T) + O(1)$$

as j varies, and likewise for $\mathrm{Sel}_{\mathcal{F}^\dagger}(T^\dagger)$ and each term on the right-hand side; the proposition follows. □

3. Bipartite Euler systems

3.1. Admissible primes.

3.1.1. Let f, N, $\wp$, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5. Then V_f is a two-dimensional E-vector space equipped with a non-degenerate, symplectic, $G_{\mathbb{Q}}$-equivariant pairing:

$$V_f \times V_f \to \wedge^2 V_f \cong E(1). \tag{19}$$

Recall that $\overline{T}_f = T_f/\pi$ is absolutely irreducible as an $\mathcal{O}[G_{\mathbb{Q}}]$-module. Since the dual lattice to T_f is also $\mathcal{O}[G_{\mathbb{Q}}]$-stable, after rescaling we may assume that (19) restricts to an $\mathcal{O}(1)$-valued pairing

$$T_f \times T_f \to \mathcal{O}(1) \tag{20}$$

which identifies T_f with $\mathrm{Hom}_{\mathcal{O}}(T_f, \mathcal{O}(1))$. We will sometimes use the condition:

(sclr) The image of the $G_{\mathbb{Q}}$ action on $\overline{T}_f$ contains a nontrivial scalar.

Fix an imaginary quadratic field $K/\mathbb{Q}$ of discriminant coprime to Np.

Definition 3.1.2. A nonconstant ultraprime $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$ is said to be **admissible** with sign $\epsilon_{\mathsf{q}} = \pm 1$ for f if $\mathrm{Frob}_{\mathsf{q}}$ has nonzero image in $\mathrm{Gal}(K/\mathbb{Q})$, $\chi(\mathrm{Frob}_{\mathsf{q}}) \not\equiv 1 \pmod p$, and T_f admits a basis of eigenvectors for $\mathrm{Frob}_{\mathsf{q}}$ with eigenvalues ϵ_{q} and $\chi(\mathrm{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$.

For example, if $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ is a complex conjugation, then q is admissible with either choice of ϵ_{q}.

3.1.3. If q is admissible with sign ϵ_{q}, then we write $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f \subset T_f$ for the eigenspace of $\mathrm{Frob}_{\mathsf{q}}$ with eigenvalue $\chi(\mathrm{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$. We abusively write q for the unique ultraprime in M_K lying over $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$, whose Frobenius is $\mathrm{Frob}^2_{\mathsf{q}}$.

Definition 3.1.4. If q is admissible with sign ϵ_{q} for f, then we define the **ordinary** local condition (with sign ϵ_{q}) as:

$$\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) = \mathrm{im}\left(\mathsf{H}^1(K_{\mathsf{q}}, \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \to \mathsf{H}^1(K_{\mathsf{q}}, T_f)\right).$$

The subscript ϵ_{q} will often be omitted (from this and future notation) when there is no risk of confusion.

Similarly, let

$$\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, W_f) = \mathrm{im}\left(\mathsf{H}^1(K_q, (\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \otimes_{\mathcal{O}} E/\mathcal{O}) \to \mathsf{H}^1(K_{\mathsf{q}}, W_f)\right).$$

Example 3.1.5. Suppose $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ is a complex conjugation and let $\{e_1, e_2\}$ be a basis of T_f such that $\mathrm{Frob}_{\mathsf{q}} e_1 = -e_1$ and $\mathrm{Frob}_{\mathsf{q}} e_2 = e_2$. Then as noted above, q is admissible with either choice of ϵ_{q}; if $\epsilon_{\mathsf{q}} = +1$, then $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f = \langle e_1 \rangle$, and if $\epsilon_{\mathsf{q}} = -1$, then $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f = \langle e_2 \rangle$. The local cohomology group $\mathsf{H}^1(K_{\mathsf{q}}, T_f)$ is free of rank 4, with a canonical decomposition into rank 2 subspaces:

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f) = \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle) \oplus \mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle).$$

The former is the ordinary local condition if $\epsilon_{\mathsf{q}} = +1$, and the latter if $\epsilon_{\mathsf{q}} = -1$. The unramified subspaces $\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, \langle e_i \rangle)$ are free $\mathcal{O}$-modules of rank one.

Proposition 3.1.6. *Let q be admissible with sign ϵ_{q}. Then $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)$ is its own exact annihilator under the local Tate pairing*

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f) \times \mathsf{H}^1(K_{\mathsf{q}}, T_f) \to \mathcal{O}$$

induced by (17) and (20).

Proof. The Frobenius $\mathrm{Frob}_{\mathsf{q}} \in G_{\mathbb{Q}}$ acts on T_f with eigenvalues $\chi(\mathrm{Frob}_{\mathsf{q}})\epsilon_{\mathsf{q}}$ and ϵ_{q}. Let $e_1, e_2 \in T_f$ be generators of the corresponding eigenspaces, so $\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f = \langle e_1 \rangle$. Then

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f) = \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle) \oplus \mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle), \tag{21}$$

and $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) = \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle)$. Since the pairing (19) is symplectic, each of the direct summands in (21) is isotropic for the pairing on $\mathsf{H}^1(K_{\mathsf{q}}, T_f)$. Now note that:

$$\mathsf{H}^1(K_{\mathsf{q}}, T_f)_{\mathrm{tors}} \subset \mathsf{H}^1(K_{\mathsf{q}}, \langle e_1 \rangle). \tag{22}$$

Indeed, in the exact sequence

$$0 \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, \langle e_2 \rangle) \to \mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle) \to \mathsf{H}^1(I_{\mathsf{q}}, \langle e_2 \rangle),$$

the last term is automatically torsion-free, and the first is as well since $\mathrm{Frob}_{\mathsf{q}}^2$ acts trivially on e_2. We claim (22) implies the proposition. Suppose $y \in \mathsf{H}^1(K_{\mathsf{q}}, T_f)$ pairs trivially with $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)$, and write

$$y = y_1 + y_2$$

in the decomposition (21). Since y_1 pairs trivially with $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)$, y_2 does as well. But y_2 also pairs trivially with $\mathsf{H}^1(K_{\mathsf{q}}, \langle e_2 \rangle)$, so y_2 lies in the kernel of the local Tate pairing, hence is a torsion class (Proposition 2.6.10), hence trivial by (22). □

3.1.7. For any finite set $\mathsf{S} \subset \mathsf{M}_K$ such that T_f is unramified outside S, and any admissible $\mathsf{q} \notin \mathsf{S}$ with sign ϵ_{q}, define a localization map

$$\begin{aligned} \mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}} : \mathsf{H}^1(K^{\mathsf{S}}/K, T_f) \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \to \\ \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f)}{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \cap \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)} = \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f / \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \approx \mathcal{O}. \end{aligned} \tag{23}$$

Define as well a residue map

$$\begin{aligned} \partial_{\mathsf{q},\epsilon_{\mathsf{q}}} : \mathsf{H}^1(K, T_f) \to \mathsf{H}^1(K_{\mathsf{q}}, T_f) \to \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f) \to \\ \frac{\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)}{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_f) \cap \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_f)} = \mathsf{H}^1(I_{\mathsf{q}}, \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f)^{\mathrm{Frob}_{\mathsf{q}}^2 = 1} \approx \mathcal{O}, \end{aligned} \tag{24}$$

where the second map is given by the projection $T_f \twoheadrightarrow (\mathrm{Frob}_{\mathsf{q}} - \epsilon_{\mathsf{q}}) T_f \simeq \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f$. We similarly define the maps $\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}}$, $\partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$ for W_f.

3.2. **Euler systems for anticyclotomic twists.**

3.2.1. Let R be a complete flat Noetherian local $\mathcal{O}$-algebra with finite residue field, equipped with an anticyclotomic character $\varphi : G_K \to R^\times$ which is trivial modulo the maximal ideal of R. We write T_φ for the anticyclotomic twist $T_f \otimes_{\mathcal{O}} R(\varphi)$, which is a countably profinite G_K-module. If q is admissible with sign ϵ_{q}, then $\varphi(\mathrm{Frob}_{\mathsf{q}}^2) = 1$, so

$$\mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) = \mathsf{H}^1(K_{\mathsf{q}}, T_f) \otimes_{\mathcal{O}} R. \tag{25}$$

We extend the ordinary local condition of the previous subsection by linearity to define the local condition $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, T_\varphi)$, and likewise the maps $\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}}, \partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$.

3.2.2. We will always suppose given a finite set $\mathsf{S} \subset \mathsf{M}_K$ and a generalized Selmer structure $(\mathcal{F}, \mathsf{S})$ for T_φ over R. For any finite set of ultraprimes T, let $\mathsf{N} = \mathsf{N}_{\mathsf{S}}$ be the set of pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ where $\mathsf{Q} \subset \mathsf{M}_{\mathbb{Q}}$ is a finite set of ultraprimes disjoint from the image of S in $\mathsf{M}_{\mathbb{Q}}$ and $\epsilon_{\mathsf{Q}} : \mathsf{Q} \to \{\pm 1\}$ is a function such that q is admissible with sign $\epsilon_{\mathsf{Q}}(\mathsf{q})$ for all $\mathsf{q} \in \mathsf{Q}$. (We will drop the subscript S when it is clear from context or when S contains only constant ultraprimes.) Given a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, we identify Q with a subset of M_K and define a generalized Selmer structure $(\mathcal{F}(\mathsf{Q}, \epsilon_{\mathsf{Q}}), \mathsf{S} \cup \mathsf{Q})$ for T_φ by the local conditions:

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{Q},\epsilon_{\mathsf{Q}})}(K_{\mathsf{s}}, T_\varphi) = \begin{cases} \mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{s}}, T_\varphi), & \mathsf{s} \notin \mathsf{Q} \\ \mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{Q}}(\mathsf{q})}(K_{\mathsf{q}}, T_\varphi), & \mathsf{s} = \mathsf{q} \in \mathsf{Q}. \end{cases} \tag{26}$$

For $\delta \in \mathbb{Z}/2\mathbb{Z}$, let $\mathsf{N}^\delta \subset \mathsf{N}$ be the collection of pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $|\mathsf{Q}| \equiv \delta \pmod 2$. Given two pairs $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$ and $\{\mathsf{Q}', \epsilon_{\mathsf{Q}'}\} \in \mathsf{N}^{\delta'}$ such that $\mathsf{Q} \cap \mathsf{Q}' = \emptyset$, write

$$\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}^{\delta+\delta'}$$

for the pair formed in the obvious way from $\mathsf{Q} \cup \mathsf{Q}'$ and the sign functions $\epsilon_{\mathsf{Q}}, \epsilon_{\mathsf{Q}'}$. The pair $\{\emptyset, \emptyset\} \in \mathsf{N}$ will be abbreviated as 1.

Similarly, suppose given $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ and an additional ultraprime $\mathsf{q} \notin \mathsf{Q}$ which is admissible with sign ϵ_{q}. Then we write $\{\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}}\} \in \mathsf{N}$ for the pair formed in the obvious way from $\mathsf{Q} \cup \{\mathsf{q}\}$ and the natural sign function.

Definition 3.2.3. A **bipartite system** (κ, λ) for $(T_\varphi, \mathcal{F}, \mathsf{S})$ of parity $\delta \in \mathbb{Z}/2\mathbb{Z}$ consists of the following data:

(1) for each pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$, a cyclic submodule

$$(\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{Q},\epsilon_{\mathsf{Q}})}(T_\varphi);$$

(2) for each pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$, a principal ideal

$$(\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset R.$$

A **bipartite Euler system** is a bipartite system satisfying the "reciprocity laws":

(1) For each $\{\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}}\} \in \mathsf{N}^{\delta+1}$,

$$\mathrm{loc}_{\mathsf{q}}((\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}}))) = (\lambda(\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}})) \subset R.$$

(2) For each $\{\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}}\} \in \mathsf{N}^\delta$,

$$\partial_{\mathsf{q}}((\kappa(\mathsf{Q}\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}}))) = (\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}})) \subset R.$$

We say (κ, λ) is **nontrivial** if there exists some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that either $\lambda(\mathsf{Q}, \epsilon_{\mathsf{Q}}) \neq 0$ or $\kappa(\mathsf{Q}, \epsilon_{\mathsf{Q}}) \neq 0$ depending on the parity of $|\mathsf{Q}| + \delta$.

We will suppress ϵ_{Q} from the notation when it is clear from context and write simply $\kappa(\mathsf{Q})$, $\lambda(\mathsf{Q})$, $\mathcal{F}(\mathsf{Q})$.

3.3. Euler systems over discrete valuation rings.

3.3.1. Suppose that R is a discrete valuation ring with uniformizer ϖ, and let $W_\varphi = T_\varphi \otimes_{\mathcal{O}} E/\mathcal{O} = W_f \otimes_{\mathcal{O}} R(\varphi)$. Let $\tau \in G_{\mathbb{Q}}$ be a complex conjugation, and let $\mathrm{Tw}(T_\varphi)$ be the module T_φ with G_K-action twisted by τ. Then there is a perfect G_K-equivariant pairing

$$(\cdot,\cdot)^\tau : T_\varphi \times \mathrm{Tw}(T_\varphi) \to R(1) \tag{27}$$

induced by

$$(x \otimes r, y \otimes s)^\tau = rs(x, y^\tau), \quad r, s \in R, \; x, y \in T_f,$$

where $(\cdot,\cdot)$ is the pairing of (20).

Using the local isomorphisms

$$\mathsf{H}^1(K_{\mathsf{s}}, \mathrm{Tw}(T_\varphi)) \cong \mathsf{H}^1(K_{\mathsf{s}^\tau}, T_\varphi)$$

of Remark 2.5.7, the pairing (27) induces local pairings

$$\mathsf{H}^1(K_{\mathsf{s}}, T_\varphi) \times \mathsf{H}^1(K_{\mathsf{s}^\tau}, T_\varphi) \to R. \tag{28}$$

The pairing (27) also induces a perfect pairing

$$W_\varphi \times \mathrm{Tw}(T_\varphi) \to R(1) \otimes_{\mathcal{O}} E/\mathcal{O} = (R \otimes \mathbb{Q}_p/\mathbb{Z}_p)(1).$$

Recall that we have fixed a generalized Selmer structure $(\mathcal{F}, \mathsf{S})$ for T_φ over R. For any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, we write $(\mathcal{F}(\mathsf{Q}), \mathsf{S}^\tau \cup \mathsf{Q})$ for the Selmer structure on W_φ which is dual to the Selmer structure $(\mathcal{F}(\mathsf{Q})^\tau, \mathsf{S}^\tau \cup \mathsf{Q})$ on $\mathrm{Tw}(T_\varphi)$ (see Remark 2.5.7).

Definition 3.3.2. We say $(\mathcal{F}, \mathsf{S})$ is **self-dual** if, for all $\mathsf{s} \in \mathsf{M}_K$, $\mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{s}}, T_\varphi)$ and $\mathsf{H}^1_{\mathcal{F}}(K_{\mathsf{s}^\tau}, T_\varphi)$ are exact annihilators under the local pairing.

Equivalently, $(\mathcal{F}, \mathsf{S})$ is self-dual if, after identifying $\mathrm{Tw}(T_\varphi)$ and $T_\varphi^\dagger = \mathrm{Hom}_{\mathcal{O}}(T_\varphi, \mathcal{O}(1))$ via (27), the local conditions given by $(\mathcal{F}^\dagger, \mathsf{S})$ and $(\mathcal{F}^\tau, \mathsf{S}^\tau)$ coincide.

3.3.3. If $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$ is admissible, then $\mathsf{q}^\tau = \mathsf{q}$; let

$$\langle \cdot,\cdot \rangle_{\mathsf{q}}^\tau : \mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) \times \mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) \to R$$

be the pairing obtained from (28) by identifying $\mathsf{H}^1(K_{\mathsf{q}^\tau}, T_\varphi) = \mathsf{H}^1(K_{\mathsf{q}}, T_\varphi)$. Then unraveling the definition of $\langle \cdot,\cdot \rangle_{\mathsf{q}}^\tau$ shows that

$$\langle c_1, c_2 \rangle_{\mathsf{q}}^\tau = \langle c_1, \mathrm{Frob}_{\mathsf{q}}\, c_2 \rangle_{\mathsf{q}}, \quad c_1, c_2 \in \mathsf{H}^1(K_{\mathsf{q}}, T_\varphi), \tag{29}$$

where

$$\langle \cdot,\cdot \rangle_{\mathsf{q}} : \mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) \times \mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) \to R$$

is the local pairing induced by (20), extending linearly using (25). (Here $\mathrm{Frob}_{\mathsf{q}}$ acts naturally on $\mathsf{H}^1(K_{\mathsf{q}}, T_\varphi) = \mathsf{H}^1(K_{\mathsf{q}}, T_f) \otimes_{\mathcal{O}} R$ since T_f is a $G_{\mathbb{Q}}$-module.)

Lemma 3.3.4. *The residual representation $\overline{T}_f$ is absolutely irreducible as an $\mathcal{O}[G_K]$-module. Moreover, if (sclr) holds, then there exists an element $z \in G_K$ that acts as a nontrivial scalar on $\overline{T}_f$.*

Proof. The discriminant of K is coprime to Np by assumption, so we have $K \cap \mathbb{Q}(\overline{T}_f) = \mathbb{Q}$. Hence the image of $G_K \to \mathrm{Aut}(T_f)$ coincides with the image of $G_{\mathbb{Q}} \to \mathrm{Aut}(T_f)$, and this implies the lemma. □

Corollary 3.3.5. *Suppose that $(\mathcal{F}, \mathsf{S})$ is self-dual. Then $(\mathcal{F}, \mathsf{S})$ is saturated, and for all $j \geq 0$, we have isomorphisms*

$$\mathrm{Sel}_{\mathcal{F}}(T_\varphi/\varpi^j) = \mathrm{Sel}_{\mathcal{F}}(W_\varphi[\varpi^j]) = \mathrm{Sel}_{\mathcal{F}}(W_\varphi)[\varpi^j].$$

Proof. Self-duality implies saturation by Proposition 2.6.10, so the first isomorphism results from Proposition 2.6.12. The second is immediate from Lemmas 3.3.4 and 2.4.12. □

Proposition 3.3.6. *Suppose that $(\mathcal{F}, \mathsf{S})$ is self-dual. Then, for each $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$:*

(1) $(\mathcal{F}(\mathsf{Q}), \mathsf{S} \cup \mathsf{Q})$ *is self-dual.*

(2) There is a non-canonical isomorphism of R-modules:

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi) \approx (R \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{r_{\mathsf{Q}}} \oplus M_{\mathsf{Q}} \oplus M_{\mathsf{Q}}$$

for some finite-length R-module M_{Q} and an integer r_{Q}.

(3) $r_{\mathsf{Q}} = \mathrm{rk}_R\, \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)$.

Proof. For (1), we need only check self-duality of the local conditions for $\mathsf{q} \in \mathsf{Q}$; this follows from Proposition 3.1.6 and (29) because $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{Q}}(\mathsf{q})}(K_{\mathsf{q}}, T_f)$ is $\mathrm{Frob}_{\mathsf{q}}$-stable for all $\mathsf{q} \in \mathsf{Q}$. For (2), it suffices to show that $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\varpi^s]/\varpi \,\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)[\varpi^{s+1}]$ is an even-dimensional vector space for all s. For this, we claim the proof of [37, Theorem 1.4.2] applies with the obvious modifications for patched Selmer groups. Indeed, we use Lemma 3.3.4 to check the hypothesis H.1 of *op. cit.*; the (patched analogue of) Hypothesis H.3, or equivalently the condition in [38, Definition 2.2.2], follows from Corollary 3.3.5 and [49, Lemma 3.7.1(i)]. Hypotheses H.2 and H.5 are not used in the proof of *loc. cit.*, and Hypothesis H.4 is clear from the discussion in (3.3.1).

Finally, (3) follows from Corollary 3.3.5, (1), and [49, Lemma 3.7.1(ii)].

□

3.3.7. For the next proposition, we require the following extension of (3.1.7). Let q be admissible with sign ϵ_{q}. Since $W_\varphi = W_f \otimes_{\mathcal{O}} R(\varphi)$, we have

$$\mathsf{H}^1(K_{\mathsf{q}}, W_\varphi) = \mathsf{H}^1(K_{\mathsf{q}}, W_f) \otimes_{\mathcal{O}} R \tag{30}$$

by the same reasoning as (25). We use (30) to define, by linearity, the subspace $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(K_{\mathsf{q}}, W_\varphi)$ and the localization maps $\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}}$, $\partial_{\mathsf{q},\epsilon_{\mathsf{q}}}$ following (3.1.7). By Proposition 3.3.6(1) combined with Proposition 2.6.12, for any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ and any $\mathsf{q} \in \mathsf{Q}$, we have

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi) = \mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, W_\varphi). \tag{31}$$

Proposition 3.3.8. *Suppose $(\mathcal{F}, \mathsf{S})$ is self-dual. For any $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}$, exactly one of the following holds:*

(1) $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) = 0$, $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) \neq 0$, *and* $r_{\mathsf{Qq}} = r_{\mathsf{Q}} + 1$. *Moreover,*

$$\begin{aligned} \lg M_{\mathsf{Qq}} &= \lg_R M_{\mathsf{Q}} - \lg_R \operatorname{coker} \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) \\ &= \lg_R M_{\mathsf{Q}} - \lg_R \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)). \end{aligned}$$

(2) $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) \neq 0$, $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) = 0$, *and* $r_{\mathsf{Qq}} = r_{\mathsf{Q}} - 1$. *Moreover,*

$$\begin{aligned} \lg_R M_{\mathsf{Qq}} &= \lg_R M_{\mathsf{Q}} + \lg_R \operatorname{coker} \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)) \\ &= \lg_R M_{\mathsf{Q}} + \lg_R \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)). \end{aligned}$$

Proof. Consider the Selmer structures $(\mathcal{F}^{\mathsf{q}}, \mathsf{S} \cup \mathsf{Qq})$ and $(\mathcal{F}_{\mathsf{q}}, \mathsf{S} \cup \mathsf{Qq})$ for T_φ, where $\mathcal{F}^{\mathsf{q}}(\mathsf{Q}) = \mathcal{F}(\mathsf{Q}) + \mathcal{F}(\mathsf{Qq})$ and $\mathcal{F}_{\mathsf{q}}(\mathsf{Q}) = \mathcal{F}(\mathsf{Q}) \cap \mathcal{F}(\mathsf{Qq})$. By Proposition 2.6.13, we have:

$$\begin{aligned} \mathrm{rk}_R \,\mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(T_\varphi) &= \mathrm{rk}_R \,\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})^\tau}(\mathrm{Tw}(T_\varphi)) + 1 \\ &= \mathrm{rk}_R \,\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(T_\varphi) + 1; \end{aligned}$$

the second equality is by (13). Moreover, because $\mathcal{F}(\mathsf{Q})$ is self-dual, Proposition 2.6.10 implies that the image of

$$\frac{\mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(T_\varphi)}{\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(T_\varphi)} \hookrightarrow \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} = \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \oplus \frac{\mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \approx R^2 \tag{32}$$

is self-annihilating under the R-bilinear local pairing induced by $\langle \cdot, \cdot \rangle^\tau_{\mathsf{q}}$, which is symmetric by (29) because the pairing (20) is alternating and $G_{\mathbb{Q}}$-equivariant. Since a two-dimensional nondegenerate quadratic space cannot contain three distinct isotropic lines, the image of (32) is contained either in the ordinary or unramified part. In other words, exactly one of $\mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi))$ and $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi))$ is nonzero, which gives the alternative of the proposition.

For the relation between M_{Q} and M_{Qq}, we suppose we are in case (1), because the two arguments are analogous. By Corollary 2.6.8, the image of

$$\frac{\mathrm{Sel}_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(W_\varphi)}{\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi)} \hookrightarrow \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)} = \frac{\mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{q}}, W_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)} \oplus \frac{\mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, W_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)} \approx (R \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}$$

is the exact annihilator of $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi))$ under the perfect induced local pairing

$$\langle \cdot, \cdot \rangle^\tau_{\mathsf{q}} : \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, T_\varphi)} \times \frac{\mathsf{H}^1_{\mathcal{F}^{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)}{\mathsf{H}^1_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(K_{\mathsf{q}}, W_\varphi)} \to R \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

This implies that $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi))$ is divisible and

$$\lg_R \mathrm{loc}_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)) = \lg_R \mathrm{coker}\, \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)). \tag{33}$$

Now, for any short exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

there is an induced exact sequence

$$0 \to \left(\frac{A \cap B_{\mathrm{div}}}{A_{\mathrm{div}}}\right) \to A_{/\mathrm{div}} \to B_{/\mathrm{div}} \to C_{/\mathrm{div}} \to 0, \tag{$*$}$$

where the subscript div denotes the maximal ϖ-divisible submodule and $M_{/\mathrm{div}} = M/M_{\mathrm{div}}$ for any R-module M. Also note that

$$\varprojlim_j \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)_{\mathrm{div}}[\varpi^j] = \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)$$

by Corollary 3.3.5 and Proposition 2.5.6, compatibly with the map ∂_{q}, and so we can identify

$$\frac{\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi) \cap \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)_{\mathrm{div}}}{\mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi)_{\mathrm{div}}} = \mathrm{coker}\, \partial_{\mathsf{q}} \left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)\right). \tag{34}$$

Consider the short exact sequences:

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi) \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi) \to \mathrm{loc}_{\mathsf{q}} \left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)\right) \to 0 \tag{35}$$

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi) \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi) \to \partial_{\mathsf{q}} \left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)\right) \to 0. \tag{36}$$

By ($*$) and (34), we obtain the exact sequences of finite-length R-modules:

$$0 \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} \to \mathrm{loc}_{\mathsf{q}} \left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)\right) \to 0 \tag{37}$$

$$0 \to \mathrm{coker}\, \partial_{\mathsf{q}} \left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)\right) \to \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)_{/\mathrm{div}} \to 0. \tag{38}$$

From this and (33), we deduce

$$\lg_R \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)_{/\mathrm{div}} = \lg_R \mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(W_\varphi)_{/\mathrm{div}} + 2 \lg_R \mathrm{loc}_{\mathsf{q}} \left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_\varphi)\right),$$

which gives the result. □

The following result will allow us to control the alternative in Proposition 3.3.8.

Theorem 3.3.9. *Let $c \in \mathsf{H}^1(K^{\mathsf{T}}/K, T_\varphi)$ be any nonzero element, where $\mathsf{T} \supset \mathsf{S}$ is a finite set. Then there are infinitely many admissible ultraprimes $\mathsf{q} \notin \mathsf{T}$, with associated signs ϵ_{q}, such that $\mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}} c \neq 0$.*

The proof is via a series of lemmas.

Lemma 3.3.10. *There is an integer j such that, for all $n \geq 0$,*

$$\varpi^j H^1(K(T_\varphi)/K, T_\varphi/\varpi^n) = 0.$$

If (sclr) holds, then we may take $j = 0$.

Proof. Let $G = \mathrm{Gal}(K(T_\varphi)/K)$, and let $Z \subset G$ be its center; Lemma 3.3.4 implies that $T_\varphi \otimes_{\mathcal{O}} E$ is absolutely irreducible as an $E[G_K]$-module, and so Z acts on T_φ by scalars. We claim:

$$Z \neq \{1\}. \tag{39}$$

Assuming (39), the lemma follows from the inflation-restriction exact sequence

$$H^1(G/Z, H^0(Z, T_\varphi/\varpi^n)) \hookrightarrow H^1(G, T_\varphi/\varpi^n) \to H^1(Z, T_\varphi/\varpi^n)$$

(and Lemma 3.3.4 for the extra assertion under (sclr)). Let us now prove (39). Let $G' = \mathrm{Gal}(K(T_f)/K)$, and let L/K be the Galois subfield of $K(T_f)$ cut out by the center $Z' = Z(G') \subset G'$. By [61, Theorem 5.5] or [54, Theorem B.5.2], Z' is nontrivial. Let E/K be the Galois extension determined by the kernel of φ; then it suffices to show that EL/L and $K(T_f)/L$ are linearly disjoint. Both EL and $K(T_f)$ are Galois over $\mathbb{Q}$, so $G_{\mathbb{Q}}$ acts on $\mathrm{Gal}(EL/L)$ and $\mathrm{Gal}(K(T_f)/L)$ by conjugation. If $\tau \in G_{\mathbb{Q}}$ is a complex conjugation, then τ acts trivially on $\mathrm{Gal}(K(T_f)/L)$ but by inversion on the pro-p-group $\mathrm{Gal}(EL/L)$, so the two groups have no nontrivial common quotient compatible with the $G_{\mathbb{Q}}$-action; hence $EL \cap K(T_f) = L$. □

Lemma 3.3.11. *Suppose given a cocycle*

$$c \in H^1(K, T_\varphi/\varpi^n)$$

such that $\varpi^j c \neq 0$*, where* j *is as in Lemma 3.3.10. Then, for any integer* $M \geq n$*, there exists a sign* $\epsilon = \pm 1$ *and infinitely many rational primes* q *such that:*

(1) q *is inert in* K *and unramified in the splitting field* $\mathbb{Q}(T_f, c)$*.*
(2) $\mathrm{Frob}_q \in \mathrm{Gal}(\mathbb{Q}(T_f)/\mathbb{Q})$ *has distinct eigenvalues* ± 1 *on* $T_f \otimes R/\varpi^M$ *(where* R *has trivial Galois action).*
(3) For any cocycle representative, $c(\mathrm{Frob}_q^2)$ *has nonzero component in the* ϵ *eigenspace for* Frob_q*.*

Proof. Abbreviate $L = K(T_\varphi/\varpi^M)$, and let $\phi \in \mathrm{Hom}_{G_K}(G_L, T_\varphi/\varpi^n)$ be the image of c under restriction; by hypothesis $\phi \neq 0$. Without loss of generality, we may suppose that the image of ϕ is contained in $T_\varphi/\varpi^n[\varpi] \simeq T_\varphi/\varpi$, which, since φ is residually trivial, is the extension of scalars $\overline{T}_f \otimes_{\mathcal{O}/\wp} R/\varpi$. Fix a complex conjugation $\tau \in G_\mathbb{Q}$. Now,

$$\mathrm{Hom}_{G_K}(G_L, \overline{T}_f \otimes R/\varpi)$$

has a natural action of $\mathrm{Gal}(K/\mathbb{Q})$, so we may assume without loss of generality that $\phi^\tau = \epsilon\phi$ for some $\epsilon \in \{\pm 1\}$. Also, since $\overline{T}_f$ is absolutely irreducible as an $\mathcal{O}[G_K]$-module by Lemma 3.3.4, there exists $g \in G_L$ such that $\phi(g)$ has nonzero component in the ϵ eigenspace of τ. Then

$$\phi(\tau g \tau g) = \tau\phi^\tau(g) + \phi(g) = \epsilon\tau\phi(g) + \phi(g)$$

has nonzero component in the ϵ eigenspace as well. Any q with Frobenius τg in $L(\phi)$ satisfies the desired conditions. □

Remark 3.3.12. If $p \geq 5$ and the image of the Galois action on T_f is sufficiently large, then we can instead use primes q such that $p \nmid q^2 - 1$, as is more common in the literature.

Proof of Theorem 3.3.9. By Lemmas 2.4.12 and 3.3.4,

$$\mathsf{H}^1(K^\mathsf{T}/K, T_\varphi)[\varpi] = 0.$$

Thus there exists some n such that the image $\bar{c}$ of c in $\mathsf{H}^1(K^\mathsf{T}/K, T_\varphi/\varpi^n)$ satisfies $\varpi^j\bar{c} \neq 0$, for some j as in Lemma 3.3.10. By definition, $\bar{c}$ is represented by a sequence of classes $c_m \in H^1(K^{T_m}/K, T_\varphi/\varpi^n)$ such that $\varpi^j c_m \neq 0$ for $\mathfrak{F}$-many m, where $(T_m)_{m\in\mathbb{N}}$ represents T. For each m, apply Lemma 3.3.11 with $M = \max\{m, n\}$ to obtain a prime $q_m \notin T_m$ and a sign ϵ_m. If $\mathsf{q} \in \mathsf{M}_\mathbb{Q}$ is the equivalence class of the sequence $(q_m)_{m\in\mathbb{N}}$, and $\epsilon_\mathsf{q} \in \mathcal{U}(\{\pm 1\}_{m\in\mathbb{N}}) \simeq \{\pm 1\}$ is the equivalence class of the sequence $(\epsilon_m)_{m\in\mathbb{N}}$, then q satisfies the theorem with the sign ϵ_q. Since there are infinitely many choices for each q_m, there are also infinitely many choices for q. □

Corollary 3.3.13. *For any* $\{\mathsf{Q}, \epsilon_\mathsf{Q}\} \in \mathsf{N}$*, there exists some* $\{\mathsf{Q}\mathsf{Q}', \epsilon_{\mathsf{Q}\mathsf{Q}'}\} \in \mathsf{N}$ *such that* $r_{\mathsf{Q}\mathsf{Q}'} = 0$*; moreover, we may choose* Q' *such that* $\mathsf{Q}' \cap \mathsf{T} = \emptyset$ *for any fixed finite set* $\mathsf{T} \subset \mathsf{M}_\mathbb{Q}$*.*

Proof. If $r_\mathsf{Q} > 0$, then Theorem 3.3.9 and Proposition 3.3.8 imply that there exists an admissible ultraprime $\mathsf{q} \notin \mathsf{T} \cup \mathsf{Q}$ with sign ϵ_q such that such that $r_{\mathsf{Q}\mathsf{q}} = r_\mathsf{Q} - 1$. The corollary follows by induction on r_Q. □

Combining Proposition 3.3.8 and Theorem 3.3.9 allows us to prove the main result of this subsection.

Theorem 3.3.14. *Suppose that* $(\mathcal{F}, \mathsf{S})$ *is self-dual and that* (κ, λ) *is a nontrivial bipartite Euler system with parity* δ *for* $(T_\varphi, \mathcal{F}, \mathsf{S})$*. Then there exists an integer* C *(possibly negative) such that:*

(1) For all $\{\mathsf{Q}, \epsilon_\mathsf{Q}\} \in \mathsf{N}^\delta$*,* r_Q *is odd,* $r_\mathsf{Q} = 1$ *if and only if* $\kappa(\mathsf{Q}) \neq 0$*, and in that case*

$$\lg_R M_\mathsf{Q} = \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))} \right) + C.$$

(2) For all $\{\mathsf{Q}, \epsilon_\mathsf{Q}\} \in \mathsf{N}^{\delta+1}$*,* r_Q *is even,* $r_\mathsf{Q} = 0$ *if and only if* $\lambda(\mathsf{Q}) \neq 0$*, and in that case*

$$\lg_R M_\mathsf{Q} = \mathrm{ord}_\varpi \lambda(\mathsf{Q}) + C.$$

In particular,

$$\delta = \mathrm{rk}_R \mathrm{Sel}_\mathcal{F}(T_\varphi) + 1 \pmod 2.$$

Proof. The proof will be in several steps.

Step 1. *If* $\lambda(\mathsf{Q}) \neq 0$ *for some* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$*, then* $r_{\mathsf{Q}} = 0$.

Proof. If $0 \neq c \in \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)$, then by Theorem 3.3.9, there exists an admissible ultraprime q with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, c \neq 0$. By Proposition 3.3.8, $\partial_{\mathsf{q}}(\kappa(\mathsf{Qq})) = 0$, which contradicts the reciprocity laws. □

Step 2. *If* $\kappa(\mathsf{Q}) \neq 0$ *for some* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta}$*, then* $r_{\mathsf{Q}} = 1$.

Proof. Choose an admissible ultraprime q with sign ϵ_{q} such that $\mathrm{loc}_{\mathsf{q}}\, \kappa(\mathsf{Q}) \neq 0$. Then by the reciprocity laws, $\lambda(\mathsf{Qq}) \neq 0$, so by Step 1 $r_{\mathsf{Qq}} = 0$. Proposition 3.3.8 implies $r_{\mathsf{Q}} = 1$. □

Step 3. *For all* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$*,* $r_{\mathsf{Q}} \equiv \delta + |\mathsf{Q}| + 1 \pmod 2$.

Proof. If $\{\mathsf{QQ}', \epsilon_{\mathsf{QQ}'}\} \in \mathsf{N}$, then by Proposition 3.3.8

$$r_{\mathsf{Q}} - r_{\mathsf{QQ}'} \equiv |\mathsf{Q}'| \pmod 2.$$

In particular, the parity of $r_{\mathsf{Q}} - |\mathsf{Q}|$ is constant as $\mathsf{Q} \in \mathsf{N}$ varies, so Steps 1 and 2 imply Step 3. □

Step 4. *Suppose* $r_{\mathsf{Q}} = 0$ *for some* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$*. Then, for all admissible ultraprimes* $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{S}$ *with sign* ϵ_{q}*,* $r_{\mathsf{Qq}} = 1$ *and*

$$\lg_R M_{\mathsf{Qq}} + \mathrm{ord}_\varpi \lambda(\mathsf{Q}) = \lg_R M_{\mathsf{Q}} + \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)}{(\kappa(\mathsf{Qq}))}\right).$$

Proof. By Step 3, $\lambda(\mathsf{Q})$ and $\kappa(\mathsf{Qq})$ are well-defined. Then Step 4 follows from Proposition 3.3.8, since by the reciprocity laws

$$\lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)}{(\kappa(\mathsf{Qq}))}\right) + \lg_R \mathrm{coker}\, \partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Qq})}(T_\varphi)) = \mathrm{ord}_\varpi \lambda(\mathsf{Q}).$$

□

The same reasoning implies:

Step 5. *Suppose that* $r_{\mathsf{Q}} = 1$ *and* $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{S}$ *is an admissible ultraprime with sign* ϵ_{q} *such that* $r_{\mathsf{Qq}} = 0$*. Then*

$$\lg_R M_{\mathsf{Qq}} + \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))}\right) = \lg_R M_{\mathsf{Q}} + \mathrm{ord}_\varpi \lambda(\mathsf{Qq}).$$

Now consider the graph $\mathcal{X}$ whose vertices are the elements of N, and where the edges are between vertices of the form $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ and $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\}$, for some admissible ultraprime q with sign ϵ_{q} (cf. [38, §2.4]). We say $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ is a **core vertex** if $r_{\mathsf{Q}} \leq 1$. The **core subgraph** $\mathcal{X}_0$ of $\mathcal{X}$ is the full subgraph on core vertices.

Step 6. *Assume* $\mathcal{X}_0$ *is path-connected. Then the theorem holds.*

Proof. For every $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathcal{X}_0$, set

$$C_{\mathsf{Q}} = \begin{cases} \lg_R M_{\mathsf{Q}} - \lg_R \left(\frac{\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi)}{(\kappa(\mathsf{Q}))}\right), & \text{if } \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta}, \\ \lg_R M_{\mathsf{Q}} - \mathrm{ord}_\varpi \lambda(\mathsf{Q}), & \text{if } \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}. \end{cases}$$

(A priori, C_{Q} could be $-\infty$.) By Steps 4 and 5, C_{Q} is constant along paths contained in $\mathcal{X}_0$. Moreover, if (κ, λ) is nontrivial, then Steps 1 and 2 imply that there exists $\mathsf{Q} \in \mathcal{X}_0$ with C_{Q} finite. Under the additional assumption that $\mathcal{X}_0$ is path-connected, the common value of C_{Q} for $\mathsf{Q} \in \mathcal{X}_0$ is the global constant C of the theorem. □

In the rest of the proof, we will establish the path-connectedness of $\mathcal{X}_0$.

Step 7. *If* $v = \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ *and* $v' = \{\mathsf{QQ}', \epsilon_{\mathsf{QQ}'}\}$ *are core vertices, then they are connected by a path in* $\mathcal{X}_0$*.*

Proof. We proceed by induction on $|\mathsf{Q}'|$, where the base case $|\mathsf{Q}'| = 1$ is trivial. If $r_{\mathsf{QQ}'/\mathsf{q}} \leq 1$ for any $\mathsf{q} \in \mathsf{Q}'$, then we may apply the inductive hypothesis, so assume otherwise. By Proposition 3.3.8, $r_{\mathsf{QQ}'} = 1$ and $\partial_{\mathsf{q}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{QQ}')}(T_\varphi)) = 0$ for all $\mathsf{q} \in \mathsf{Q}'$. Hence

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{QQ}')}(T_\varphi) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_\varphi).$$

Then, by Theorem 3.3.9 and Proposition 3.3.8, there exists an admissible ultraprime $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{Q}' \cup \mathsf{S}$ with sign ϵ_{q} such that $r_{\mathsf{Qq}} = r_{\mathsf{QQ}'\mathsf{q}} = 0$. For any ultraprime $\mathsf{q}' \in \mathsf{Q}'$, $\left\{\mathsf{QQ}'\mathsf{q}/\mathsf{q}', \epsilon_{\mathsf{QQ}'\mathsf{qq}'}|_{\mathsf{QQ}'\mathsf{q}/\mathsf{q}'}\right\} \in \mathsf{N}$ is a core vertex, which is connected to v' in $\mathcal{X}_0$. By the inductive hypothesis, $\left\{\mathsf{QQ}'\mathsf{q}/\mathsf{q}', \epsilon_{\mathsf{QQ}'\mathsf{qq}'}|_{\mathsf{QQ}'\mathsf{q}/\mathsf{q}'}\right\}$ is also connected to the core vertex $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\}$, hence to v, by a path in $\mathcal{X}_0$. This completes the inductive step. □

Step 8. *If $v = \{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ is a core vertex and $\mathsf{T} \subset \mathsf{M}_{\mathbb{Q}}$ is any finite set, then there exists a core vertex $v' = \{\mathsf{Q}', \epsilon_{\mathsf{Q}'}\}$ such that v and v' are connected by a path in $\mathcal{X}_0$ and $\mathsf{Q}' \cap \mathsf{T} = \emptyset$.*

Proof. By iterating, it suffices to assume that $\mathsf{Q} \cap \mathsf{T}$ consists of exactly one ultraprime $\mathsf{q} \in \mathsf{Q}$. If $r_{\mathsf{Q}/\mathsf{q}} \leq 1$, then the conclusion is obvious, so suppose otherwise. As in the proof of Step 7, choose an admissible ultraprime $\mathsf{q}' \notin \mathsf{Q} \cup \mathsf{S} \cup \mathsf{T}$ with associated sign $\epsilon_{\mathsf{q}'}$ such that $r_{\mathsf{Q}\mathsf{q}'} = 0$, which implies $r_{\mathsf{Q}\mathsf{q}'/\mathsf{q}} = 1$. The core vertex $v' = \left\{\mathsf{Q}\mathsf{q}'/\mathsf{q}, \epsilon_{\mathsf{Q}\mathsf{q}'}|_{\mathsf{Q}\mathsf{q}'/\mathsf{q}}\right\}$ has the desired properties. □

Finally, we have:

Step 9. *The core subgraph $\mathcal{X}_0$ is path-connected.*

Proof. Let $\{\mathsf{Q}_1, \epsilon_{\mathsf{Q}_1}\}$ and $\{\mathsf{Q}_2, \epsilon_{\mathsf{Q}_2}\}$ be two core vertices; we wish to show they are connected by a path in $\mathcal{X}_0$. Without loss of generality, by Step 8, we may assume $\mathsf{Q}_1 \cap \mathsf{Q}_2 = \emptyset$. (This step is necessary because the sign functions ϵ_{Q_1} and ϵ_{Q_2} need not agree on $\mathsf{Q}_1 \cap \mathsf{Q}_2$.) Consider $\{\mathsf{Q}_1\mathsf{Q}_2, \epsilon_{\mathsf{Q}_1\mathsf{Q}_2}\} \in \mathsf{N}$. This may not be a core vertex, but, by Corollary 3.3.13, there exists $\{\mathsf{Q}_3, \epsilon_{\mathsf{Q}_3}\} \in \mathsf{N}$ such that $\{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3, \epsilon_{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3}\}$ is a core vertex. We may then conclude by Step 6. □

□

Proposition 3.3.15. *In the setting of Theorem 3.3.14, there exists a constant $C' \geq 0$ depending on $|\mathsf{S}|$, T_f, and the ramification index of $R/\mathcal{O}$, but not on φ, such that $C \geq -C'$. If (sclr) holds, then we may take $C' = 0$.*

Proof. By Theorem 3.3.14, it suffices to show that there exists a constant C' with the desired dependencies and a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $r_{\mathsf{Q}} = 0$ and $\lg_R M_{\mathsf{Q}} \leq C'$. Inspecting the proof of Lemma 3.3.10, we first note that the constant j in Lemma 3.3.11 can be chosen to depend only on T_f and the ramification index of $R/\mathcal{O}$, and we can take $j = 0$ under (sclr).

Moreover, if k is the residue field of R, then $d = \dim_k \mathsf{H}^1(K^{\mathsf{S}}/K, W_\varphi[\varpi])$ is also bounded with a bound depending only on $|\mathsf{S}|$ and T_f, by the proof of Lemma 2.4.5. We now construct a sequence $\{\mathsf{Q}_i, \epsilon_{\mathsf{Q}_i}\}$ recursively (starting from $\mathsf{Q}_1 = 1$) by the following rules:

- If $r_{\mathsf{Q}_i} = 0$ and the exponent of $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi) \neq 0$ is $n_i > (i+2) \cdot j$, then choose any $\mathsf{q}_{i+1} \notin \mathsf{Q}_i$ with sign $\epsilon_{\mathsf{q}_{i+1}}$ such that the exponent of $\mathrm{loc}_{\mathsf{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi))$ is at least $n_i - j$.
- If $r_{\mathsf{Q}_i} > 0$, then choose any $\mathsf{q}_{i+1} \notin \mathsf{Q}_i$ with sign $\epsilon_{\mathsf{q}_{i+1}}$ such that
$$\lg_R \operatorname{coker}(\mathrm{loc}_{\mathsf{q}_{i+1}} \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(T_\varphi)) \leq j.$$

These choices are possible by Lemma 3.3.11 and Corollary 3.3.5, cf. the proof of Theorem 3.3.9. In either of the above two cases, set $\left\{\mathsf{Q}_{i+1}, \epsilon_{\mathsf{Q}_{i+1}}\right\} = \left\{\mathsf{Q}_i\mathsf{q}_{i+1}, \epsilon_{\mathsf{Q}_i\mathsf{q}_{i+1}}\right\}$; if neither holds, then end the construction. For each i, let r'_{Q_i} be the minimal number of generators of the torsion R-module $\varpi^{ij} M_{\mathsf{Q}_i}$, which is also the dimension of the k-vector spaces $\varpi^{ij} M_{\mathsf{Q}_i} / \varpi^{ij+1} M_{\mathsf{Q}_i}$ and $(\varpi^{ij} M_{\mathsf{Q}_i})[\varpi]$.

Claim. In the first case of the construction, $r'_{\mathsf{Q}_{i+1}} < r'_{\mathsf{Q}_i}$. In the second case, $r'_{\mathsf{Q}_{i+1}} \leq r'_{\mathsf{Q}_i}$.

Before proving the claim, we show it implies the proposition. After $r_1 \leq d$ steps (where the inequality holds by Corollary 3.3.5 and Proposition 3.3.6), we alternate between the two cases of the construction by Proposition 3.3.8, and so the claim implies that the construction must halt after at most $2r'_1 \leq 2d$ more steps. Hence for some $i \leq 3d$, $r_{\mathsf{Q}_i} = 0$ and $\varpi^{(i+2)j} M_{\mathsf{Q}_i} = 0$. In this case,
$$\lg_R M_{\mathsf{Q}_i} \leq (i+2)j \dim_k \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)[\varpi] \leq (3d+2)j(d+3d),$$
the last inequality by the reasoning of [38, Corollary 2.2.10]. Since d and j have bounds depending only on $|\mathsf{S}|$, T_f, and the ramification index of R, and we may take $j = 0$ under (sclr), the proposition follows.

Now let us prove the claim, for which we abbreviate $A \coloneqq \mathrm{Sel}_{\mathcal{F}_{\mathsf{q}_{i+1}}(\mathsf{Q}_i)}(W_\varphi)_{/\operatorname{div}}$. Start with the first case. From (35), we have an exact sequence

$$\mathrm{Hom}_{R/\varpi^{n_i}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi), A) \to \mathrm{Hom}_{R/\varpi^{n_i}}(A, A) \to \mathrm{Ext}^1_{R/\varpi^{n_i}}(\mathrm{loc}_{\mathsf{q}_{i+1}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)\right), A), \tag{40}$$

where the last term is ϖ^j-torsion because $\mathrm{loc}_{\mathsf{q}_{i+1}}\left(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)\right)$ is a cyclic R/ϖ^{n_i}-module of length at least $n_i - j$. Hence there is a map $f : \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi) \to A$ such that the composite $A \hookrightarrow \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi) \xrightarrow{f} A$ is multiplication by ϖ^j. Let B be the image of the resulting map

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi) \xrightarrow{f\oplus \mathrm{loc}_{\mathfrak{q}_{i+1}}} A \oplus \mathrm{loc}_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)),$$

so that

$$\dim_k \varpi^{ij} B/\varpi^{ij+1} B \leq 2r'_{\mathsf{Q}_i}.$$

Now, an elementary calculation shows that B contains $\varpi^j A \oplus \varpi^j \mathrm{loc}_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi))$, so

$$\dim_k \varpi^{(i+1)j} A/\varpi^{(i+1)j+1} A + 1 \leq \dim_k \varpi^{ij} B/\varpi^{ij+1} B \leq 2r'_{\mathsf{Q}_i}.$$

(Here we are using that $\varpi^{(i+1)j} \mathrm{loc}_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)) \neq 0$, which holds because $\mathrm{loc}_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi))$ is cyclic of length at least $n_i - j > (i+1)j$.) On the other hand, $\varpi^{(i+1)j} \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i\mathfrak{q}_{i+1})}(W_\varphi)_{/\mathrm{div}}$ is a quotient of $\varpi^{(i+1)j} A$ by (38), so we have

$$2r'_{\mathsf{Q}_{i+1}} \leq \dim_k \varpi^{(i+1)j} A/\varpi^{(i+1)j+1} A \leq 2r'_{\mathsf{Q}_i} - 1,$$

which proves the claim in this case.

We now consider the second case of the construction. Analogously to (37), (38), in this case we have exact sequences:

$$0 \to A \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i\mathfrak{q}_{i+1})}(W_\varphi)_{/\,\mathrm{div}} \to \partial_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i\mathfrak{q}_{i+1})}(W_\varphi)) \to 0, \tag{41}$$

$$0 \to \mathrm{coker}\,\mathrm{loc}_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(T_\varphi)) \to A \to \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i)}(W_\varphi)_{/\,\mathrm{div}} \to 0. \tag{42}$$

By (42),

$$\dim_k \varpi^{ij} A/\varpi^{ij+1} A \leq 2r'_{\mathsf{Q}_i} + 1.$$

Since $\partial_{\mathfrak{q}_{i+1}}(\mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i\mathfrak{q}_{i+1})}(W_\varphi))$ is cyclic and ϖ^j-torsion by the analogue of (33), the exact sequence (41) shows that $\varpi^{(i+1)j} \mathrm{Sel}_{\mathcal{F}(\mathsf{Q}_i q_{i+1})}(W_\varphi)_{/\,\mathrm{div}}$ injects into $\varpi^{ij} A$, so we conclude $2r'_{\mathsf{Q}_{i+1}} \leq 2r'_{\mathsf{Q}_i} + 1$; hence $r'_{\mathsf{Q}_{i+1}} \leq r'_{\mathsf{Q}_i}$, as claimed. □

3.4. **Euler systems over Λ.** For this subsection, we assume:

$$a_p(f) \notin \wp. \tag{ord}$$

3.4.1. Let K_∞/K be the anticyclotomic $\mathbb{Z}_p$-extension, and let Λ be the anticyclotomic Iwasawa algebra $\mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!]$ with canonical character

$$\Psi : G_K \to \Lambda^\times.$$

If γ is a topological generator of $\mathrm{Gal}(K_\infty/K)$, then as a ring $\Lambda = \mathcal{O}[\![T]\!]$ where $T = \Psi(\gamma) - 1$. For each height-one prime $\mathfrak{P} \subset \Lambda$ with $\mathfrak{P} \neq (\wp)$, let $S_\mathfrak{P}$ be the integral closure of $\Lambda/\mathfrak{P}$ in its field of fractions, so that Ψ induces a character $G_K \to \Lambda^\times \to S_\mathfrak{P}^\times$. We write $T_\mathfrak{P}$ for the twist $T_f \otimes_\mathcal{O} S_\mathfrak{P}(\Psi)$, $W_\mathfrak{P}$ for $T_\mathfrak{P} \otimes \mathbb{Q}_p/\mathbb{Z}_p = T_\mathfrak{P} \otimes_\mathcal{O} E/\mathcal{O}$, and $\mathbf{T}_f$ for the interpolated twist $T_f \otimes_\mathcal{O} \Lambda(\Psi)$. Also let $\mathbf{W}_f = \mathbf{T}_f^*$ be the Cartier dual with Λ action twisted by the canonical involution ι, so that for each $\mathfrak{P}$ there is a natural map

$$W_\mathfrak{P} \to \mathbf{W}_f$$

of $\Lambda[G_K]$-modules (see, e.g., [37, §2]).

3.4.2. For each place $v|p$ of K, there is a free rank-one direct summand $\mathrm{Fil}_v^+ T_f \subset T_f$ on which $I_v \subset G_K$ acts through the cyclotomic character; let $\mathrm{gr}_v T_f = T_f/\mathrm{Fil}_v^+ T_f$ be the quotient. Let $\mathsf{S} \subset \mathsf{M}_K$ be the set of ultraprimes $\mathsf{v} = \underline{v}$ lying over $v|Np$, and define the Selmer structure $(\mathcal{F}_\Lambda, \mathsf{S})$ for $\mathbf{T}_f$ as follows:

$$\mathsf{H}^1_{\mathcal{F}_\Lambda}(K_\mathsf{v}, \mathbf{T}_f) = \begin{cases} \mathrm{im}\left(H^1(K_v, \mathrm{Fil}_v^+ T_f \otimes \Lambda) \to H^1(K_v, \mathbf{T}_f)\right), & \mathsf{v} = \underline{v}, v|p, \\ H^1(K_v, \mathbf{T}_f), & \mathsf{v} = \underline{v}, v|N\infty,\, v = v^\tau, \\ \mathsf{H}^1_{\mathrm{unr}}(K_\mathsf{v}, \mathbf{T}_f), & \text{otherwise.} \end{cases} \tag{43}$$

Remark 3.4.3. If $q||N$ is inert in K, then a direct calculation shows that the whole cohomology group $H^1(K_q, \mathbf{T}_f)$ is ordinary in the sense of [38, §3.1], so (43) is consistent with the Selmer structure defined in *loc. cit.* under the conditions therein.

3.4.4. For each height-one prime $\mathfrak{P} \subset \Lambda$ with $\mathfrak{P} \neq (\wp)$, we also define the Selmer structure $(\mathcal{F}_{\mathfrak{P}}, \mathsf{S})$ for $T_{\mathfrak{P}}$ by:

$$\mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}}(K_{\mathsf{v}}, T_{\mathfrak{P}}) = \begin{cases} \ker\left(H^1(K_v, T_{\mathfrak{P}}) \to H^1(K_v, \mathrm{gr}_v\, T_f \otimes S_{\mathfrak{P}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right), & \mathsf{v} = \underline{v}, v|p, \\ \ker\left((\mathsf{H}^1(K_{\mathsf{v}}, T_{\mathfrak{P}}) \to \mathsf{H}^1(I_{\mathsf{v}}, T_{\mathfrak{P}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right), & \text{otherwise.} \end{cases} \tag{44}$$

This is well-defined because $\mathsf{H}^1(I_{\mathsf{v}}, T_{\mathfrak{P}})$ is p-torsion-free when $T_{\mathfrak{P}}$ is unramified at v.

As in (3.3.1), for all $\mathsf{Q} \in \mathsf{N}$ the Selmer structure $(\mathcal{F}_\Lambda(\mathsf{Q}), \mathsf{S})$ for $\mathbf{T}_f$ induces a dual Selmer structure $(\mathcal{F}_\Lambda(\mathsf{Q})^*, \mathsf{S})$ for $\mathbf{W}_f$.

Proposition 3.4.5. *The Selmer structures $(\mathcal{F}_{\mathfrak{P}}, \mathsf{S})$ are self-dual for all $\mathfrak{P} \neq (\wp)$. Moreover, for all $\mathfrak{P} \neq (\wp)$ and for all $\mathsf{Q} \in \mathsf{N}$, the natural map induces well-defined homomorphisms:*

$$\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) \otimes_\Lambda \Lambda/\mathfrak{P} \to \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(T_{\mathfrak{P}}), \;\; \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(W_{\mathfrak{P}}) \to \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)[\mathfrak{P}].$$

The first map is injective. Moreover, there is a finite set of height-one primes Σ_Λ of Λ such that, for all $\mathfrak{P} \notin \Sigma_\Lambda$, both maps have finite kernels and cokernels whose cardinalities are bounded by a constant depending on T_f and $[S_{\mathfrak{P}} : \Lambda/\mathfrak{P}]$, but not on Q or on $\mathfrak{P}$ itself.

Proof. The self-duality of $\mathcal{F}_{\mathfrak{P}}$ follows from the self-duality of the usual ordinary (resp. unramified) local condition on $H^1(K_v, T_{\mathfrak{P}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ for each $v|p$ (resp. $v|N$).

By the proof of [38, Proposition 3.3.1] and the references therein, for the rest of the proposition it suffices to show the following:

Claim. The inclusion $\mathbf{T}_f/\mathfrak{P}\mathbf{T}_f \hookrightarrow T_{\mathfrak{P}}$ induces maps

$$\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_{\mathsf{v}}, \mathbf{T}_f) \otimes \Lambda/\mathfrak{P} \to \mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(K_{\mathsf{v}}, T_{\mathfrak{P}}), \;\; \mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(K_{\mathsf{v}}, W_{\mathfrak{P}}) \to \mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(K_{\mathsf{v}}, \mathbf{W}_f)[\mathfrak{P}]$$

with kernels and cokernels having bounds of the desired sort, for $\mathsf{v} = \underline{v}$ with $v|N$, $v = v^\tau$, and for $\mathsf{v} = \mathsf{q} \in \mathsf{Q}$.

For $\mathsf{v} = \mathsf{q} \in \mathsf{Q}$,

$$\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_{\mathsf{q}}, \mathbf{T}_f) = \mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_f) \otimes \Lambda, \;\; \mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(K_{\mathsf{q}}, T_{\mathfrak{P}}) = \mathsf{H}^1_{\mathrm{ord}}(K_{\mathsf{q}}, T_f) \otimes S_{\mathfrak{P}},$$

so we clearly have local maps with kernel and cokernels bounded as desired (and similarly for $\mathbf{W}_f$ and $W_{\mathfrak{P}}$). So suppose that $\mathsf{v} = \underline{v}$ with $v|N$ and $v = v^\tau$. Then φ is trivial on G_v, so

$$H^1(K_v, \mathbf{T}_f) = H^1(K_v, T_f) \otimes \Lambda, \;\; H^1(K_v, T_{\mathfrak{P}}) = H^1(K_v, T_f) \otimes S_{\mathfrak{P}}.$$

In particular, $H^1(K_v, T_{\mathfrak{P}})$ is finite, so

$$\mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}}(K_{\mathsf{v}}, T_{\mathfrak{P}}) = \mathsf{H}^1(K_{\mathsf{v}}, T_{\mathfrak{P}}).$$

Hence the desired local map $\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_{\mathsf{v}}, \mathbf{T}_f) \otimes \Lambda/\mathfrak{P} \to \mathsf{H}^1_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(K_{\mathsf{v}}, T_{\mathfrak{P}})$ is well-defined and injective, and its cokernel is identified with $H^1(K_v, T_f) \otimes S_{\mathfrak{P}}/(\Lambda/\mathfrak{P})$, which clearly has a bound of the desired sort. The argument for the second map in the claim is similar. □

3.4.6. Recall that, for any finitely generated Λ-module M, there exists a unique Λ-module M' of the form $\Lambda^r \oplus \bigoplus \Lambda/\mathfrak{P}_i^{e_i}$ such that M admits a map to M' with finite kernel and cokernel, where $\mathfrak{P}_i$ are height-one primes; we denote this relationship by $M \sim M'$. The characteristic ideal $\mathrm{char}_\Lambda(M)$ is zero if $r \geq 1$, and equal to $\prod \mathfrak{P}_i^{e_i}$ otherwise. The following easy lemma is implicit in [49, p. 66].

Lemma 3.4.7. *Let $\mathfrak{P} \subset \Lambda$ be a height-one prime. Then there exists an integer d and a sequence of distinct height-one primes $\mathfrak{P}_m$ such that, for all finitely generated torsion Λ-modules M,*

$$\mathrm{lg}_{\mathcal{O}}(M/\mathfrak{P}_m) = md\, \mathrm{ord}_{\mathfrak{P}}\, \mathrm{char}_\Lambda(M) + O(1)$$

as m varies (holding M fixed). Moreover $[S_{\mathfrak{P}_m} : \Lambda/\mathfrak{P}_m]$ is constant for large enough m, and if $\mathfrak{P} \neq (\wp)$, then the rings $\Lambda/\mathfrak{P}_m$ are abstractly isomorphic.

Proof. If $\mathfrak{P} \neq (\wp)$ is generated by a distinguished polynomial $f \in \Lambda$, and π is a uniformizer for $\mathcal{O}$, then we may take $\mathfrak{P}_m = f + \pi^m$ (for sufficiently large m) and $d = [S_{\mathfrak{P}} : \mathcal{O}]$. If $\mathfrak{P} = (\wp)$, then we may take $\mathfrak{P}_m = T^m + \pi$ and $d = 1$. □

Proposition 3.4.8. *For all* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$*, we have*

$$\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)^\vee \sim \Lambda^{r_{\mathsf{Q}}} \oplus M_{\mathsf{Q}} \oplus M_{\mathsf{Q}}$$

for some torsion Λ*-module* M_{Q}*, where*

$$r_{\mathsf{Q}} = \mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f).$$

Here, for a topological $\mathcal{O}$-module M, $M^\vee$ denotes the Pontryagin dual.

Proof. This follows from Propositions 3.3.6 and 3.4.5 exactly as in the proof of [37, Theorem 2.2.10(b)]. □

Theorem 3.4.9. *Suppose that* $\{\boldsymbol{\kappa}, \boldsymbol{\lambda}\}$ *is a nontrivial bipartite Euler system with parity* δ *for the triple* $(\mathbf{T}_f, \mathcal{F}_\Lambda, \mathsf{S})$*. Then there exists a nonzero fractional ideal* $I \subset \Lambda \otimes \mathbb{Q}_p$ *such that:*

(1) *For all* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^\delta$*,* r_{Q} *is odd,* $r_{\mathsf{Q}} = 1$ *if and only if* $\boldsymbol{\kappa}(\mathsf{Q}) \neq 0$*, and in that case*

$$\mathrm{char}_\Lambda(M_{\mathsf{Q}}) \cdot I = \mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f)}{(\boldsymbol{\kappa}(\mathsf{Q}))}\right).$$

(2) *For all* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$*,* r_{Q} *is even,* $r_{\mathsf{Q}} = 0$ *if and only if* $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$*, and in that case*

$$\mathrm{char}_\Lambda(M_{\mathsf{Q}}) \cdot I = (\boldsymbol{\lambda}(\mathsf{Q})).$$

In particular,

$$\delta = \mathrm{rk}_R \mathrm{Sel}_{\mathcal{F}}(\mathbf{T}_f) + 1 \pmod 2.$$

If (sclr) holds, then $I \subset \Lambda$*.*

Proof. Let $\mathfrak{P} \subset \Lambda$ be any height-one prime other than $(\wp)$; via the natural maps $\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T}_f) \to \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}}(T_{\mathfrak{P}})$ and $\Lambda \to S_{\mathfrak{P}}$, the Euler system $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ defines an Euler system $(\kappa_{\mathfrak{P}}, \lambda_{\mathfrak{P}})$ of parity δ for the triple $(T_{\mathfrak{P}}, \mathcal{F}_{\mathfrak{P}}, \mathsf{S})$. In particular, Theorem 3.3.14 applies.

By Lemmas 2.4.12 and 3.3.4, $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f)$ is Λ-torsion-free. Hence for fixed Q, by Proposition 3.4.5, $\boldsymbol{\kappa}(\mathsf{Q}) \neq 0$ if and only if $\kappa_{\mathfrak{P}}(\mathsf{Q}) \neq 0$ for all but finitely many $\mathfrak{P}$. Similarly, $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$ if and only if $\lambda_{\mathfrak{P}}(\mathsf{Q}) \neq 0$ for all but finitely many $\mathfrak{P}$. Because

$$\mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = \mathrm{rk}_{S_{\mathfrak{P}}} \mathrm{Sel}_{\mathcal{F}_{\mathfrak{P}}(\mathsf{Q})}(T_{\mathfrak{P}})$$

for all but finitely many $\mathfrak{P}$ by Proposition 3.4.5, the claims about r_{Q} follow from Theorem 3.3.14.

For any $\mathfrak{P}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\delta+1}$ such that $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$, by Proposition 3.4.5 and Lemma 3.4.7 we have

$$\begin{aligned} e_{\mathfrak{P}}(\mathsf{Q}) &\coloneqq \mathrm{ord}_{\mathfrak{P}}(\boldsymbol{\lambda}(\mathsf{Q})) - \mathrm{ord}_{\mathfrak{P}} \mathrm{char}_\Lambda(M_{\mathsf{Q}}) \\ &= \lim_{m\to\infty} \frac{\mathrm{lg}_{\mathcal{O}}(S_{\mathfrak{P}_m}/\lambda_{\mathfrak{P}_m}(\mathsf{Q})) - \mathrm{lg}_{\mathcal{O}} M_{\mathsf{Q},\mathfrak{P}_m}}{md}. \end{aligned}$$

Applying Theorem 3.3.14, this quantity does not depend on $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ (as long as $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$); it is also clearly zero for almost all $\mathfrak{P}$, so that $\prod_{\mathfrak{P}} \mathfrak{P}^{e_{\mathfrak{P}}}$ defines a fractional ideal I of Λ satisfying (2). The same calculation shows that I satisfies (1) as well. Then Proposition 3.3.15 shows that $e_{\mathfrak{P}} \geq 0$ if $\mathfrak{P} \neq (\wp)$, and also for $\mathfrak{P} = (\wp)$ under (sclr), which shows the desired integrality properties for I and completes the proof. □

4. Geometry of modular Jacobians

4.1. **Purely toric reduction of semistable abelian varieties.**

4.1.1. Fix a prime ℓ, and let A be an abelian variety over $\mathbb{Q}_\ell$ whose Néron model $\mathcal{A}_{/\mathbb{Z}_\ell}$ is semistable with purely toric reduction. We denote by $A_{\mathbb{F}_\ell}$ the special fiber, by $A^0_{\mathbb{F}_\ell}$ the neutral connected component, and by $\mathcal{X}_\ell(A)$ the character group of $A^0_{\mathbb{F}_\ell}$, which is a free $\mathbb{Z}$-module of finite rank with an action of $\mathrm{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$. Also let $\mathcal{X}_\ell(A)^\vee = \mathrm{Hom}(\mathcal{X}_\ell(A), \mathbb{Z})$ be the $\mathbb{Z}$-dual. We recall the following basic result from the theory of rigid analytic uniformizations.

Proposition 4.1.2. (1) *With notation as above, there is a canonical* $G_{\mathbb{Q}_\ell}$*-equivariant exact sequence:*

$$0 \to \mathcal{X}_\ell(A^\vee) \to \mathcal{X}_\ell(A)^\vee \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell^\times \to A(\overline{\mathbb{Q}}_\ell) \to 0.$$

(2) *Suppose that $A^0_{\mathbb{F}_\ell}$ splits over $\mathbb{F}_{\ell^n}$. Taking* $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_{\ell^n})$*-invariants in (1) gives an exact sequence fitting into a commutative diagram of* $\mathrm{Gal}(\mathbb{Q}_{\ell^n}/\mathbb{Q}_\ell)$*-modules:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}_\ell(A^\vee) & \longrightarrow & \mathcal{X}_\ell(A)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell^n}^\times & \longrightarrow & A(\mathbb{Q}_{\ell^n}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{ord}_\ell} & & \downarrow{\scriptstyle \mathrm{sp}} & & \\ 0 & \longrightarrow & \mathcal{X}_\ell(A^\vee) & \xrightarrow{j} & \mathcal{X}_\ell(A)^\vee & \longrightarrow & \Phi_\ell(A) & \longrightarrow & 0. \end{array}$$

Here, j is the monodromy pairing of [34, §9]*, and $\Phi_\ell(A)$ is the component group of $A_{\mathbb{F}_\ell}$.*

(3) *For any $p \neq \ell$, there is a canonical exact sequence of $G_{\mathbb{Q}_\ell}$-modules:*

$$0 \to \mathcal{X}_\ell(A)^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_p(1) \to T_pA \to \mathcal{X}_\ell(A^\vee) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 0.$$

Proof. Let T be the algebraic torus over $\mathbb{Z}_\ell$ with character group $\mathcal{X}_\ell(A)$; let $T_{\mathbb{F}_\ell} \subset T$ be the special fiber and $\widehat{T}$ the formal completion of T along $T_{\mathbb{F}_\ell}$. By [33, Theorem 3.6], the isomorphism $T_{\mathbb{F}_\ell} \cong A^0_{\mathbb{F}_\ell}$ lifts uniquely to an isomorphism between $\widehat{T}$ and the formal completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ along $A^0_{\mathbb{F}_\ell}$. Then by [5, Theorem 1.2], this isomorphism $\widehat{T} \cong \widehat{\mathcal{A}}$ extends uniquely to a surjection of rigid analytic groups $T^{\mathrm{rig}} \to A^{\mathrm{rig}}$, whose kernel M is a lattice in T^{rig}. By [5, Theorem 2.1] and étale descent, the rigid analytic torus with character group M uniformizes $(A^\vee)^{\mathrm{rig}}$; hence by [5, Proposition 6.10], M is canonically identified with $\chi_\ell(A^\vee)$, and so we have an exact sequence of rigid analytic groups over $\mathbb{Q}_\ell$:

$$0 \to \chi_\ell(A^\vee) \to T^{\mathrm{rig}} \to A^{\mathrm{rig}} \to 0.$$

Taking $\overline{\mathbb{Q}}_\ell$-points gives the exact sequence in (1). For (2), the top row of the diagram is exact since $H^1(\mathbb{Q}_{\ell^n}, \mathcal{X}_\ell(A^\vee)) = \mathrm{Hom}(G_{\mathbb{Q}_{\ell^n}}, \mathcal{X}_\ell(A^\vee)) = 0$. The commutativity of the leftmost square is [20, Theorem 2.1], and to establish the commutativity on the right it suffices to show that the image of $\mathcal{X}_\ell(A)^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell^n}^\times$ in $A(\mathbb{Q}_{\ell^n}) = \mathcal{A}(\mathbb{Z}_{\ell^n})$ is exactly those points that reduce to the neutral connected component of $A^0_{\mathbb{F}_\ell}$. But this is clear because $\mathcal{X}_\ell(A)^\vee \otimes \mathbb{Z}_{\ell^n}^\times = \widehat{T}(\mathbb{Z}_{\ell^n})$ maps isomorphically to $\widehat{\mathcal{A}}(\mathbb{Z}_{\ell^n})$ by construction, and by definition $\widehat{\mathcal{A}}(\mathbb{Z}_{\ell^n}) \subset \mathcal{A}(\mathbb{Z}_{\ell^n})$ is the set of points that reduce to $A^0_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^n})$.

For (3), apply the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}_\ell(A^\vee) & \longrightarrow & \mathcal{X}_\ell(A)^\vee \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell^\times & \longrightarrow & A(\overline{\mathbb{Q}}_\ell) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \\ 0 & \longrightarrow & \mathcal{X}_\ell(A^\vee) & \longrightarrow & \mathcal{X}_\ell(A)^\vee \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell^\times & \longrightarrow & A(\overline{\mathbb{Q}}_\ell) & \longrightarrow & 0 \end{array}$$

and take inverse limits in n. □

4.2. **Multiplicity one.**

4.2.1. Let N_1 and N_2 be coprime positive integers, with N_2 squarefree. Consider the Hecke algebra $\mathbb{T} = \mathbb{T}_{N_1,N_2}$ generated over $\mathbb{Z}$ by operators T_ℓ for all primes $\ell \nmid N = N_1N_2$ and U_ℓ for all $\ell | N$, acting on the cusp forms of weight two and level $\Gamma_0(N)$ which are new at all factors $\ell | N_2$. If I is the kernel of the projection $\mathbb{T}_{N,1} \to \mathbb{T}$, then we set

$$J_{\min}^{N_1,N_2} \coloneqq J_0(N)/IJ_0(N), \tag{45}$$

an abelian variety over $\mathbb{Q}$ with a (faithful) action of $\mathbb{T}$. If N_1, N_2 are clear from context, we will omit the superscript.

4.2.2. For any maximal ideal $\mathfrak{m} \subset \mathbb{T}$ and any $\mathbb{T}$-module M, let $M_{\mathfrak{m}}$ denote the $\mathfrak{m}$-adic completion. If A is an abelian variety with an action of $\mathbb{T}$, the $\mathfrak{m}$-adic Tate module is $T_{\mathfrak{m}}A \coloneqq (T_pA)_{\mathfrak{m}}$, where p is the residue characteristic of $\mathfrak{m}$.

Proposition 4.2.3. *Suppose $\mathfrak{m}$ is non-Eisenstein with residue characteristic $p \nmid 2N$. Then $T_{\mathfrak{m}}J_{\min}$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$.*

Proof. Throughout the proof, we will also view $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ as a maximal ideal of $\mathbb{T}_{N,1}$ via pullback. By [63, Theorem 5.2(b)], $J_0(N)[\mathfrak{m}]$ is a two-dimensional vector space over $\mathbb{T}_{N,1}/\mathfrak{m}$. Now, by [35, Proposition 2.2], the dual map to the projection $\phi : J_0(N) \to J_{\min}$ identifies $J_{\min}^\vee$ with the neutral connected component of $J_0(N)[I]$ (notation as in (45)); in particular, $\phi^\vee$ is injective, so $T_\mathfrak{m} J_0(N) \to T_\mathfrak{m} J_{\min}$ is surjective. Hence $J_{\min}[\mathfrak{m}]$ is a $\mathbb{T}_{N_1,N_2}/\mathfrak{m}$-vector space of dimension at most two (in particular, exactly two because $\mathfrak{m}$ is non-Eisenstein), so the proposition follows as in [74, Corollary 2, p.332]. □

4.2.4. Now suppose that A is an abelian variety over $\mathbb{Q}$ with faithful $\mathbb{T}$-action, admitting a $\mathbb{T}$-equivariant isogeny to $J_{\min}$. Then the Néron model of A over $\mathbb{Z}_\ell$ has purely toric reduction for all $\ell | N_2$, and we will apply the notations of §4.1.

Proposition 4.2.5 (Helm). *Let $\mathfrak{m} \subset \mathbb{T}$ be non-Eisenstein of residue characteristic $p \nmid 2N$. Then the natural maps induce $\mathbb{T}_\mathfrak{m}$-module isomorphisms:*

$$T_\mathfrak{m} J_{\min} \otimes_{\mathbb{T}_\mathfrak{m}} \mathrm{Hom}(J_{\min}, A)_\mathfrak{m} \xrightarrow{\sim} T_\mathfrak{m} A,$$

$$\mathcal{X}_\ell(J_{\min}^\vee)_\mathfrak{m} \otimes_{\mathbb{T}_\mathfrak{m}} \mathrm{Hom}(J_{\min}, A)_\mathfrak{m} \xrightarrow{\sim} \mathcal{X}_\ell(A^\vee)_\mathfrak{m}, \ \ \forall \ell | N_2.$$

Here, all Hom-sets are understood to be $\mathbb{T}$-equivariant morphisms, and tensor products are taken modulo $\mathbb{Z}$-torsion.

Proof. This follows from [35, Corollary 4.10, Proposition 4.14]. Note that [35] uses contravariant Tate modules, so it is necessary to dualize to recover the covariant formulation. □

We record the following elementary lemma for later use.

Lemma 4.2.6. *Let $\mathcal{X} = \mathcal{X}_\ell(J_{\min}^\vee)_\mathfrak{m}$ for some $\ell | N_2$ and $\mathfrak{m} \subset \mathbb{T}$, where $\mathfrak{m}$ is non-Eisenstein of residue characteristic $p \nmid 2N$. If the associated residual representation $\overline{\rho}_\mathfrak{m}$ is ramified at ℓ or if $p \nmid \ell - 1$, then $\mathcal{X}$ is free of rank one over $\mathbb{T}_\mathfrak{m}$. In general, there exist $\mathbb{T}_\mathfrak{m}$-module maps*

$$\phi_i : \mathcal{X} \to \mathbb{T}_\mathfrak{m}, \ \ \psi_i : \mathbb{T}_\mathfrak{m} \to \mathcal{X}, \ \ i = 1, 2$$

such that

$$\phi_i \circ \psi_i = \psi_i \circ \phi_i = t_i \in \mathbb{T}_\mathfrak{m} \subset \mathrm{End}(\mathcal{X})$$

and

$$t_1 + t_2 = \ell - 1 \in \mathbb{T}_\mathfrak{m}.$$

Proof. If $\ell - 1$ is a p-adic unit, or if $\overline{\rho}_\mathfrak{m}$ is ramified at ℓ, then this follows from [35, Lemma 6.5]. In general, abbreviate $\mathcal{X}^\vee = \mathrm{Hom}_\mathbb{Z}(\mathcal{X}_\ell(J_{\min}^\vee), \mathbb{Z})_\mathfrak{m}$. By Proposition 4.1.2(3), we have an exact sequence of $\mathbb{T}_\mathfrak{m}[G_{\mathbb{Q}_\ell}]$-modules

$$0 \to \mathcal{X}^\vee(1) \to T_\mathfrak{m} J_{\min} \xrightarrow{\pi} \mathcal{X} \to 0; \tag{46}$$

the action of $G_{\mathbb{Q}_\ell}$ on $\mathcal{X}$ is unramified and Frobenius acts as U_ℓ [63, Proposition 3.8], which is a constant ± 1 because the residue characteristic of $\mathfrak{m}$ is $p > 2$.

By Proposition 4.2.3, $T_\mathfrak{m} J_{\min}$ is free of rank two over $\mathbb{T}_\mathfrak{m}$. Choose a basis $\{e_1, e_2\}$ for $T_\mathfrak{m} J_{\min}$, which identifies $\wedge^2 T_\mathfrak{m} J_{\min} \cong \mathbb{T}_\mathfrak{m}$ by the generator $e_1 \wedge e_2$. Let

$$\langle \cdot, \cdot \rangle : T_\mathfrak{m} J_{\min} \times T_\mathfrak{m} J_{\min} \to \wedge^2 T_\mathfrak{m} J_{\min} \cong \mathbb{T}_\mathfrak{m}$$

be the resulting alternating pairing, so that

$$y = \langle e_1, y \rangle e_2 - \langle e_2, y \rangle e_1 \tag{47}$$

for all $y \in T_\mathfrak{m} J_{\min}$. Define maps

$$\widetilde{\phi}_i : T_\mathfrak{m} J_{\min} \to \mathbb{T}_\mathfrak{m}, \ \ i = 1, 2$$
$$\widetilde{\phi}_1 : y \mapsto \langle y, (F - U_\ell) e_2 \rangle$$
$$\widetilde{\phi}_2 : y \mapsto \langle y, (F - U_\ell) e_1 \rangle,$$

where $F \in G_{\mathbb{Q}_\ell}$ is any lift of Frobenius. We first claim that the maps $\widetilde{\phi}_i$ factor through π. Since $\mathbb{T}_\mathfrak{m}$ is p-torsion-free, it suffices to check this after inverting p. It follows from (46) that, on $T_\mathfrak{m} J_{\min} \otimes \mathbb{Q}_p$, F acts semisimply with distinct eigenvalues U_ℓ and ℓU_ℓ, and each eigenspace is isotropic for $\langle \cdot, \cdot \rangle$ because the action of $G_{\mathbb{Q}_\ell}$ on $\wedge^2 T_\mathfrak{m} J_{\min}$ is through the cyclotomic character. Since $(F - U_\ell) y$ lies in the ℓU_ℓ-eigenspace of F for all $y \in T_{\mathfrak{m}} J_{\min} \otimes \mathbb{Q}_p$, the maps $\widetilde{\phi}_i \otimes \mathbb{Q}_p$ factor through the projection onto the U_ℓ-eigenspace, hence through $\pi \otimes \mathbb{Q}_p$ by the exact sequence (46). So indeed each $\widetilde{\phi}_i$ descends to a $\mathbb{T}_{\mathfrak{m}}$-module map $\phi_i : \mathcal{X} \to \mathbb{T}_{\mathfrak{m}}$. Now define maps

$$\begin{aligned} \psi_i &: \mathbb{T}_{\mathfrak{m}} \to \mathcal{X}, \quad i = 1, 2 \\ \psi_1 &: 1 \mapsto U_\ell \pi(e_1) \\ \psi_2 &: 1 \mapsto -U_\ell \pi(e_2). \end{aligned}$$

We claim that ψ_i and ϕ_i satisfy the conclusion of the lemma. One readily calculates:

$$\begin{aligned} \phi_1 \circ \psi_1(1) &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \\ \psi_1 \circ \phi_1(\pi(e_1)) &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \pi(e_1) \\ \psi_1 \circ \phi_1(\pi(e_2)) &= U_\ell \langle e_2, (F - U_\ell) e_2 \rangle \pi(e_1) \\ &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \pi(e_2) - U_\ell (F - U_\ell) \pi(e_2) \\ &= U_\ell \langle e_1, (F - U_\ell) e_2 \rangle \pi(e_2), \end{aligned}$$

where in the last two steps we have used (47) and the fact that $F = U_\ell$ on $\mathcal{X}$. Similarly,

$$\phi_2 \circ \psi_2 = \psi_2 \circ \phi_2 = -U_\ell \langle e_2, (F - U_\ell) e_1 \rangle,$$

and

$$U_\ell \langle e_1, (F - U_\ell) e_2 \rangle - U_\ell \langle e_2, (F - U_\ell) e_1 \rangle = \operatorname{tr}_{T_{\mathfrak{m}} J_{\min}} U_\ell (F - U_\ell) = \ell - 1.$$

□

4.3. **Shimura curves.**

4.3.1. If $\nu(N_2)$ is even, then fix a maximal order $\mathcal{O}_B$ of B and let $X_{N_1,N_2,\mathbb{Z}[\frac{1}{N}]}$ be the smooth projective Shimura curve over $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ described in [3, §5.1]. In particular, for $N_2 > 1$ and for any field k of residue characteristic not dividing N, the points $X_{N_1,N_2}(k) := X_{N_1,N_2,\mathbb{Z}[\frac{1}{N}]}(k)$ parametrize isomorphism classes of triples (A, ι, C), where A is an abelian surface over k, ι is an embedding $\mathcal{O}_B \hookrightarrow \operatorname{End}_k(A)$, and $C \subset A[N_1]$ is a sub-group scheme of order N_1^2 which is stable and cyclic for the action of $\mathcal{O}_B$. For $N_2 = 1$, $X_{N,1,\mathbb{Z}[\frac{1}{N}]}$ is the usual modular curve $X_0(N)$, and the preceding moduli interpretation applies to the open modular curve $Y_0(N) \subset X_0(N)$. We write $X_{N_1,N_2} := X_{N_1,N_2,\mathbb{Q}}$ for the generic fiber.

4.3.2. For all $\ell \nmid N_2$, we have the usual Hecke correspondences T_ℓ ($\ell \nmid N$) or U_ℓ ($\ell | N_1$) on $X_{N_1,N_2,\mathbb{Z}[\frac{1}{N}]} \times X_{N_1,N_2,\mathbb{Z}[\frac{1}{N}]}$; for $\ell | N_2$, we have the involution U_ℓ of $X_{N_1,N_2,\mathbb{Z}[\frac{1}{N}]}$, whose action on the complex fiber is given by the double coset operator of [18, p. 873]. By [35, Theorem 2.3], these correspondences induce, by Picard functoriality, a faithful action of $\mathbb{T}_{N_1,N_2}$ on the Jacobian $J^{N_1,N_2} := \operatorname{Jac}(X_{N_1,N_2})$. When N_1 and N_2 are understood, we abbreviate $J = J^{N_1,N_2}$. By [35, Corollary 2.4], there is a noncanonical Hecke-equivariant isogeny $J \to J_{\min}$.

4.3.3. If $q \nmid N$ is a prime, then we have the two natural degeneracy maps $\delta_{q,1}, \delta_{q,2} : X_{N_1 q, N_2} \to X_{N_1,N_2}$ defined on the level of the moduli problems by by $\delta_{q,1}(A, \iota, C) = (A, \iota, C[N_1])$ and $\delta_{q,2}(A, \iota, C) = (A/C[q], \iota, C/C[q])$. Consider the following Taylor-Wiles hypothesis on the residual representation $\overline{\rho}_{\mathfrak{m}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}/\mathfrak{m})$ associated to $\mathfrak{m}$:

$$\text{(TW)} \qquad \text{if } p = 3, \text{ then } \overline{\rho}_{\mathfrak{m}} \text{ is absolutely irreducible over } \mathbb{Q}(\sqrt{-3}).$$

If $\mathbb{T}^{(q)}_{N_1 q, N_2}$ is the subalgebra of $\mathbb{T}_{N_1 q, N_2}$ generated by all Hecke operators except for U_q, then a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ also defines a maximal ideal $\mathfrak{m}^{(q)}$ of $\mathbb{T}^{(q)}_{N_1,N_2}$ by pullback along the projection $\mathbb{T}^{(q)}_{N_1 q,N_2} \to \mathbb{T}_{N_1,N_2}$.

Proposition 4.3.4 (Ihara's Lemma). *Let $q \nmid N$ be any prime, and let $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying (TW). Then the natural map*

$$\delta_{q,1}^* + \delta_{q,2}^* : J^{N_1,N_2}[\mathfrak{m}]^{\oplus 2} \to J^{N_1 q, N_2}[\mathfrak{m}^{(q)}]$$

is injective.

Proof. Let $\ell \nmid 6Npq$ be a prime such that $p \nmid \ell - 1$ and $T_\ell^2 - (\ell+1)^2 \notin \mathfrak{m}$; infinitely many such primes exist by [26, Lemma 2] and (TW). Let $X_{N_1,N_2}(\ell)$ be the Shimura curve defined analogously to X_{N_1,N_2} but with an additional with $\Gamma_1(\ell)$ level structure, and similarly for $X_{N_1q,N_2}(\ell)$; let $J^{N_1,N_2}(\ell)$ and $J^{N_1q,N_2}(\ell)$ be the corresponding Jacobians. The forgetful projections $X_{N_1,N_2}(\ell) \xrightarrow{\gamma} X_{N_1\ell,N_2} \xrightarrow{\delta_{\ell,1}} X_{N_1,N_2}$ and $X_{N_1q,N_2}(\ell) \to X_{N_1q\ell,N_2} \to X_{N_1q,N_2}$ induce a commutative diagram of degeneracy maps:

$$\begin{array}{ccc} J^{N_1,N_2}[p]^{\oplus 2} & \xrightarrow{\delta_{q,1}^*+\delta_{q,2}^*} & J^{N_1q,N_2}[p] \\ \downarrow{\scriptstyle (\delta_{\ell,1}^*)^{\oplus 2}} & & \downarrow \\ J^{N_1\ell,N_2}[p]^{\oplus 2} & \xrightarrow{\alpha} & J^{N_1q\ell,N_2}[p] \\ \downarrow{\scriptstyle (\gamma^*)^{\oplus 2}} & & \downarrow \\ J^{N_1,N_2}(\ell)[p]^{\oplus 2} & \xrightarrow{\beta} & J^{N_1q,N_2}(\ell)[p]. \end{array}$$

By [27, §3, Theorem 2] (and the end of [26] for the case $p = 3$), every Jordan-Hölder constituent of the $\overline{\mathbb{F}}_p[G_{\mathbb{Q}}]$-module $(\ker\beta) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is one-dimensional. The same is true of $\ker\alpha$ since the map γ^* is injective (as γ has degree $\ell - 1$, which is prime to p). On the other hand, every Jordan-Hölder constituent of $J^{N_1,N_2}[\mathfrak{m}] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is an irreducible two-dimensional representation because $\mathfrak{m}$ is non-Eisenstein; so it suffices to show that $\delta_{\ell,1}^* : J^{N_1,N_2}[\mathfrak{m}] \to J^{N_1\ell,N_2}[\mathfrak{m}^{(\ell)}]$ is injective. Indeed, the composite

$$J^{N_1,N_2}[\mathfrak{m}]^{\oplus 2} \xrightarrow{\delta_{\ell,1}^*\oplus\delta_{\ell,2}^*} J^{N_1\ell,N_2}[\mathfrak{m}^{(\ell)}] \xrightarrow{\delta_{\ell,1,*}\oplus\delta_{\ell,2,*}} J^{N_1,N_2}[\mathfrak{m}]^{\oplus 2}$$

is given by the matrix

$$\begin{pmatrix} \ell+1 & T_\ell \\ T_\ell & \ell+1 \end{pmatrix},$$

cf. [27, p. 447], which is injective because $T_\ell^2 - (\ell+1)^2 \notin \mathfrak{m}$; so a fortiori $\delta_{\ell,1}^* : J^{N_1,N_2}[\mathfrak{m}] \to J^{N_1\ell,N_2}[\mathfrak{m}^{(\ell)}]$ is injective, as desired. □

Theorem 4.3.5 (Helm). *Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying (TW). Then there is an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules:*

$$\mathrm{Hom}(J_{\min}, J)_{\mathfrak{m}} \simeq \bigotimes_{\ell | N_2} \mathcal{X}_\ell(J_{\min}^\vee)_{\mathfrak{m}},$$

modulo $\mathbb{Z}$-torsion on the right-hand side. Here, the tensor products are over $\mathbb{T}_{\mathfrak{m}}$.

Proof. If $p \neq 3$, this is [35, Theorem 8.7]. However, as explained in [35, Remark 8.12], the assumption $p \neq 3$ is used only in the level-raising arguments of [35, §7]: in particular, the application of [27, Theorem A] in the proof of [35, Lemma 7.1], and the application of Ihara's Lemma in the proof of [35, Proposition 7.2]. Under condition (TW), the first result still applies even when $p = 3$ by [26, Theorem 1]; and the necessary case of Ihara's Lemma is given by Proposition 4.3.4 above. □

4.4. **Shimura sets.**

4.4.1. Let $B = B_{N_2}$ be the quaternion algebra over $\mathbb{Q}$ ramified at the prime factors of N_2 (and possibly ∞). Recall from [62, p. 369] that an oriented Eichler order (R, ϕ) of level N_1 in B is an Eichler order R of level N_1 equipped with a local orientation ϕ_ℓ for each $\ell | N$. If $\ell | N_1$, then ϕ_ℓ is the data of a maximal order $R_\ell \subset B \otimes \mathbb{Q}_\ell$ containing $R \otimes \mathbb{Z}_\ell$. If $\ell | N_2$, then ϕ_ℓ is the data of a homomorphism from R to a fixed field $\mathbb{F}_{\ell^2}$ of cardinality ℓ^2. In particular, for a fixed R, there are exactly two choices of local orientation ϕ_ℓ for each $\ell | N$. An isomorphism of oriented Eichler orders (R, ϕ) and (R', ϕ') is an automorphism of B (necessarily inner) carrying R to R', compatibly with the orientations.

4.4.2. If $\nu(N_2)$ is odd, then we define the Shimura set X_{N_1,N_2} to be the set of isomorphism classes of oriented Eichler orders of level N_1 in B. Because $B^\times(\mathbb{A}_f)$ has a natural transitive action on the set of oriented Eichler orders of level N_1, choosing an oriented Eichler order (R,ϕ) as a base point identifies X_{N_1,N_2} with the finite double coset space

$$B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}_f)/\widehat{R}^\times. \tag{48}$$

When N_1 and N_2 are clear from context, the subscripts on X_{N_1,N_2} may be omitted. We will write $\mathbb{Z}[X_{N_1,N_2}]^0$ for the set of $\sum a_i y_i \in \mathbb{Z}[X_{N_1,N_2}]$ with $\sum a_i = 0$.

4.4.3. The set $X \times X$ carries Hecke correspondences T_ℓ ($\ell \nmid N$) and U_ℓ ($\ell | N$), given in the double coset description as in [18, p. 873]. We let these correspondences act on the set $\mathbb{Z}[X]^0$ by Picard functoriality, i.e. by identifying $\mathbb{Z}[X]$ with the set of formal 0-cycles $\mathrm{Hom}_{\mathrm{set}}(X,\mathbb{Z})$. By the explicit description of the Jacquet-Langlands correspondence in [27, p. 459], we obtain in this way a faithful action of $\mathbb{T}_{N_1,N_2}$ on $\mathbb{Z}[X]^0$. The analogue of Theorem 4.3.5 is:

Theorem 4.4.4. *Let $\mathfrak{m} \subset \mathbb{T}$ be a non-Eisenstein maximal ideal of residue characteristic $p \nmid 2N$ satisfying (TW). Then there is an isomorphism of $\mathbb{T}_\mathfrak{m}$-modules:*

$$\mathbb{Z}[X]^0_\mathfrak{m} \simeq \bigotimes_{\ell | N_2} \mathcal{X}_\ell(J^\vee_{\min})_\mathfrak{m},$$

modulo $\mathbb{Z}$-torsion on the right-hand side. Here, the tensor products are taken over $\mathbb{T}_\mathfrak{m}$.

Proof. Choose any prime $q|N_2$, so that $\nu(N_2/q)$ is even. Let $\mathbb{T}' = \mathbb{T}_{N_1q,N_2/q}$, and write $\mathfrak{m}$ as well for the maximal ideal of $\mathbb{T}'$ induced by the map $\mathbb{T}' \to \mathbb{T}$.

Applying Theorem 4.3.5 to the pair $N_1q, N_2/q$, we obtain an isomorphism of $\mathbb{T}'_\mathfrak{m}$-modules (modulo $\mathbb{Z}$-torsion)

$$\mathrm{Hom}(J^{N_1q,N_2/q}_{\min}, J^{N_1q,N_2/q})_\mathfrak{m} \simeq \bigotimes_{\ell | N_2/q} \mathcal{X}_\ell(J^{N_1q,N_2/q,\vee}_{\min}). \tag{49}$$

By [35, Corollary 5.3, Lemma 8.2], this implies an isomorphism of $\mathbb{T}_\mathfrak{m}$-modules

$$\mathrm{Hom}(J^{N_1,N_2}_{\min}, J^{N_1q,N_2/q}_{q\text{-new}})_\mathfrak{m} \simeq \bigotimes_{\ell | N_2/q} \mathcal{X}_\ell(J^{N_1,N_2,\vee}_{\min}), \tag{50}$$

where $J^{N_1q,N_2/q}_{q\text{-new}}$ is the q-new quotient of $J^{N_1q,N_2/q}$ in the sense of [35, p. 66]. Then, by Proposition 4.2.5, we have

$$\mathcal{X}_q(J^{N_1q,N_2/q,\vee}_{q\text{-new}})_\mathfrak{m} \simeq \mathcal{X}_q(J^{N_1,N_2,\vee}_{\min})_\mathfrak{m} \otimes \bigotimes_{\ell | N_2/q} \mathcal{X}_\ell(J^{N_1,N_2,\vee}_{\min})_\mathfrak{m}. \tag{51}$$

By [3, Proposition 5.3], $\mathcal{X}_q(J^{N_1q,N_2/q,\vee})$ is identified with $\mathbb{Z}[X_{N_1,N_2}]^0$. It remains to show that the inclusion $J^{N_1q,N_2/q,\vee}_{q\text{-new}} \hookrightarrow J^{N_1q,N_2/q,\vee}$ induces an isomorphism on character groups at q. Indeed, since the projection $J^{N_1q,N_2/q} \to J^{N_1q,N_2/q}_{q\text{-new}}$ has connected kernel, by [21, Theorem 8.2] the induced map on character groups is surjective:

$$\mathcal{X}_q(J^{N_1q,N_2/q,\vee}) \twoheadrightarrow \mathcal{X}_q(J^{N_1q,N_2/q,\vee}_{q\text{-new}}). \tag{52}$$

After tensoring both sides with $\mathbb{Q}$, (52) is an isomorphism because the q-old isogeny factors of $J^{N_1q,N_2/q,\vee}$ have good reduction at q. Since the source of (52) is a free $\mathbb{Z}$-module, the surjection is an isomorphism. □

4.5. **Special fibers of Shimura curves.**

4.5.1. We again suppose $\nu(N_2)$ is even, and fix an orientation on the maximal order $\mathcal{O}_B \subset B = B_{N_2}$ from (4.3.1). In this subsection, we will recall – following [62] – the geometry of the special fiber of the canonical model of X_{N_1,N_2} over $\mathbb{Z}_q$ in two cases: $q \nmid N$, and $q|N_2$.

Proposition 4.5.2. *Suppose $\nu(N_2)$ is even, and fix a prime $q \nmid N_1N_2$. Then:*

1. *The supersingular locus $X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss} = X_{N_1,N_2}(\mathbb{F}_{q^2})^{ss}$ is canonically identified with X_{N_1,N_2q}. This identification is compatible with all the Hecke correspondences T_ℓ ($\ell \nmid Nq$) and U_ℓ ($\ell | N$); moreover, the action of Frob_q on $X_{N_1,N_2}(\mathbb{F}_{q^2})^{ss}$ coincides with the action of U_q on X_{N_1,N_2q}.*

(2) If $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ *is a non-Eisenstein maximal ideal of residue characteristic* $p \nmid 2Nq$ *satisfying (TW), then the Abel-Jacobi map induces a surjective composite*

$$\underbrace{\mathbb{Z}[X_{N_1,N_2q}]^0 \to J^{N_1,N_2}(\mathbb{F}_{q^2})_{\mathfrak{m}} \to H^1(\mathbb{F}_{q^2}, T_{\mathfrak{m}}J^{N_1,N_2})}_{\varphi} \tag{53}$$

such that $\varphi \circ U_q = \mathrm{Frob}_q \circ \varphi$.

Note in (1) that the correspondence U_q on $X_{N_1,N_2q} \times X_{N_1,N_2q}$ is the graph of the involution that reverses the local orientation at q, so it makes sense to refer to the action of U_q on X_{N_1,N_2q} itself.

Proof. The first part is proven in [62, Theorem 3.4], but we recall the construction for use below in Proposition 4.7.12. If (A,ι,C) is a point of $X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss}$, then $\mathrm{End}^0(A,\iota)$ is isomorphic to B_{N_2q}, and $R \coloneqq \mathrm{End}(A,\iota,C) \subset \mathrm{End}^0(A,\iota)$ is an Eichler order of level N_1. Moreover, R has a natural orientation at all primes dividing $\ell|Nq$, which we now recall. For $\ell|N_1$, the local orientation is determined by the inclusion $R \subset \mathrm{End}(A,\iota)$, where the latter is a maximal order.

For $\ell|N_2$, if $\mathfrak{m}_\ell \subset \mathcal{O}_B$ is the unique maximal ideal of residue characteristic ℓ, then $A[\mathfrak{m}_\ell]$ is a vector space of dimension one over $\mathcal{O}_B/\mathfrak{m}_\ell \simeq \mathbb{F}_{\ell^2}$, where the isomorphism is chosen according to the orientation of $\mathcal{O}_B$. The action of R on $A[\mathfrak{m}_\ell]$ therefore defines a homomorphism $R \to \mathbb{F}_{\ell^2}$, which we take to be the orientation of R at ℓ.

Finally, the Lie algebra of A is a $\overline{\mathbb{F}_{q^2}}$-vector space of dimension 2, on which R acts by scalars valued in $\mathbb{F}_{q^2}$, cf. [62, p. 24]. This defines a map $R \to \mathbb{F}_{q^2}$, which we take to be the local orientation at q. Thus for every $(A,\iota,C) \in X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss}$, we have described an oriented Eichler order of level N_1 in B_{N_2q}, well-defined up to the choice of isomorphism $\mathrm{End}^0(A,\iota) \simeq B_{N_2q}$, i.e. up to $B_{N_2q}^\times(\mathbb{Q})$-conjugacy. This describes a map

$$X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss} \to X_{N_1,N_2q},$$

and [62, Theorem 3.4] shows that this map is an isomorphism.

The Hecke compatibility for operators coprime to q is clear from the construction. We can also see that replacing (A,ι,C) by its Frobenius twist has the effect of switching the orientation of $\mathrm{End}(A,\iota,C)$ at q, which is precisely the action of U_q on X_{N_1,N_2q}. In particular, Frob_q^2 acts trivially on $X_{N_1,N_2}(\overline{\mathbb{F}_{q^2}})^{ss}$, so all the supersingular points are in fact defined over $\mathbb{F}_{q^2}$, which shows (1).

For (2), the identity $\varphi \circ U_q = \mathrm{Frob}_q \circ \varphi$ follows from (1). We claim the surjectivity follows from the proof of [47, Proposition 4.8] (replacing the Shimura curve therein with its compactification when $N_2 = 1$), although *op. cit.* imposes the additional assumptions [47, Assumption 4.1(1)-(5)] on p and $\overline{\rho}_{\mathfrak{m}}$.[2] Indeed, these conditions are used only thrice in the proof of [47, Proposition 4.8]: first, to establish Ihara's Lemma [47, Lemma 4.7], which in our case is Proposition 4.3.4; second, to deduce that $H^0_{\text{ét}}(X_{N_1,N_2,\overline{\mathbb{F}}_q}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} = 0$, which only requires that $\mathfrak{m}$ is non-Eisenstein; and, third, to control the action of the Hecke operator denoted $\mathrm{S}_{\mathfrak{p}}$ in [47, p. 2100], which is unnecessary in our context since all our Shimura curves have $\Gamma_0(N_1)$ level structure.

□

4.5.3. Now suppose instead that $q|N_2$. The Shimura curve X_{N_1,N_2} has a canonical, semistable integral model $X_{N_1,N_2,\mathbb{Z}_q}$ over $\mathrm{Spec}\,\mathbb{Z}_q$. We denote by $X^{\pm}_{N_1,N_2/q}$ the set $X_{N_1,N_2/q} \times \{\pm\}$.

Proposition 4.5.4. *The set of irreducible geometric components of the special fiber* $X_{N_1,N_2,\mathbb{F}_q}$ *of* $X_{N_1,N_2,\mathbb{Z}_q}$ *is canonically identified with* $X^{\pm}_{N_1,N_2/q}$*. Each component is defined over* $\mathbb{F}_{q^2}$*, and the Frobenius action switches the sign without changing the value in* $X_{N_1,N_2/q}$*.*

This identification is compatible with all the Hecke correspondences T_ℓ *(*$\ell \nmid N$*) and* U_ℓ *(*$\ell|N$, $\ell \neq q$*); moreover, the induced action of* U_q *on* $\mathbb{Z}[X^{\pm}_{N_1,N_2/q}] \cong \mathbb{Z}[X_{N_1,N_2/q}]^2$ *is given by the matrix*

$$\begin{pmatrix} T_q & -1 \\ q & 0 \end{pmatrix}.$$

[2] In [47, Definition 4.5], it is also assumed that $q^2 \not\equiv 1 \pmod{p}$, but this is not used in the proof of Proposition 4.8 of *op. cit.*

Proof. This follows from [62, Theorem 5.4], but once again we recall the construction for use in Proposition 4.7.13 below. First, [62, Theorem 5.3] identifies the set of irreducible components with the set of so-called pure triples (A, ι, C), where A is a superspecial abelian surface over $\mathbb{F}_{q^2}$ with an embedding $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ and C is a $\Gamma_0(N_1)$-level structure. The purity condition means that $\mathcal{O}_B$ acts on the 2-dimensional $\mathbb{F}_{q^2}$-vector space $\mathrm{Lie}\, A$ via scalars, i.e. through a homomorphism $\mathcal{O}_B \to \mathbb{F}_{q^2}$. Since $\mathcal{O}_B$ is given with an orientation at q, we say a pure triple is of type $+$ if this homomorphism agrees with the orientation, and of type $-$ otherwise.

Now for any pure triple (A, ι, C), $\mathrm{End}^0(A, \iota)$ is isomorphic to $B_{N_2/q}$, and $\mathrm{End}(A, \iota, C)$ is canonically an oriented Eichler order of level N_1 in $\mathrm{End}^0(A, \iota)$. (The orientation is defined as in the proof of Proposition 4.5.2.) Thus we have a well-defined map from the set of irreducible components to $X^{\pm}_{N_1,N_2/q}$, sending a pure triple (A, ι, C) of type $\delta \in \{\pm\}$ to $(\mathrm{End}(A, \iota, C), \delta)$. This map is an isomorphism by [62, Theorem 4.13], and the Frobenius action is described in the following remark of *op. cit.* The Hecke equivariance is clear away from q, and the action of U_q is described in [3, Proposition 5.8(2)]. □

4.5.5. Let Φ be the component group of the special fiber of the Néron model of J^{N_1,N_2} over $\mathbb{Z}_q$.

Proposition 4.5.6. *Suppose $\mathfrak{m} \subset \mathbb{T}_{N_1 q, N_2}$ is a non-Eisenstein maximal ideal. Then we have an isomorphism*

$$\frac{\mathbb{Z}[X^{\pm}_{N_1,N_2/q}]^0_{\mathfrak{m}}}{(U_q^2 - 1)} \xrightarrow{\sim} \Phi_{\mathfrak{m}} \tag{54}$$

with the following property: for all degree-zero divisors $y = \sum a_i y_i$ on $X_{N_1,N_2/q,\mathbb{Q}_q}$ such that each y_i lies in $X_{N_1,N_2/q}(\mathbb{Q}_{q^2})$ and reduces to a smooth point of $X_{N_1,N_2/q,\mathbb{F}_{q^2}}$ lying on the component $\overline{y}_i$, (54) sends the class of $\sum a_i \overline{y}_i$ to the image in $\Phi_{\mathfrak{m}}$ of $[y] \in J^{N_1,N_2/q}(\mathbb{Q}_{q^2})$.

Proof. This is [3, Propositions 5.13, 5.14]. □

4.6. **Geometric level raising.**

4.6.1. Let f, N, $\wp$, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5. We also fix a factorization $N = N_1 N_2$, where N_1 and N_2 are coprime, and N_2 is squarefree. If Q and Q' are coprime squarefree positive integers, then we abbreviate $\mathbb{T}^Q_{Q'} = \mathbb{T}_{N_1 Q, N_2 Q'}$, omitting any superscript or subscript which is equal to 1. From now on, we will abbreviate $T_j := T_f/\pi^j$ and $\mathcal{O}_j := \mathcal{O}/\pi^j$ for any integer $j \geq 1$.

Definition 4.6.2. We say a prime $q \nmid 2pN$ is ***j*-admissible** with sign $\epsilon_q = \pm 1$ if $a_q(f) \equiv \epsilon_q(q+1) \pmod{\pi^j}$ and $q \not\equiv 1 \pmod{p}$. In this case, T_j has a unique subspace $\mathrm{Fil}^+_{q,\epsilon_q} T_j$, free of rank one over $\mathcal{O}_j$, on which Frob_q acts as $q\epsilon_q$. We will omit the subscript ϵ_q when there is no risk of confusion. We say q is **weakly admissible** with sign ϵ_q if it is j-admissible with sign ϵ_q for some $j \geq 1$. A **weakly admissible pair** $\{Q, \epsilon_Q\}$ is an ordered pair of a squarefree number Q and a function $\epsilon_Q : \{q|Q\} \to \{\pm 1\}$ such that q is weakly admissible with sign $\epsilon_Q(q)$ for all $q|Q$. If $\{Q, \epsilon_Q\}$ is a weakly admissible pair, then for all $q|Q$, there is a unique root $u_q \in \mathcal{O}$ of the polynomial $y^2 - y a_q(f) + q$ such that $u_q \equiv \epsilon_Q(q) \pmod{\wp}$. We view $\mathcal{O}$ as a $\mathbb{T}^Q$-algebra by letting U_q act through u_q, and letting the other Hecke operators act through their eigenvalues on f; let $\mathfrak{m}_Q^{\epsilon_Q}$ be the associated maximal ideal (we will usually drop the superscript). Finally, a weakly admissible pair $\{Q, \epsilon_Q\}$ is called ***j*-level-raising** if

$$\mathrm{lg}_{\mathcal{O}} \left(\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \right) \geq j.$$

Remark 4.6.3. If $\{Q, \epsilon_Q\}$ is j-level-raising, then each $q|Q$ is j-admissible with sign $\epsilon_Q(q)$. Indeed, in $\mathbb{T}_Q$ we have $U_q^2 = 1$ for each $q|Q$, but U_q acts on $\mathcal{O}$ by the unique root of $y^2 - y a_q(f) + q$ congruent to $\epsilon_Q(q)$; hence $a_q(f) = \epsilon_Q(q)(q+1)$ in $\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$.

4.6.4. In light of the structural similarity of Theorems 4.3.5 and 4.4.4, let

$$M_Q = \begin{cases} \mathrm{Hom}(J^{N_1,N_2 Q}_{\min}, J^{N_1,N_2 Q}), & \nu(N_2 Q) \text{ even}, \\ \mathbb{Z}[X_{N_1,N_2 Q}]^0, & \nu(N_2 Q) \text{ odd}. \end{cases} \tag{55}$$

Lemma 4.6.5. *Suppose $\{Q, \epsilon_Q\}$ is a weakly admissible pair, and let*

$$C = \sum_{\substack{\ell | N_2 \\ \overline{T}_f \text{ unram at } \ell}} \mathrm{ord}_\pi(\ell - 1).$$

Then there exists an $\mathcal{O}$-module map

$$M_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$$

with kernel and cokernel annihilated by π^C; in particular, $\pi^C(M_Q \otimes_{\mathbb{T}^Q} \mathcal{O})$ is cyclic of length at least $\lg_{\mathcal{O}}(\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}) - 2C$.

Proof. We may assume that $\mathfrak{m}_Q \subset \mathbb{T}^Q$ descends to $\mathbb{T}_Q$. Now, by Theorem 4.3.5 or Theorem 4.4.4 (depending on $\nu(N_2 Q)$), we have

$$M_{Q,\mathfrak{m}_Q} \simeq \otimes_{\ell | N_2 Q} \mathcal{X}_\ell \left(J_{\min}^{N_1, N_2 Q, \vee} \right)_{\mathfrak{m}_Q},$$

modulo $\mathbb{Z}$-torsion on the right. Lemma 4.2.6 implies that there exists a collection of $\mathbb{T}_Q$-module maps

$$\phi_i : M_{Q,\mathfrak{m}_Q} \to \mathbb{T}_{Q,\mathfrak{m}_Q}, \quad \psi_i : \mathbb{T}_{Q,\mathfrak{m}_Q} \to M_{Q,\mathfrak{m}_Q}, \quad i = 1, \ldots, 2^{\nu(N_2)}$$

such that

$$\phi_i \circ \psi_i = \psi_i \circ \phi_i = t_i \in \mathbb{T}_{Q,\mathfrak{m}_Q} \subset \mathrm{End}(M_{Q,\mathfrak{m}_Q})$$

and

$$t_1 + \ldots + t_{2^{\nu(N_2)}} = \prod_{\substack{\ell | N_2 \\ \overline{T}_f \text{ unram at } \ell}} (\ell - 1) \in \mathbb{T}_{Q,\mathfrak{m}_Q}.$$

Since $\mathcal{O}$ is a discrete valuation ring, we may choose some i such that the image of t_i in $\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$ divides π^C. Then ϕ_i and ψ_i induce $\mathcal{O}$-module maps

$$M_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}, \;\; \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O} \to M_Q \otimes_{\mathbb{T}^Q} \mathcal{O}$$

whose composition in either direction is multiplication by a divisor of π^C, which implies the result. □

The following corollary is not needed for geometric level raising, but will be used later in the construction of bipartite Euler systems.

Corollary 4.6.6. *Let $\{Q, \epsilon_Q\}$ be a weakly admissible pair that is $(j+2C)$-level raising, and suppose $\nu(N_2 Q)$ is even. Then there exists a map of $\mathbb{T}^Q[G_{\mathbb{Q}}]$-modules*

$$T_{\mathfrak{m}_Q} J^{N_1, N_2 Q} \to T_j \tag{56}$$

that factors through $T_{j+C} \to T_j$ and is surjective after $\mathcal{O}$-linearization, and this map is unique up to multiplication by a unit scalar.

Proof. Write $\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O} = \mathcal{O}/\pi^M$ for some $M \geq j + 2C$. Note that

$$T_{\mathfrak{m}_Q} J_{\min}^{N_1, N_2 Q} \otimes_{\mathbb{T}^Q} \mathcal{O} \cong T_M := T_f / \pi^M; \tag{57}$$

indeed, they are both free of rank two over $\mathcal{O}/\pi^M$ by Proposition 4.2.3, so (57) follows from [14, Théorème 1] and the Eichler-Shimura relation

$$\mathrm{Frob}_\ell^2 - T_\ell \, \mathrm{Frob}_\ell + \ell = 0 \text{ on } T_{\mathfrak{m}_Q} J_{\min}^{N_1, N_2 Q}, \; \forall \ell \nmid Np.$$

By (57) and Proposition 4.2.5, we have

$$\begin{aligned} T_{\mathfrak{m}_Q} J^{N_1, N_2 Q} \otimes_{\mathbb{T}^Q} \mathcal{O} &\cong M_Q \otimes_{\mathbb{T}_Q} \left(T_{\mathfrak{m}_Q} J_{\min}^{N_1, N_2 Q} \otimes_{\mathbb{T}^Q} \mathcal{O} \right) \\ &\cong M_Q \otimes_{\mathbb{T}_Q} T_M. \end{aligned}$$

The corollary now follows from Lemma 4.6.5 and the absolute irreducibility of $\overline{T}_f$.

□

Theorem 4.6.7. *Assume $\overline{T}_f$ satisfies (TW).*

(1) If $\{Qq, \epsilon_{Qq}\}$ is a weakly admissible pair, then

$$\lg_{\mathcal{O}} (\mathbb{T}_{Qq} \otimes_{\mathbb{T}^{Qq}} \mathcal{O}) \geq \lg_{\mathcal{O}} \left(\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{(a_q(f) - \epsilon_{Qq}(q)(q+1))} \right) - C,$$

where C is the number of Lemma 4.6.5.

(2) If $\{Q, \epsilon_Q\}$ is a weakly admissible pair such that q is j-admissible with sign $\epsilon_Q(q)$ for all $q | Q$, then $\{Q, \epsilon_Q\}$ is $(j - \nu(Q) \cdot C)$-level-raising.

Proof. (2) follows from (1) by induction, so we prove (1). In the proof, we will abbreviate $\epsilon_q := \epsilon_{Qq}(q)$. There are two cases, depending on the parity of $\nu(N_2Q)$.

Case 1. *$\nu(N_2Q)$ is even.*

Let us abbreviate $J^Q = J^{N_1,N_2Q}$ and $J^Q_{\min} = J^{N_1,N_2Q}_{\min}$. Consider the composite

$$\frac{M_{Qq}}{(U_q-\epsilon_q)} \twoheadrightarrow \frac{H^1\left(\mathbb{F}_{q^2}, T_{\mathfrak{m}_Q}J^Q\right)}{(\mathrm{Frob}_q - \epsilon_q)} \simeq M_{Q,\mathfrak{m}_Q} \otimes_{\mathbb{T}_{Q,\mathfrak{m}_Q}} \frac{T_{\mathfrak{m}_Q}J^Q_{\min}}{(\mathrm{Frob}_q - \epsilon_q)}$$

induced from Proposition 4.5.2(2) and Proposition 4.2.5. Since $T_{\mathfrak{m}_Q}J^Q_{\min}$ is free of rank two over $\mathbb{T}_{Q,\mathfrak{m}_Q}$ and Frob_q acts with the characteristic polynomial $\mathrm{Frob}_q^2 - T_q\,\mathrm{Frob}_q + q$ (whose roots are distinct modulo $\mathfrak{m}_Q$), we may fix an identification

$$\frac{T_{\mathfrak{m}_Q}J^Q_{\min}}{(\mathrm{Frob}_q - \epsilon_q)} \simeq \frac{\mathbb{T}_{Q,\mathfrak{m}_Q}}{(T_q - \epsilon_q(q+1))}$$

of $\mathbb{T}_{Q,\mathfrak{m}_Q}$-modules. Using the Hecke compatibility from Proposition 4.5.2(1), we obtain a map

$$M_{Qq} \otimes_{\mathbb{T}^{Qq}} \mathcal{O} \twoheadrightarrow M_Q \otimes_{\mathbb{T}^Q} \frac{\mathcal{O}}{(a_q(f) - \epsilon_q(q+1))},$$

hence (by Lemma 4.6.5) a map

$$M_{Qq} \otimes_{\mathbb{T}^{Qq}} \mathcal{O} \to \mathbb{T}_Q \otimes_{\mathbb{T}^Q} \frac{\mathcal{O}}{(a_q(f) - \epsilon_q(q+1))}$$

with cokernel annihilated by π^C. Since the action of $\mathbb{T}^{Qq}$ on M_{Qq} factors through $\mathbb{T}_{Qq}$, we conclude

$$\mathrm{lg}_{\mathcal{O}}\left(\mathbb{T}_{Qq} \otimes_{\mathbb{T}^{Qq}} \mathcal{O}\right) \geq \mathrm{lg}_{\mathcal{O}}\left(\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{(a_q(f) - \epsilon_q(q+1))}\right) - C.$$

Case 2. *$\nu(N_2Q)$ is odd.*

By Proposition 4.5.6, the action of $\mathbb{T}^q_{Q,\mathfrak{m}_{Qq}}$ on

$$M_{Q,\mathfrak{m}_Q} \otimes_{\mathbb{T}_Q} \left(\mathbb{T}_Q^2 / \operatorname{im}\begin{pmatrix} T_q - \epsilon_q & -1 \\ q & -\epsilon_q \end{pmatrix}\right),$$

with U_q acting by ϵ_q, factors through $\mathbb{T}_{Qq,\mathfrak{m}_{Qq}}$. Hence the action of $\mathbb{T}^q_Q$ on

$$A := M_Q \otimes_{\mathbb{T}^Q} \frac{\mathcal{O}}{(a_q(f) - \epsilon_q(q+1))}$$

likewise factors through $\mathbb{T}_{Qq}$ (again with U_q acting by ϵ_q). By Lemma 4.6.5, A has a $\mathbb{T}^q_Q$-module map to

$$\frac{\mathbb{T}_Q \otimes_{\mathbb{T}^Q} \mathcal{O}}{(a_q(f) - \epsilon_q(q+1))}$$

with cokernel annihilated by π^C, from which the result follows as in Case 1. □

Remark 4.6.8. If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ in the notation of (3.2.2), then for $\mathfrak{F}$-many n there is a corresponding weakly admissible pair $\{Q_n, \epsilon_{Q_n}\}$, where $(Q_n)_{n\in\mathbb{N}}$ is a sequence representing Q. To be precise, if $\mathsf{Q} = \{\mathsf{q}_1, \ldots, \mathsf{q}_r\}$, we choose sequences $(q_i^n)_{n\in\mathbb{N}}$ representing each q_i; for $\mathfrak{F}$-many n, the product $Q_n = q_1^n \cdots q_r^n$, equipped with sign function $\epsilon_{Q_n}(q_i^n) = \epsilon_{\mathsf{Q}}(\mathsf{q}_i)$, forms a weakly admissible pair $\{Q_n, \epsilon_{Q_n}\}$. It follows from the definition of N and from the theorem that, for any $j \geq 0$, there exist $\mathfrak{F}$-many n such that the pair $\{Q_n, \epsilon_{Q_n}\}$ is j-level-raising. We say that a sequence of weakly admissible pairs $\{Q_n, \epsilon_{Q_n}\}$ (defined for $\mathfrak{F}$-many n) represents the pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ if it is obtained from this construction for some choice of representatives $(q_i^n)_{n\in\mathbb{N}}$.

4.7. **CM points.**

4.7.1. Let us now fix an imaginary quadratic field $K \subset \overline{\mathbb{Q}}$, and a positive integer $N = N^+N^-$ such that every prime factor of N^+ is split in K, and N^- is a squarefree product of primes inert in K. Let $B = B_{N^-}$ be the quaternion algebra over $\mathbb{Q}$ ramified at the factors of N^-, and possibly ∞. For each $\ell | N$, we have a fixed embedding $K \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. If $\ell | N^+$, this determines a distinguished prime $\mathfrak{l}$ of K above ℓ. We write $\mathfrak{l}^c$ for its conjugate. If $\ell | N^-$, this determines a distinguished isomorphism $\mathcal{O}_K/\ell \simeq \mathbb{F}_{\ell^2}$. Finally, for each positive integer m, let $\mathcal{O}_{m,K} \subset \mathcal{O}_K$ be the order of conductor m.

4.7.2. Assume $\nu(N^-)$ is **even**, and again fix an orientation on the maximal order $\mathcal{O}_B$ of B from (4.3.1).

Definition 4.7.3. Let m be a positive integer coprime to N, and write $K[m]$ for the ring class field of conductor m. The set of (positively oriented) CM points of conductor m,

$$\mathrm{CM}_{N^+,N^-}(m) \subset X_{N^+,N^-}(K[m]),$$

is the set of triples (A, ι, C) over $K[m]$ admitting an isomorphism $\mathcal{O}_{m,K} \xrightarrow{\sim} \mathrm{End}(A, \iota, C)$ such that:

(1) The action of $\mathcal{O}_{m,K}$ on the $K[m]$-vector space $\mathrm{Lie}\, A$ is through the natural inclusion $\mathcal{O}_{m,K} \subset K \subset K[m]$.
(2) For all primes $\ell | N^+$, $\mathcal{O}_{m,K} \otimes \mathbb{Z}_\ell = \mathcal{O}_{K,\mathfrak{l}} \times \mathcal{O}_{K,\mathfrak{l}^c}$ acts on the ℓ-primary component $C_\ell \subset C$ through the projection to $\mathcal{O}_{K,\mathfrak{l}}$.
(3) For all primes $\ell | N^-$, let $\mathfrak{m}_\ell \subset \mathcal{O}_B$ be the unique maximal ideal of residue characteristic ℓ. Then $A[\mathfrak{m}_\ell](\overline{K[m]})$ is a rank-one vector space over $\mathcal{O}_B/\mathfrak{m}_\ell \simeq \mathbb{F}_{\ell^2}$, where the isomorphism comes from the orientation of $\mathcal{O}_B$ at ℓ. We require that the action of $\mathcal{O}_{m,K}/\ell = \mathcal{O}_K/\ell$ on this vector space correspond to our distinguished isomorphism $\mathcal{O}_K/\ell \simeq \mathbb{F}_{\ell^2}$.

4.7.4. If A is an abelian variety, any element $\gamma \in (\mathrm{End}(A) \otimes \mathbb{A}_f)^\times$ defines an abelian variety A_γ with a map $f : A_\gamma \to A$ in the isogeny category of abelian varieties such that $f_*(T_\ell A_\gamma) = \gamma_\ell T_\ell A$ for all ℓ. In this way, we obtain a canonical action of

$$\mathrm{Pic}\, \mathcal{O}_{m,K} = K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{m,K}^\times$$

on $\mathrm{CM}_{N^+,N^-}(m)$. We denote by

$$\mathrm{rec} : \mathrm{Gal}(K[m]/K) \to K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{m,K}^\times \tag{58}$$

the reciprocity map of class field theory, normalized so that uniformizers correspond to geometric Frobenius elements.

Proposition 4.7.5. *(1) Via the reciprocity map, the action of* $\mathrm{Gal}(K[m]/K)$ *on* $\mathrm{CM}_{N^+,N^-}(m)$ *agrees with the action of* $K^\times \backslash \mathbb{A}_{f,K}^\times / \widehat{\mathcal{O}}_{m,K}^\times$ *described above.*
(2) $\mathrm{CM}_{N^+,N^-}(m)$ *is a torsor under the action of* $\mathrm{Gal}(K[m]/K)$.

Proof. Part (1) follows from Shimura's reciprocity law. For (2), see the discussion in [80, p. 55]; it is an elementary exercise using the complex uniformization of X_{N^+,N^-} to see that our definition of the positively oriented CM points of conductor m agrees with the adelic description given in *loc. cit.* (Recall that any $\mathbb{C}$-valued point of X_{N^+,N^-} admitting extra endomorphisms by $\mathcal{O}_{m,K}$ is automatically defined over $K[m]$.) ☐

4.7.6. Now assume that $\nu(N^-)$ is **odd**. In this case, we fix an embedding $K \hookrightarrow B$.

Definition 4.7.7. Suppose m is coprime to N. Then $\mathrm{CM}_{N^+,N^-}(m)$ is defined as the set of isomorphism classes of oriented Eichler orders (R, ϕ) of B of level N^+ such that:

(1) $R \cap K = \mathcal{O}_{m,K}$.
(2) For all primes $\ell | N^+$, let $R_1 \supset R \otimes \mathbb{Z}_\ell$ be the maximal order determined by ϕ_ℓ, and choose an isomorphism $R_1 \simeq M_2(\mathbb{Z}_\ell)$. If $\ell^k || N$, then $R \otimes \mathbb{Z}_\ell \subset M_2(\mathbb{Z}_\ell)$ is the stabilizer of a subgroup $C \subset (\mathbb{Z}/\ell^k)^2$ with $C \simeq \mathbb{Z}/\ell^k$, and we require that the multiplication action of $\mathcal{O}_{m,K}$ on C is given by the projection $\mathcal{O}_{m,K}/\ell^k \twoheadrightarrow \mathcal{O}_{m,K}/\mathfrak{l}^k$.
(3) For all primes $\ell | N^-$, let $\mathfrak{m}_\ell \subset R$ be the unique maximal ideal of residue characteristic ℓ. Then we require that the isomorphism $\mathcal{O}_{m,K}/\ell = R/\mathfrak{m}_\ell \simeq \mathbb{F}_{\ell^2}$ determined by ϕ_ℓ agrees with the fixed isomorphism $\mathcal{O}_K/\ell \simeq \mathbb{F}_{\ell^2}$ chosen above.

Here the equivalence relation is conjugation by $K^\times$. (Properties (1)-(3) are not stable under conjugation by $B^\times(\mathbb{Q})$.)

Notice that $\mathrm{CM}_{N^+,N^-}(m)$ comes equipped with a natural projection map

$$\mathrm{CM}_{N^+,N^-}(m) \to X_{N^+,N^-}. \tag{59}$$

4.7.8. We define an action of $\mathrm{Gal}(K[m]/K)$ on $\mathrm{CM}_{N^+,N^-}(m)$ as follows: for $\sigma \in \mathrm{Gal}(K[m]/K)$ and $(R,\phi) \in \mathrm{CM}_{N^+,N^-}(m)$, let $\sigma \cdot (R,\phi)$ be the Eichler order $\mathrm{rec}(\sigma)\widehat{R}\,\mathrm{rec}(\sigma)^{-1} \cap B(\mathbb{Q})$, with the induced orientation. (Here rec is as in (58).)

Proposition 4.7.9. *Suppose m is coprime to N. Then $\mathrm{CM}_{N^+,N^-}(m)$ is a torsor for $\mathrm{Gal}(K[m]/K)$.*

Proof. It is clear that $\mathrm{Gal}(K[m]/K)$ acts with trivial stabilizers on $\mathrm{CM}_{N^+,N^-}(m)$, so we will show transitivity. Let (R,ϕ) and (R',ϕ') be two elements of $\mathrm{CM}_{N^+,N^-}(m)$. Since the orientations ϕ and ϕ' are determined by properties (2) and (3) of the definition of $\mathrm{CM}_{N^+,N^-}(m)$, it suffices to show that there exists $k \in \mathbb{A}_{f,K}^\times$ such that
$$k\widehat{R}'k^{-1} = \widehat{R}.$$
We do this by working locally at all primes ℓ. To ease notation, abbreviate
$$R_\ell = R\otimes\mathbb{Z}_\ell,\ R'_\ell = R'\otimes\mathbb{Z}_\ell,\ B_\ell = B\otimes\mathbb{Q}_\ell,\ \mathcal{O}_{m,K,\ell} = \mathcal{O}_{m,K}\otimes\mathbb{Z}_\ell,\ K_\ell = K\otimes\mathbb{Q}_\ell.$$
Suppose first that $\ell \nmid N$. Because $R\cap K = R'\cap K = \mathcal{O}_{m,K}$, [22, Lemma 6.2] implies there exists $k_\ell \in K_\ell^\times$ such that $k_\ell R'_\ell k_\ell^{-1} = R_\ell$. For $\ell | N^-$, the maximal order in B_ℓ is unique, so $R'_\ell = R_\ell$. For $\ell|N^+$, let $j = \mathrm{ord}_\ell(N^+)$, and fix an isomorphism $B_\ell \cong M_2(\mathbb{Q}_\ell)$ that identifies $\mathcal{O}_{K,\ell}$ with
$$\begin{pmatrix} \mathbb{Z}_\ell & 0 \\ 0 & \mathbb{Z}_\ell \end{pmatrix} \subset M_2(\mathbb{Q}_\ell).$$
Eichler orders of level ℓ^j in B_ℓ are all of the form $\mathrm{End}(L_1)\cap\mathrm{End}(L_2)$, where $L_1 \subset L_2$ are lattices in $\mathbb{Q}_\ell^2$ with $L_2/L_1 \simeq \mathbb{Z}/\ell^j\mathbb{Z}$. If $\mathrm{End}(L)$ contains $\mathcal{O}_{m,K,\ell}$, then L is of the form $\mathbb{Z}_\ell \oplus \ell^n\mathbb{Z}_\ell$ (up to homothety), for some $n\in\mathbb{Z}$. The possible Eichler orders of level ℓ^j containing $\mathcal{O}_{m,K,\ell}$ are therefore
$$\begin{pmatrix} \mathbb{Z}_\ell & \ell^{-n}\mathbb{Z}_\ell \\ \ell^{n+j}\mathbb{Z}_\ell & \mathbb{Z}_\ell \end{pmatrix},\ n\in\mathbb{Z}.$$
These are evidently all conjugate by diagonal matrices, so we may choose $k_\ell \in K_\ell^\times$ such that $k_\ell R'_\ell k_\ell^{-1} = R_\ell$. Setting $k = \prod_{\ell\nmid N^-} k_\ell$, we have
$$k\widehat{R}'k^{-1}\cap B(\mathbb{Q}) = \widehat{R}.$$
□

Remark 4.7.10. We will soon be varying N^- (keeping K and N^+ fixed). The choices made in the definition of the CM points – i.e. the oriented maximal order $\mathcal{O}_B$ if $\nu(N^-)$ is even, and the embedding $K\hookrightarrow B$ if $\nu(N^-)$ is odd – will be considered to be fixed, once and for all, for each possible N^-.

4.7.11. In the remainder of this section, we recall the geometric ingredients for the explicit reciprocity laws originally studied in [3].

Proposition 4.7.12. *Suppose $\nu(N^-)$ is even. Let m be coprime to N, and let $q\nmid Nm$ be a prime inert in K, with $\mathfrak{q}$ a prime of $K[m]$ above q. Then there is an isomorphism $t_{N^+,N^-,q} : \mathrm{CM}_{N^+,N^-}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-q}(m)$ of $\mathrm{Gal}(K[m]/K)$-torsors fitting into a commutative diagram:*

$$\begin{array}{ccccccc}
\mathrm{CM}_{N^+,N^-}(m) & \hookrightarrow & & & & & X_{N^+,N^-}(K[m]) \\
\downarrow t_{N^+,N^-,q} & & & & & & \downarrow \mathrm{Red}_{\mathfrak{q}} \\
\mathrm{CM}_{N^+,N^-q}(m) & \longrightarrow & X_{N^+,N^-q} & \xrightarrow{\sim} & X_{N^+,N^-}(\mathbb{F}_{q^2})^{ss} & \hookrightarrow & X_{N^+,N^-}(\mathbb{F}_{q^2}).
\end{array}$$

Proof. Choose any $(A,\iota,C) \in \mathrm{CM}_{N^+,N^-}(m)$, and let (A_0,ι_0,C_0) denote its reduction modulo $\mathfrak{q}$. Since q is inert in K, A_0 is supersingular. Moreover we have a distinguished action $\mathcal{O}_{m,K}\hookrightarrow \mathrm{End}(A_0,\iota_0,C_0)$ coming from the reduction of the complex multiplication. Choose an isomorphism $\mathrm{End}^0(A_0,\iota_0,C_0)\simeq B_{N^-q}$ identifying the corresponding embedding $K\hookrightarrow \mathrm{End}^0(A_0,\iota_0,C_0)$ with our fixed inclusion $K\hookrightarrow B_{N^-q}$. The choice of this isomorphism is unique up to $K^\times$-conjugacy. Therefore $\mathrm{End}(A_0,\iota_0,C_0)$ yields a well-defined point of $\mathrm{CM}_{N^+,N^-q}(m)$ – note that conditions (2) and (3) of Definition 4.7.7 are satisfied by conditions (2) and (3) of Definition 4.7.3, where the orientation on $\mathrm{End}(A_0,\iota_0,C_0)$ is defined in the proof of Proposition 4.5.2. This defines the map $t_{N^+,N^-,q}$, and it is Galois-equivariant by Proposition 4.7.5(1). Since the $\mathrm{Gal}(K[m]/K)$-action is simply transitive on both $\mathrm{CM}_{N^+,N^-}(m)$ and $\mathrm{CM}_{N^+,N^-q}(m)$, t_{N^+,N^-q} is automatically an isomorphism. The commutativity of the diagram follows from the construction in the proof of Proposition 4.5.2. □

Proposition 4.7.13. *Suppose $\nu(N^-)$ is odd. Let m be coprime to N, and let $q \nmid 2Nm$ be a prime inert in K, with $\mathfrak{q}$ a prime of $K[m]$ above q. Then every point of $\mathrm{CM}_{N^+,N^-q}(m)$ lies in $X_{N^+,N^-q}(K[m])^{sm}$, the subset of points which reduce modulo $\mathfrak{q}$ to smooth points of the special fiber. Moreover, there is an isomorphism $s_{N^+,N^-,q} : \mathrm{CM}_{N^+,N^-q}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-}$ of $\mathrm{Gal}(K[m]/K)$-torsors fitting into a commutative diagram:*

$$\begin{array}{ccc} \mathrm{CM}_{N^+,N^-q}(m) & \longrightarrow & X_{N^+,N^-q}(K[m])^{sm} \\ \downarrow{\scriptstyle s_{N^+,N^-,q}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\mathfrak{q}}} \\ \mathrm{CM}_{N^+,N^-}(m) & \xrightarrow{(59)\times\{+\}} & X^{\pm}_{N^+,N^-}. \end{array}$$

Proof. That each point of $\mathrm{CM}_{N^+,N^-q}(m)$ has smooth reduction modulo $\mathfrak{q}$ follows from [3, p. 55]. For the rest, let (A,ι,C) be a point of $\mathrm{CM}_{N^+,N^-q}(m)$, and (A_0,ι_0,C_0) its reduction modulo $\mathfrak{q}$. Since (A_0,ι_0,C_0) is a nonsingular $\mathcal{O}_{K[m]}/\mathfrak{q}$-valued point of the special fiber, by [62, Proposition 4.4, Theorem 5.3], there is a unique $\mathcal{O}_{B_{N^-q}}$-stable subgroup scheme $H \subset A_0$ which is isomorphic to α_q; since H is unique, it is automatically $\mathcal{O}_{m,K}$-stable as well. Let $\bar{\iota}_0$ and $\overline{C}_0$ denote the induced $\mathcal{O}_{B_{N^-q}}$-action and $\Gamma_0(N^+)$-structure on A_0/H. Then $(A_0/H,\bar{\iota}_0,\overline{C}_0)$ is a pure triple over $\mathcal{O}_{K[m]}/\mathfrak{q} \simeq \mathbb{F}_{q^2}$ in the notation of the proof of Proposition 4.5.4, and the irreducible component of the special fiber of X_{N^+,N^-q} containing (A_0,ι_0,C_0) is parameterized by the q-Frobenius twist $(A_0/H,\bar{\iota}_0,\overline{C}_0)^{(q)}$. As in the proof of Proposition 4.7.12, the induced $\mathcal{O}_{m,K}$-action on $(A_0/H)^{(q)}$ allows us to view $\mathrm{End}(A_0/H,\bar{\iota}_0,\overline{C}_0)$ as a point of $\mathrm{CM}_{N^+,N^-}(m)$, and the resulting map $\mathrm{CM}_{N^+,N^-q}(m) \to \mathrm{CM}_{N^+,N^-}(m)$ is then an isomorphism of $\mathrm{Gal}(K[m]/K)$-torsors.

To finish the proof, we must show that $(A_0/H,\bar{\iota}_0,\overline{C}_0)^{(q)}$ is pure of type $+$, or equivalently that $(A_0/H,\bar{\iota}_0,\overline{C}_0)$ is pure of type $-$. By [62, Proposition 4.7], it suffices to show that H is of type $+$ in the following sense: if $\mathcal{M}$ is the Dieudonné module of $A_0[q^\infty]$, then H corresponds to a submodule $\mathcal{N} \subset \mathcal{M}$ containing $(F,V)\mathcal{M}$; we wish to show that $\mathcal{O}_{B_{N^-q}}$ acts on the one-dimensional $\overline{\mathbb{F}_{q^2}}$-vector space $\mathcal{M}/\mathcal{N}$ by the map $\mathcal{O}_{B_{N^-q}} \to \mathbb{F}_{q^2} \subset \overline{\mathbb{F}_{q^2}}$ determined by the fixed orientation.

Let $\mathcal{A}$ denote the Néron model of A over $\operatorname{Spec}\mathcal{O}_{K[m],\mathfrak{q}}$, with special fiber A_0. Now, $\mathcal{M}/F\mathcal{M}$ is dual to $\operatorname{Lie} A_0 = \operatorname{Lie}\mathcal{A} \otimes \mathcal{O}_{K[m]}/\mathfrak{q}$, and $\mathcal{O}_{m,K}$ acts on $\operatorname{Lie}\mathcal{A}$ by the canonical embedding $\mathcal{O}_{m,K} \hookrightarrow \mathcal{O}_{K[m],\mathfrak{q}}$ (using the orientation condition of Definition 4.7.3(1)). Hence $\mathcal{O}_{m,K}$ acts on $\mathcal{M}/F\mathcal{M}$ via the reduction map to $\mathcal{O}_{m,K}/q \simeq \mathbb{F}_{q^2}$. Because $\mathcal{M}/F\mathcal{M}$ surjects onto $\mathcal{M}/\mathcal{N}$, it then suffices to show that the actions of $\mathcal{O}_{B_{N^-q}}/\mathfrak{m}_q$ and $\mathcal{O}_{m,K}/q$ on $\mathcal{M}/\mathcal{N}$ coincide under the fixed composite isomorphism $\mathcal{O}_{B_{N^-q}}/\mathfrak{m}_q \simeq \mathbb{F}_{q^2} \simeq \mathcal{O}_{m,K}/q$. For this, it suffices to show the same compatibility for the actions on the finite flat group scheme $\mathcal{A}[\mathfrak{m}_q]$; these actions coincide over the generic fiber of $\operatorname{Spec}\mathcal{O}_{K[m],\mathfrak{q}}$ by Definition 4.7.3(3), and so the desired compatibility follows from [60, Corollaire 3.3.6] under the assumption $q \neq 2$.

□

5. Construction of bipartite Euler systems

5.1. The CM class construction.

5.1.1. Fix a quadratic imaginary field $K \subset \overline{\mathbb{Q}}$, and let f, N, $\wp$, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5, such that $\overline{T}_f$ satisfies (TW). We assume that N admits a factorization $N = N^+N^-$, where all $\ell | N^+$ are split in K, and N^- is a squarefree product of primes inert in K. We continue the notation of §4.6 (using $N_1 = N^+$ and $N_2 = N^-$). Fix an integer m which is coprime to N, and let $G_m = \mathrm{Gal}(K[m]/K)$. If $q \nmid m$ is a prime inert in K, we fix a prime $\mathfrak{q}$ of $K[m]$ above q; for instance, this can be done by choosing an embedding $K[m] \hookrightarrow \overline{\mathbb{Q}}$. If q is j-admissible with sign ϵ_q and inert in K, we define the ordinary subspace:

$$H^1_{\mathrm{ord},\epsilon_q}(K_q,T_j) = \mathrm{im}\left(H^1(K_q,\mathrm{Fil}^+_{q,\epsilon_q} T_j) \to H^1(K_q,T_j)\right). \tag{60}$$

Using the map obtained from Shapiro's Lemma (e.g. [71, §3.1.2])

$$\mathrm{Res}_{\mathfrak{q}} : H^1(K[m],T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, H^1(K_q,T_j)), \tag{61}$$

we also have maps:

$$\partial_{q,\epsilon_q} : H^1(K[m],T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, H^1(I_q,\mathrm{Fil}^+_q T_j)) \approx \mathcal{O}_j[G_m],$$
$$\mathrm{loc}_{q,\epsilon_q} : H^1(K[m]^\Sigma/K[m],T_j) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, H^1_{\mathrm{unr}}(K_q,T_j/\mathrm{Fil}^+_q T_j)) \approx \mathcal{O}_j[G_m], \quad \Sigma \subset M_{\mathbb{Q}} \text{ finite}, q \notin \Sigma,$$

defined as in (23), (24).

For notational convenience, for this section only we temporarily denote by $\mathcal{N}_j$ the collection of weakly admissible pairs $\{Q, \epsilon_Q\}$ which are $(j+2C)$-level-raising, and such that all primes $q|Q$ are odd and inert in K. Let Σ be the set of primes of $\mathbb{Q}$ dividing $Np\infty$, and for any squarefree integer Q, let Σ_Q be the set of primes of $\mathbb{Q}$ dividing $NpQ\infty$.

Construction 5.1.2. *For all $\{Q, \epsilon_Q\} \in \mathcal{N}_j$, there exist maps (well-defined to a unit scalar):*

$$\kappa_j(\cdot, Q, \epsilon_Q) : \mathrm{CM}_{N^+,N^-Q}(m) \to H^1(K[m]^{\Sigma_Q}/K[m], T_j), \qquad \nu(N^-Q) \text{ even},$$
$$\lambda_j(\cdot, Q, \epsilon_Q) : \mathrm{CM}_{N^+,N^-Q}(m) \to \mathrm{Hom}_{\mathrm{Set}}(G_m, \mathcal{O}_j) = \mathcal{O}_j[G_m], \qquad \nu(N^-Q) \text{ odd},$$

compatible under the natural reduction maps for $j' \le j$, $\mathrm{Gal}(K[m]/K)$-equivariant, and such that the following properties hold.

1. *If $\{Q, \epsilon_Q\} \in \mathcal{N}_j$ where $\nu(N^-Q)$ is even, then for all $q|Q$ and all $y \in \mathrm{CM}_{N^+,N^-Q}(m)$,*
$$\mathrm{Res}_{\mathfrak{q}}(\kappa_j(y, Q, \epsilon_Q)) \in H^1_{\mathrm{ord},\epsilon_Q(q)}(K_q, T_j).$$
2. *If $\{Qq, \epsilon_{Qq}\}, \{Q, \epsilon_Q\} \in \mathcal{N}_j$ where $\epsilon_Q = \epsilon_{Qq}|_Q$ and $\nu(N^-Qq)$ is even, then there is an isomorphism $i : H^1(I_q, \mathrm{Fil}^+_{q,\epsilon_{Qq}(q)} T_j) \simeq \mathcal{O}_j$ such that, for all $y \in \mathrm{CM}_{N^+,N^-Qq}(m)$,*
$$i\left(\partial_{q,\epsilon_{Qq}(q)} \kappa_j(y, Qq, \epsilon_{Qq})\right) = \lambda_j(s_{N^+,N^-Q,q}(y), Q, \epsilon_Q).$$
Here $s_{N^+,N^-Q,q} : \mathrm{CM}_{N^+,N^-Qq}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-Q}(m)$ is the map of Proposition 4.7.13.
3. *If $\{Qq, \epsilon_{Qq}\}, \{Q, \epsilon_Q\} \in \mathcal{N}_j$ where $\epsilon_Q = \epsilon_{Qq}|_Q$ and $\nu(N^-Qq)$ is odd, then there is an isomorphism $i : H^1_{\mathrm{unr}}(K_q, T_j/\mathrm{Fil}^+_{q,\epsilon_{Qq}(q)} T_j) \simeq \mathcal{O}_j$ such that, for all $y \in \mathrm{CM}_{N^+,N^-Q}(m)$,*
$$i\left(\mathrm{loc}_{q,\epsilon_{Qq}(q)}(\kappa_j(y, Q, \epsilon_Q))\right) = \lambda_j(t_{N^+,N^-Q,q}(y), Q, \epsilon_Q).$$
Here $t_{N^+,N^-Q,q} : \mathrm{CM}_{N^+,N^-Q}(m) \xrightarrow{\sim} \mathrm{CM}_{N^+,N^-Qq}(m)$ is the map of Proposition 4.7.12.

Proof. The specifications ϵ_Q will be dropped to ease notation. We fix throughout a prime $\ell_0 \nmid Nmp$ such that $a_{\ell_0}(f) - \ell_0 - 1$ is a unit in $\mathcal{O}$.

Suppose first that $\nu(N^-Q)$ is odd. By Lemma 4.6.5, there is a unique map (up to scalars) $M_Q \to \mathcal{O}_j$ of $\mathbb{T}^Q$-modules that factors through $\mathcal{O}_{j+C}$ and is surjective after $\mathcal{O}$-linearization. For $y \in \mathrm{CM}_{N^+,N^-Q}(m)$ and $g \in G_m$, we define $\lambda_j(y, Q)(g)$ to be the image of gy by the composite map

$$\mathrm{CM}_{N^+,N^-Q}(m) \to X_{N^+,N^-Q} \xrightarrow{T_{\ell_0} - \ell_0 - 1} M_Q \to \mathcal{O}_j.$$

(The notation M_Q was defined in (55).)

Now suppose that $\nu(N^-Q)$ is even. For each $y \in \mathrm{CM}_{N^+,N^-Q}(m)$, $(T_{\ell_0} - \ell_0 - 1)y$ is a degree zero divisor on X_{N^+,N^-Q}, and its image in the Jacobian J^Q is defined over $K[m]$. Let

$$d(y, Q) \in H^1(K[m]^{\Sigma_Q}/K[m], T_{\mathfrak{m}_Q} J^Q)$$

be the Kummer image. We define $\kappa_j(y, Q)$ to be the image of $d(y, Q)$ under the map

$$H^1(K[m]^{\Sigma_Q}/K[m], T_{\mathfrak{m}_Q} J^Q) \to H^1(K[m]^{\Sigma_Q}/K[m], T_j)$$

induced by Corollary 4.6.6.

We now establish properties (1)-(3).

(1) Let $\mathcal{X} = \mathcal{X}_q(J^Q)$ be the character group as in (4.1.1), and let $\mathcal{X}^\vee := \mathrm{Hom}(\mathcal{X}, \mathbb{Z})$. Because J^Q is Hecke-equivariantly isogenous to $J^{N^+,N^-Q}_{\min}$ over $\mathbb{Q}$ by [35, Corollary 2.4], $\mathcal{X} \otimes \mathbb{Q}$ is isomorphic to $\mathcal{X}_q(J^{N^+,N^-Q}_{\min}) \otimes \mathbb{Q}$ as a $\mathbb{T}_Q[\mathrm{Frob}_q]$-module; in particular, Frob_q acts on $\mathcal{X}$ as U_q by [63, Proposition 3.8]. Since $\mathcal{X}^\vee \otimes \overline{\mathbb{Q}}_q^\times$ is p-divisible and $\mathrm{Frob}_q^2 = 1$ on $\mathcal{X}$, the exact sequence in Proposition 4.1.2(1) induces a $\mathrm{Gal}(K_q/\mathbb{Q}_q)$-equivariant commutative diagram of Kummer maps

$$\begin{array}{ccc} \varprojlim \left(\mathcal{X}^\vee \otimes K_q^\times\right)/p^n & \xrightarrow{\sim} & H^1(K_q, \mathcal{X}^\vee \otimes \mathbb{Z}_p(1)) \\ \downarrow\!\!\downarrow & & \downarrow \\ \varprojlim \left(J^Q(K_q)\right)/p^n & \longrightarrow & H^1(K_q, T_p J^Q), \end{array}$$

where the right vertical map is induced by the injection of $\mathbb{T}_Q[G_{\mathbb{Q}_q}]$-modules $\mathcal{X}^\vee \otimes \mathbb{Z}_p(1) \hookrightarrow T_p J^Q$ from Proposition 4.1.2(3). Completing at $\mathfrak{m}_Q$, we conclude that the image of the Kummer map $J^Q(K_q)_{\mathfrak{m}_Q} \to H^1(K_q, T_{\mathfrak{m}_Q} J^Q)$ coincides with the image of the map $H^1(K_q, \mathcal{X}^\vee_{\mathfrak{m}_Q}(1)) \to H^1(K_q, T_{\mathfrak{m}_Q} J^Q)$ coming from the short exact sequence of $\mathbb{T}_{Q,\mathfrak{m}_Q}[G_{\mathbb{Q}_q}]$-modules

$$0 \to \mathcal{X}^\vee_{\mathfrak{m}_Q}(1) \to T_{\mathfrak{m}_Q} J^Q \to \mathcal{X}_{\mathfrak{m}_Q} \to 0.$$

On the other hand, $\mathcal{X}^\vee_{\mathfrak{m}_Q}(1)$ and $\mathcal{X}_{\mathfrak{m}_Q}$ have Frob_q-action through the scalars $q\epsilon_q$ and ϵ_q, respectively, where $\epsilon_q = \epsilon_Q(q)$. Hence the $\mathcal{O}$-span of the image of $\mathcal{X}^\vee_{\mathfrak{m}_Q}(1)$ under the map of $\mathbb{T}^Q_{\mathfrak{m}_Q}[G_\mathbb{Q}]$-modules $T_{\mathfrak{m}_Q} J^Q \to T_j$ is $\mathrm{Fil}_q^+ T_j$, and the claim follows.

(2) By the reasoning of (1), we have a map

$$\alpha : J^{Qq}(K_q)_{\mathfrak{m}_{Qq}} \to H^1_{\mathrm{ord}}(K_q, T_j) \to H^1(I_q, \mathrm{Fil}_q^+ T_j)$$

such that $\partial_q(\kappa_j(y, Qq))(g) = \alpha((T_{\ell_0} - \ell_0 - 1)gy)$ for $y \in \mathrm{CM}_{N^+, N^- Qq}$; also, α factors through $H^1(I_q, \mathrm{Fil}_q^+ T_{j+C})$ and is surjective after $\mathcal{O}$-linearization.

Let Φ be the component group of the special fiber of the Néron model of J^{Qq} over $\mathbb{Z}_q$. Applying the snake lemma to the diagram in Proposition 4.1.2(2), we see that the specialization map $\mathrm{Sp}_q : J^{Qq}(K_q) \to \Phi$ is surjective, and the pro-p part of its kernel has Frob_q-action through $-U_q$ since $p \nmid q-1$. Meanwhile, $H^1(I_q, \mathrm{Fil}_q^+ T_{j+C})$ has Frob_q-action through $\epsilon_q = \epsilon_{Qq}(q)$; hence α factors as $\overline{\alpha} \circ \mathrm{Sp}_q$, for a map $\overline{\alpha} : \Phi_{\mathfrak{m}_{Qq}} \to H^1(I_q, \mathrm{Fil}_q^+ T_j) \approx \mathcal{O}_j$ that factors through $\mathcal{O}_{j+C}$ and is surjective after $\mathcal{O}$-linearization.

By Proposition 4.5.6 and the formula for the U_q-action in Proposition 4.5.4, restricting the map (54) to the "+" components defines a surjection $\beta : M_{Qq, \mathfrak{m}_{Qq}} \twoheadrightarrow \Phi_{\mathfrak{m}_{Qq}}$. Applying Propositions 4.5.6 and 4.7.13, we also have

$$\beta\left((T_{\ell_0} - \ell_0 - 1) s_{N^+, N^- Q, q}(y)\right) = \mathrm{Sp}_q\left((T_{\ell_0} - \ell_0 - 1)y\right)$$

for all $y \in \mathrm{CM}_{N^+, N^- Qq}$. Since Lemma 4.6.5 shows that $\overline{\alpha} \circ \beta$ coincides up to a unit scalar with the map $M_{Qq, \mathfrak{m}_{Qq}} \to \mathcal{O}_j$ used to construct $\lambda_j(y, Qq)$, (2) follows.

(3) By Proposition 4.7.12 and the local-global compatibility of the Abel-Jacobi map,

$$\mathrm{loc}_q \kappa_j(y, Q)(g) \in H^1_{\mathrm{unr}}(\mathbb{Q}_{q^2}, T_j / \mathrm{Fil}_q^+ T_j) = H^1(\mathbb{F}_{q^2}, T_j / \mathrm{Fil}_q^+ T_j)$$

is the image of $t_{N^+, N^- Q, q}(gy)$ under the composite map

$$\begin{aligned} \mathrm{CM}_{N^+, N^- Qq}(m) \to X_{N^+, N^- Qq} \xrightarrow{T_{\ell_0} - \ell_0 - 1} M_{Qq} \xrightarrow{(53)} H^1(\mathbb{F}_{q^2}, T_{\mathfrak{m}_Q} J^Q) \to \\ H^1(\mathbb{F}_{q^2}, T_{j+C}) \twoheadrightarrow H^1(\mathbb{F}_{q^2}, T_j) \twoheadrightarrow H^1(\mathbb{F}_{q^2}, T_j / \mathrm{Fil}_q^+ T_j). \end{aligned} \tag{62}$$

We claim it suffices to show the composite

$$\psi : M_{Qq} \to H^1(\mathbb{F}_{q^2}, T_{j+C} / \mathrm{Fil}_q^+ T_{j+C}) \approx \mathcal{O}_{j+C}$$

is equivariant for the full Hecke algebra $\mathbb{T}^q_{Q, \mathfrak{m}_{Qq}}$; indeed, by Lemma 4.6.5, the composition of ψ with the reduction map $\mathcal{O}_{j+C} \to \mathcal{O}_j$ will therefore coincide (up to a unit scalar) with the map used to construct $\lambda_j(y, Qq)$, which will give (3). The equivariance for all Hecke operators prime to q is clear, so we wish to show

$$\psi \circ U_q = \epsilon_q \psi, \tag{63}$$

where again $\epsilon_q = \epsilon_{Qq}(q)$. Now, the map $\varphi : M_{Qq} \twoheadrightarrow H^1(\mathbb{F}_{q^2}, T_{\mathfrak{m}_Q} J^Q)$ satisfies $\varphi \circ U_q = \mathrm{Frob}_q \circ \varphi$ by Proposition 4.5.2(2). Also, the maps

$$H^1(\mathbb{F}_{q^2}, T_{\mathfrak{m}_Q} J^Q) \to H^1(\mathbb{F}_{q^2}, T_{j+C}) \twoheadrightarrow H^1(\mathbb{F}_{q^2}, T_{j+C} / \mathrm{Fil}_q^+ T_{j+C})$$

are maps of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$-modules, and the latter map projects onto the ϵ_q eigenspace for Frob_q. Hence (63) holds, as desired.

□

Remark 5.1.3. In the special case $Q = 1$, Construction 5.1.2 can be described more simply. First, let A_f be a representative of the isogeny class of abelian varieties of GL_2-type associated to f; we can and do assume $\mathrm{End}_{\mathbb{Q}}(A_f) = \mathcal{O}_f$. Then T_f is realized as $T_pA_f \otimes_{\mathcal{O}_f \otimes \mathbb{Z}_p} \mathcal{O}$.

When $\nu(N^-)$ is even, there is a modular parametrization $\varphi : J^{N^+,N^-} \to A_f$; we may assume without loss of generality that the induced map $\varphi_* : T_{\mathfrak{m}}J^{N^+,N^-} \to T_f$ is surjective after $\mathcal{O}$-linearization, and that the map $T_{\mathfrak{m}}J^{N^+,N^-} \to T_j$ used in the construction of $\kappa_j(\cdot, 1)$ is the reduction of φ_* modulo π^j. Then for all $y \in \mathrm{CM}_{N^+,N^-}(m)$, $\kappa_j(y, 1)$ is the image of $\varphi((T_{\ell_0} - \ell_0 - 1)y)$ under the Kummer map $A_f(K[m]) \to H^1(K[m], A_f[\wp^j]) = H^1(K[m], T_j)$.

Similarly, when $\nu(N^-)$ is odd, there is a Hecke-equivariant map $\varphi : \mathbb{Z}[X_{N^+,N^-}]^0 \to \mathcal{O}$ corresponding to the realization of f as a quaternionic modular form on $B_{N^-}^\times$. Without loss of generality, φ is surjective after $\mathcal{O}$-linearization, and $\lambda_j(y, 1)(g) = \varphi((T_{\ell_0} - \ell_0 - 1)gy) \pmod{\pi^j}$ for all $y \in \mathrm{CM}_{N^+,N^-}(m)$ and $g \in \mathrm{Gal}(K[m]/K)$.

5.2. p-adic interpolation.

5.2.1. Suppose for this subsection that:

(spl) $$p \text{ splits in } K$$

and

(ord) $$a_p(f) \notin \wp.$$

We denote by $K_m \subset K[p^m]$ the mth layer of the anticyclotomic $\mathbb{Z}_p$-extension K_∞/K.

Proposition 5.2.2. *Suppose Q is a squarefree product of primes inert in K. Then there exists a sequence $y(m) \in \mathrm{CM}_{N^+,N^-Q}(p^m)$ such that*

$$T_py(m) = \mathrm{tr}_{K[p^{m+1}]/K[p^m]}\, y(m + 1) + y(m - 1), \quad \forall m \geq 1,$$

as formal sums of points on X_{N^+,N^-Q}.

Proof. This is a standard calculation, but we give a sketch of the proof for lack of a precise reference. The set

$$\mathrm{CM}_{N^+,N^-Q}(p^\infty) := \cup_{m\geq 0} \mathrm{CM}_{N^+,N^-Q}(p^m)$$

carries a natural Hecke correspondence T_p, compatible with the action of $\mathrm{Gal}(K[p^\infty]/K)$ and with the map $\mathrm{CM}_{N^+,N^-Q}(p^\infty) \to X_{N^+,N^-Q}$. If y is a CM point of conductor p^m with $m \geq 1$, then (since p is split in K) T_py contains a CM point of conductor p^{m-1}, and another of conductor p^{m+1}. Since T_py is fixed by $\mathrm{Gal}(K[p^{m+1}]/K[p^m])$, the proposition follows formally. □

5.2.3. Suppose given any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, and let $\{Q_n, \epsilon_{Q_n}\}$ be a representative sequence of weakly admissible pairs as in Remark 4.6.8, with each Q_n a squarefree product of primes inert in K. For each n, let $y(m)_n \in \mathrm{CM}_{N^+,N^-Q_n}(p^m)$ be a sequence of CM points which are compatible in the sense of Proposition 5.2.2.

Since $T_p \notin \mathfrak{m}$, Hensel's Lemma implies that the Hecke algebras $\mathbb{T}^{Q_n}_{\mathfrak{m}_{Q_n}}$ contain a (unique) element $u \notin \mathfrak{m}_{Q_n}$ such that $u^2 - uT_p + p = 0$. Let $\alpha_p \in \mathcal{O}^\times$ be the image of u.

5.2.4. We now suppose that $|\mathsf{Q}| + \nu(N^-)$ is even. Adopting the notation of Construction 5.1.2, it follows from the compatibility relation of the $y(m)_n$ that the classes

$$d(m, Q_n) := \mathrm{Cores}_{K[p^m]/K_m}\left(u^{-m+1}d(y(m)_n, Q_n) - u^{-m}\,\mathrm{Res}_{K[p^m]/K[p^{m-1}]}\, d(y(m - 1)_n, Q_n)\right)$$

are compatible under the corestriction maps

$$H^1(K_m, T_{\mathfrak{m}_{Q_n}}J^{Q_n}) \to H^1(K_{m-1}, T_{\mathfrak{m}_{Q_n}}J^{Q_n}).$$

Let $\kappa_j(m, Q_n)$ be the image of $d(m, Q_n)$ under the map of Corollary 4.6.6; this is well-defined for $\mathfrak{F}$-many n depending on j, and the classes $\kappa_j(m, Q_n)$ are compatible under corestriction. Let $\mathsf{S} \subset \mathsf{M}_K$ be the set of constant ultraprimes $\underline{v}$ such that $v|Np\infty$, and, for each m, let $(\mathsf{S} \cup \mathsf{Q})_m \subset \mathsf{M}_{K_m}$ be the preimage of $\mathsf{S} \cup \mathsf{Q}$ under the projection $\mathsf{M}_{K_m} \twoheadrightarrow \mathsf{M}_K$. We let

$$\boldsymbol{\kappa}(\mathsf{Q}) \in \varprojlim_{m,j} \mathsf{H}^1(K_m^{(\mathsf{S}\cup\mathsf{Q})_m}/K_m, T_j) \simeq \mathsf{H}^1(K^{\mathsf{S}\cup\mathsf{Q}}, T_f \otimes \Lambda(\Psi))$$

be the class represented by the family $\kappa_j(m, Q_n)$, where the isomorphism follows from Shapiro's Lemma.

5.2.5. Similarly, if $|\mathsf{Q}| + \nu(N^-)$ is odd, the elements

$$\lambda_j(m, Q_n) \coloneqq \alpha_p^{-m+1}\lambda_j(y(m)_n, Q_n) - \alpha_p^{-m}\lambda_j(y(m-1)_n, Q_n) \in \mathcal{O}_j[\mathrm{Gal}(K[p^m]/K)]$$

are compatible under the natural projection maps

$$\mathcal{O}_j[\mathrm{Gal}(K[p^m]/K)] \to \mathcal{O}_j[\mathrm{Gal}(K[p^{m-1}]/K)].$$

Applying the projection $\mathrm{Gal}(K[p^m]/K) \twoheadrightarrow \mathrm{Gal}(K_m/K)$ for each m, we then obtain an element

$$\boldsymbol{\lambda}(\mathsf{Q}) \in \varprojlim_{m,j} \mathcal{U}\left(\{\mathcal{O}_j[\mathrm{Gal}(K_m/K)]\}_{n\in\mathbb{N}}\right) \simeq \mathcal{O}[\![\mathrm{Gal}(K_\infty/K)]\!] \simeq \Lambda.$$

Remark 5.2.6. If $\nu(N^-)$ is even, then $\boldsymbol{\kappa}(1)$ is the usual interpolated Heegner class, which appears in slightly more restricted contexts in [8, 37, 57]. If $\nu(N^-)$ is odd, then $\boldsymbol{\lambda}(1)$ is the usual anticyclotomic p-adic L-function, e.g. the one denoted Θ_∞ in [19].

5.2.7. Recall the Selmer structure $(\mathcal{F}_\Lambda, \mathsf{S})$ for $\mathbf{T}_f \coloneqq T_f \otimes \Lambda$ defined in (3.4.2).

Proposition 5.2.8. *The pair $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ is a bipartite Euler system with parity $\nu(N^-)$ for the triple $(\mathbf{T}_f, \mathcal{F}_\Lambda, \mathsf{S})$. Moreover, either $\boldsymbol{\kappa}(1)$ or $\boldsymbol{\lambda}(1)$ is nontrivial, depending on the parity of $\nu(N^-)$.*

Proof. We first show that $\boldsymbol{\kappa}(\mathsf{Q})$ lies in $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f)$ for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$, or equivalently that $\boldsymbol{\kappa}(\mathsf{Q})$ lies in $\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_\mathsf{v}, \mathbf{T}_f)$ for all v. If $\mathsf{v} \notin \mathsf{S} \cup \mathsf{Q}$, this is clear; if $\mathsf{v} = \mathsf{q} \in \mathsf{Q}$, it follows from Construction 5.1.2(1); and if $\mathsf{v} = \underline{v}$ with $v|N$, it is automatic because $\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_\mathsf{v}, \mathbf{T}_f) = \mathsf{H}^1(K_\mathsf{v}, \mathbf{T}_f)$ (by definition if $v|N^-$, and by [65, Corollary B.3.4] if $v|N^+$).

So we verify the local condition for $\mathsf{v} = \underline{v}$ with $v|p$. If Q is represented by the sequence $(Q_n)_{n\in\mathbb{N}}$, let $\mathrm{Fil}_v^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n} \subset T_{\mathfrak{m}_{Q_n}} J^{Q_n}$ be the maximal $\mathbb{T}_{Q_n,\mathfrak{m}_{Q_n}}$-stable subspace on which I_v acts by the cyclotomic character (adopting the notation of Construction 5.1.2 and if necessary restricting our attention to $\mathfrak{F}$-many n). As in [18, Proposition 4.7], it suffices to show that, for all m and n and a fixed extension of v to K_∞, the image $d_{n,m}$ of the class $d(m, Q_n)$ under the composite

$$H^1(K_m, T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m,v}, T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}_v^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n})$$

is trivial. Since $d(m, Q_n)$ is a $\mathbb{T}_{Q_n,\mathfrak{m}_{Q_n}}$-linear combination of Kummer images over K_m, by [4, Example 3.11] and [52, Proposition 12.5.8] $d_{n,m}$ lies in the kernel of

$$H^1(K_{m,v}, T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}_v^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}) \to H^1(K_{m,v}, \mathbb{Q}_p \otimes T_{\mathfrak{m}_{Q_n}} J^{Q_n} / \mathrm{Fil}_v^+ T_{\mathfrak{m}_{Q_n}} J^{Q_n}).$$

Since the classes $d_{n,m}$ are corestriction-compatible as m varies, the argument of [39, Proposition 2.4.5] shows that indeed $d_{n,m} = 0$ for all n, m.

The explicit reciprocity laws are a consequence of Construction 5.1.2(2,3), and the nonvanishing of either $\boldsymbol{\kappa}(1)$ or $\boldsymbol{\lambda}(1)$ (according to the parity of $\nu(N^-)$) is due to the work of Cornut and Vatsal [22, Theorem 1.10] and Vatsal [76, Theorem 1.1]; see [76, §2.3] for the relation of the cited theorem to $\boldsymbol{\lambda}(1)$. ☐

Remark 5.2.9. In fact, Vatsal proves the stronger result that $\boldsymbol{\lambda}(1) \not\equiv 0 \pmod{\pi}$ when $\nu(N^-)$ is odd. Indeed, this follows from [76, Proposition 4.7], combined with the independence of the choice of Heegner points described in Remark 3.7 of *op. cit.* The constant ν in [76, Proposition 4.7] is trivial because $\overline{T}_f$ is absolutely irreducible.

5.3. Kolyvagin classes.

5.3.1. Before defining the Kolyvagin classes in patched cohomology, we begin by recalling a calculation explained in [31, §3]. Throughout this subsection, we assume

$$\text{(disc)} \qquad \mathrm{disc}(K) \neq -3, -4,$$

or equivalently $\mathcal{O}_K^\times = \{\pm 1\}$.

Let m be a squarefree product of primes ℓ inert in K with $(m, Np) = 1$. Under the condition (disc), we have

$$\mathrm{Gal}(K[m]/K[1]) \simeq \prod_{\ell | m} \mathrm{Gal}(K[\ell]/K[1]);$$

each $\mathrm{Gal}(K[\ell]/K[1])$ is cyclic of order $\ell+1$. For a place λ of $K[m]$ over some $\ell|m$, extend λ to a place of $\overline{\mathbb{Q}}$ and let $\mathrm{Frob}_\lambda \in G_{\mathbb{Q}}$ be a lift of Frobenius. Also let $\sigma_\lambda \in I_\lambda \subset G_{K[1]}$ be an element whose image generates $\mathrm{Gal}(K[\ell]/K[1])$. Recall the Kolyvagin derivative operators [31, p. 239]:

$$D_\ell = \sum_{i=1}^{\ell} i\sigma_\lambda^i \in \mathbb{Z}[\mathrm{Gal}(K[\ell]/K[1])], \quad D_m = \prod_{\ell|m} D_\ell.$$

Finally, let Q be a squarefree product of primes inert in K that is coprime to Nmp, and choose a CM point $y(m,Q) \in \mathrm{CM}_{N^+,N^-Q}(m)$. We define

$$P(m,Q) = D_m y(m,Q),$$

viewed as a formal sum of points on X_{N^+,N^-Q}.

Proposition 5.3.2. *For any $\ell|m$, there exists a CM point $y(m/\ell,Q) \in \mathrm{CM}_{N^+,N^-Q}(m/\ell)$ such that:*

(1) $(\sigma_\lambda - 1)P(m,Q) = (\ell+1)D_{m/\ell}\,y(m,Q) - T_\ell P(m/\ell,Q)$.

(2) If $\nu(N^-Q)$ is even, then

$$D_{m/\ell}\,y(m,Q) \equiv \mathrm{Frob}_\lambda P(m/\ell,Q) \pmod{\lambda}.$$

Proof. This is proved in [31, p. 240] in the modular curve case; the same reasoning applies to Shimura curves by [53, Proposition 4.13]. The argument for (1) formally applies to Shimura sets as well, along the lines of Proposition 5.2.2. ☐

Fix ℓ_0 as in the proof of Construction 5.1.2, and let

$$P'(m,Q) = (T_{\ell_0} - \ell_0 - 1)P(m,Q).$$

Proposition 5.3.3. *Let $\mathfrak{m}_Q \subset \mathbb{T}_Q = \mathbb{T}_{N^+,N^-Q}$ be non-Eisenstein, and let $I_m \subset \mathbb{T}_Q$ be the ideal generated by $\ell+1$ and T_ℓ for all $\ell|m$. Then if $\nu(N^-Q)$ is even:*

(1) Restriction induces an isomorphism

$$\mathrm{Res}_m : H^1(K[1], T_{\mathfrak{m}_Q} J^Q/I_m) \xrightarrow{\sim} H^1(K[m], T_{\mathfrak{m}_Q} J^Q/I_m)^{\mathrm{Gal}(K[m]/K[1])}.$$

(2) The Kummer image $d(m,Q)$ of $P'(m,Q)$ in $H^1(K[m], T_{\mathfrak{m}_Q} J^Q/I_m)$ lies in the image of Res_m.

(3) If $c(m,Q) = \mathrm{Cores}_{K[1]/K}\,\mathrm{Res}_m^{-1} d(m,Q)$, then for all $\ell|m$ and any choices of representatives,

$$c(m,Q)(\sigma_\lambda) = \mathrm{Frob}_\lambda^{-1} c(m/\ell,Q)(\mathrm{Frob}_\lambda^2).$$

(4) The class $c(m,Q)$ is unramified at any place $v \nmid NpmQ\infty$.

Proof. First note that the residual representation $\overline{\rho}_{\mathfrak{m}_Q}$ associated to $\mathfrak{m}_Q$ has no $G_{K[m]}$-invariants by the same argument as Lemma 3.3.4. Hence (1) follows from the inflation-restriction exact sequence as in [31, p. 241]. Also, (2) is immediate from Proposition 5.3.2, and (4) is clear from the construction. For (3), we modify the argument of [50, Proposition 4.4]. First of all, both σ_λ and Frob_λ^2 act trivially on $T_{\mathfrak{m}_Q} J^Q/I_m$, so the assertion is independent of the choice of cocycle representatives for $c(m,Q)$ and $c(m/\ell,Q)$. Also, it suffices to check the corresponding statement for $c'(m,Q) = \mathrm{Res}_m^{-1} d(m,Q)$ and $c'(m/\ell,Q)$.

Fix division points $\frac{P'(m,Q)}{\ell+1}$ and $\frac{P'(m/\ell,Q)}{\ell+1}$ in $J^Q(\overline{K})$. For any $g \in G_{K[1]}$, $g \cdot d(m,Q) = d(m,Q)$, so there exists $A_g \in T_{\mathfrak{m}_Q} J^Q/I_m$ such that

$$(h-1)A_g = (h-1)(g-1)\frac{P'(m,Q)}{\ell+1} \in T_{\mathfrak{m}_Q} J^Q/I_m, \quad \forall h \in G_{K[m]}. \tag{64}$$

Since $G_{K[m]}$ has no fixed points on $T_{\mathfrak{m}_Q} J^Q/I_m$, (64) uniquely determines A_g, and $g \mapsto A_g$ is a cocycle representing $c'(m,Q)$. We wish to compute A_{σ_λ}. By Proposition 5.3.2(1),

$$(\sigma_\lambda - 1)\frac{P'(m,Q)}{\ell+1} = (T_{\ell_0} - \ell_0 - 1)D_{m/\ell}\,y(m,Q) - T_\ell \frac{P'(m/\ell,Q)}{\ell+1} + T$$

for a uniquely determined torsion point $T \in J^Q[\ell+1]$, and it follows that A_{σ_λ} is the image of T. Since the $\overline{K}$-points of $J^Q[\ell+1]$ have distinct reduction modulo λ, Proposition 5.3.2(2) shows that we can also characterize T as the unique point in $J^Q[\ell+1]$ congruent to

$$(T_\ell - \mathrm{Frob}_\lambda(\ell+1))\frac{P'(m/\ell,Q)}{\ell+1} \pmod{\lambda}.$$

Since $\mathrm{Frob}_\lambda^2 - T_\ell \mathrm{Frob}_\lambda + \ell = 0$ on $J^Q(k_\lambda)$, where $k_\lambda \cong \overline{\mathbb{F}_\ell}$ is the residue field, T is also the unique point in $J^Q[\ell+1]$ congruent modulo λ to

$$(\mathrm{Frob}_\lambda - \mathrm{Frob}_\lambda^{-1})\frac{P'(m/\ell, Q)}{\ell+1} = \mathrm{Frob}_\lambda^{-1} d(m/\ell, Q)(\mathrm{Frob}_\lambda^2) = \mathrm{Frob}_\lambda^{-1} c'(m/\ell, Q)(\mathrm{Frob}_\lambda^2);$$

this shows (3). □

Definition 5.3.4. For a squarefree product m of primes inert in K and coprime to N, let $I_m(f) \subset \mathcal{O}$ be the ideal generated by $a_\ell(f)$ and $\ell + 1$ for all $\ell | m$. Suppose given $\{Q, \epsilon_Q\} \in \mathcal{N}_j$ (notation as before Construction 5.1.2), with $(m, Q) = 1$ and $v_\wp(I_m(f)) \geq j$. If $\nu(N^- Q)$ is even, then the Kolyvagin class

$$\bar{c}_j(m, Q) \in H^1(K^{\Sigma_{Qm}}/K, T_j) \tag{65}$$

is defined to be the image of $c(m, Q)$ under the map $T_{\mathfrak{m}_Q} J^Q / I_m \to T_j$ given by Corollary 4.6.6. If $\nu(N^- Q)$ is odd, then Construction 5.1.2, extended linearly to formal sums of CM points, defines an element

$$\lambda_j(P(m, Q), Q) \in \mathcal{O}_j[\mathrm{Gal}(K[m]/K)].$$

Because $v_\wp(I_m(f)) \geq j$, $\lambda_j(P(m, Q), Q)$ is constant on cosets of $\mathrm{Gal}(K[m]/K[1])$ by Proposition 5.3.2(1) and therefore descends to

$$\lambda'_j(m, Q) \in \mathcal{O}_j[\mathrm{Gal}(K[1]/K)]. \tag{66}$$

The Kolyvagin element is then defined as:

$$\lambda_j(m, Q) = \mathrm{tr}_{K[1]/K} \lambda'_j(m, Q) \in \mathcal{O}_j. \tag{67}$$

When $m = Q = 1$, then j can be arbitrarily large, giving a class $\bar{c}_\infty(1, 1) \in H^1(K, T_f)$ or an element $\lambda_\infty(1, 1) \in \mathcal{O}$.

Remark 5.3.5. When $Q = N^- = 1$ and $\mathcal{O}_f = \mathbb{Z}$, this agrees with Kolyvagin's original construction described in [45, §1].

5.3.6. We now consider the local properties of the classes $\bar{c}_j(m, Q)$ at places $v | p$ of K. Recall from Remark 5.1.3 the abelian variety A_f such that $T_f = T_p A_f \otimes_{\mathcal{O}_f \otimes \mathbb{Z}_p} \mathcal{O}$. For any finite extension L of $\mathbb{Q}_p$, define

$$H^1_f(L, T_f) \coloneqq \mathrm{im}\left(A_f(L) \otimes_{\mathcal{O}_f} \mathcal{O} \to H^1(L, T_f)\right). \tag{68}$$

Also, for any $j \geq 1$, let

$$\begin{aligned} H^1_f(L, T_j) &\coloneqq \mathrm{im}\left(H^1_f(L, T_f) \to H^1(L, T_j)\right) \\ &= \mathrm{im}\left(A_f(L) \otimes \mathcal{O}/\wp^j \to H^1(L, T_j)\right), \end{aligned}$$

which is also the kernel of the composite

$$H^1(L, T_j) = H^1(L, A_f[\wp^j]) \to H^1(L, A_f(\overline{L})).$$

Proposition 5.3.7. *Let $v | p$ be a place of K. For m, Q, and j as in Definition 5.3.4, $\bar{c}_j(m, Q)_v$ lies in $H^1_f(K_v, T_j)$.*

Proof. Extend v to a place of $K[m]$. By [51, Proposition 3.8] and inflation-restriction (cf. the proof of [31, Proposition 6.2(1)]), the natural map $H^1(K_v, A_f(\overline{K})) \to H^1(K[m]_v, A_f(\overline{K}))$ is injective, so it is enough to show that

$$\mathrm{Res}_{K[m]_v} \bar{c}_j(m, Q) \in H^1_f(K[m]_v, T_j).$$

Since $\mathrm{Res}_{K[m]} c(m, Q) = [K[1] : K] d(m, Q)$, and $d(m, Q)$ is defined as the Kummer image of $P'(m, Q)$, it suffices to show:

Claim. The image of the composite map

$$J^Q(K[m]_v) \to H^1(K[m]_v, J^Q[p^j]) \xrightarrow{\beta} H^1(K[m]_v, T_j) \tag{69}$$

lies in $H^1_f(K[m]_v, T_j)$.

Here β is induced by the map of $G_{\mathbb{Q}}$-modules

$$J^Q[p^j] = T_p J^Q/p^j \twoheadrightarrow T_{\mathfrak{m}_Q} J^Q/p^j \xrightarrow{(56)} T_j. \tag{70}$$

Let $\mathcal{J}^Q$ and $\mathcal{A}_f$ be the Néron models of J^Q and A_f, respectively, over $\operatorname{Spec} \mathcal{O}_{K[m]_v}$. By [60, Corollaire 3.3.6], the map (70) of $G_{K[m]_v}$-modules $J^Q[p^j] \to T_j$ extends to a map of finite flat group schemes $\mathcal{J}^Q[p^j] \to \mathcal{A}_f[\wp^j]$ over $\operatorname{Spec} \mathcal{O}_{K[m]_v}$; moreover (69) fits into a commutative diagram

$$\begin{array}{ccccc} \mathcal{J}^Q(\mathcal{O}_{K[m]_v}) & \longrightarrow & H^1_{\mathrm{fppf}}(\operatorname{Spec} \mathcal{O}_{K[m]_v}, \mathcal{J}^Q[p^j]) & \longrightarrow & H^1_{\mathrm{fppf}}(\operatorname{Spec} \mathcal{O}_{K[m]_v}, \mathcal{A}_f[\wp^j]) \\ \| & & \downarrow & & \downarrow \\ J^Q(K[m]_v) & \longrightarrow & H^1(K[m]_v, J^Q[p^j]) & \longrightarrow & H^1(K[m]_v, T_j). \end{array}$$

Here H^1_{fppf} refers to sheaf cohomology computed in the fppf topology. By [32, Lemma 7], the image of the right vertical arrow is $H^1_f(K[m]_v, T_j)$, and the claim follows. ☐

5.3.8. Recall that ϵ_f is the global root number of f. For applications to the parity conjecture, we will require the following:

Proposition 5.3.9. *Let m be a squarefree product of primes inert in K and suppose $j \le v_\wp(I_m(f))$. If $\nu(N^-)$ is even, then $\bar{c}_j(m,1)$ lies in the $\epsilon_f \cdot (-1)^{\nu(m)+1}$-eigenspace for the action of the generator $\tau \in \operatorname{Gal}(K/\mathbb{Q})$. If $\nu(N^-)$ is odd and $\lambda_j(m,1) \neq 0$, then $\epsilon_f = (-1)^{\nu(m)}$.*

Proof. Since f is a newform of level N, the maps $\varphi : J^{N^+,N^-} \to A_f$ or $\varphi : \mathbb{Z}[X_{N^+,N^-}]^0 \to \mathcal{O}$ from Remark 5.1.3 satisfy

$$\varphi \circ w_N = \epsilon_f \cdot (-1)^{\nu(N^-)+1} \cdot \varphi, \tag{71}$$

where $w_N = \prod_{\ell | N^+} w_\ell \prod_{\ell | N^-} U_\ell$ is the Atkin-Lehner involution; the minus signs appear because the local root number of f at $\ell | N^-$ is the *negative* of the U_ℓ eigenvalue for the quaternionic modular form on $B^\times_{N^-}$ corresponding to f.

Now suppose $\nu(N^-)$ is even. Choose a lift $\tau \in \operatorname{Gal}(K[m]/\mathbb{Q})$; then, for all $y(m) \in \mathrm{CM}_{N^+,N^-}(m)$, we claim that $w_N \tau y(m)$ lies in $\mathrm{CM}_{N^+,N^-}(m)$ as well. Indeed, this is clear from Definition 4.7.3: applying $w_N\tau$ reverses all the orientation conditions (1)-(3), but then we can replace the action of $\mathcal{O}_{m,K}$ with its complex conjugate so the conditions are again satisfied. Since $\bar{c}_j(m,1)$ is independent of the choice of $y(m)$ by Proposition 5.3.3(2) and the transitivity of the G_K-action on $\mathrm{CM}_{N^+,N^-}(m)$, the calculation in [31, Proposition 5.4] applies to show (using (71) and Remark 5.1.3) that $\tau \bar{c}_j(m,1) = -\epsilon_f \cdot (-1)^{\nu(m)} \bar{c}_j(m,1)$, as desired.

The case when $\nu(N^-)$ is odd is similar: if $y(m) \in \mathrm{CM}_{N^+,N^-}(m)$ is represented by a pair (R,ϕ) satisfying Definition 4.7.7 for the fixed embedding $K \hookrightarrow B$, then $w_N y(m)$ is represented by the pair (R, ϕ^{op}), with all orientations reversed. But (R,ϕ^{op}) satisfies Definition 4.7.7 for the embedding $K \xrightarrow{\tau} K \hookrightarrow B$, which is conjugate to $K \hookrightarrow B$ by an element of $B^\times(\mathbb{Q})$, so $w_N y(m)$ lies in the image of $\mathrm{CM}_{N^+,N^-}(m) \to X_{N^+,N^-}$. Applying formally the calculations in [31, Proposition 5.4], it follows from (71) and Remark 5.1.3 that $\lambda_j(m,1) = \epsilon_f \cdot (-1)^{\nu(m)} \lambda_j(m,1)$, which gives the claim. ☐

Definition 5.3.10. An ultraprime $\mathsf{l} \in \mathsf{M}_{\mathbb{Q}}$ is called Kolyvagin-admissible if

$$\mathrm{Frob}_{\mathsf{l}} \in \operatorname{Gal}(K(T_f)/\mathbb{Q})$$

is a complex conjugation. We will also abusively write l for the unique corresponding ultraprime in M_K. A Kolyvagin-admissible set is a finite set of Kolyvagin-admissible ultraprimes, and the collection of all Kolyvagin-admissible sets is denoted K. We will use multiplicative notation for the Kolyvagin-admissible sets, i.e. if $\mathsf{m}, \mathsf{n} \in \mathsf{K}$ with $\mathsf{m} \cap \mathsf{n} = \emptyset$, we write mn for the union $\mathsf{m} \cup \mathsf{n}$. Similarly, if $\mathsf{l} \notin \mathsf{m}$ is Kolyvagin-admissible, we write ml for $\mathsf{m} \cup \{\mathsf{l}\}$. We write 1 for $\emptyset \in \mathsf{K}$.

5.3.11. If l is Kolyvagin-admissible, then the local cohomology

$$\mathsf{H}^1(K_{\mathsf{l}}, T_f)$$

is free of rank four over $\mathcal{O}$, and carries a natural action of the complex conjugation $\tau \in \mathrm{Gal}(K/\mathbb{Q})$. It has a canonical splitting of the finite-singular exact sequence:

$$\mathsf{H}^1(K_{\mathsf{l}}, T_f) = \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f) \oplus \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f),$$

defined as follows. If the sequence $(\ell_n)_{n\in\mathbb{N}}$ represents l, then for any j and for $\mathfrak{F}$-many n, Frob_{ℓ_n} acts as complex conjugation on T_j, and

$$H^1_{\mathrm{tr}}(K_{\ell_n}, T_j) = \ker\left(H^1(K_{\ell_n}, T_f/\pi^j) \to H^1(K[\ell_n]_{\lambda_n}, T_f/\pi^j)\right)$$

is isomorphic to $H^1(I_{\ell_n}, T_j)^{\mathrm{Frob}^2_{\ell_n}=1}$, where λ_n is a prime of $K[\ell_n]$ over ℓ_n. Then

$$\mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f) = \varprojlim \mathcal{U}\left(\left\{H^1_{\mathrm{tr}}(K_{\ell_n}, T_j)\right\}_{n\in\mathbb{N}}\right) \subset \mathsf{H}^1(K_{\mathsf{l}}, T_f)$$

is our transverse subspace. We denote by $\mathrm{loc}_{\mathsf{l}}^{\pm}$ and $\partial_{\mathsf{l}}^{\pm}$ the composites $\mathsf{H}^1(K, T_f) \to \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f)^{\pm}$ and $\mathsf{H}^1(K, T_f) \to \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f)^{\pm}$, respectively, where the superscript $\pm$ refers to the Frobenius eigenspace with eigenvalue ± 1. The codomain of each is free of rank one over $\mathcal{O}$.

Let $\mathsf{S} \subset \mathsf{M}_K$ be the set of constant ultraprimes $\underline{v}$ such that $v|Np\infty$. We will consider the Kolyvagin-transverse Selmer structure $(\mathcal{F}(\mathsf{m}), \mathsf{S}\cup\mathsf{m})$ on T_f, for any $\mathsf{m} \in \mathsf{K}$:

$$\mathsf{H}^1_{\mathcal{F}(\mathsf{m})}(K_{\mathsf{v}}, T_f) = \begin{cases} \ker\left(H^1(K_v, T_f) \to \frac{H^1(K_v, V_f)}{H^1_f(K_v, V_f)}\right), & \mathsf{v} = \underline{v} \in \mathsf{S}, \\ \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f), & \mathsf{v} = \mathsf{l} \in \mathsf{m}, \\ \mathsf{H}^1_{\mathrm{unr}}(K_{\mathsf{v}}, T_f), & \text{otherwise}. \end{cases} \tag{72}$$

Here $H^1_f(K_v, V_f)$ is the Bloch-Kato local condition on $V_f = T_f \otimes \mathbb{Q}_p$; when $v|p$, the notation is consistent with (68) by [4, Examples 3.10.1, 3.11]. Note that $(\mathcal{F}(\mathsf{m}), \mathsf{S}\cup\mathsf{m})$ is a self-dual Selmer structure by the self-duality of $H^1_f(K_v, V_f)$ – the transverse local conditions at m are self-dual by [49, Proposition 1.3.2]. If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}\cup\mathsf{m}}$, then we denote by $(\mathcal{F}(\mathsf{m}, \mathsf{Q}), \mathsf{S}\cup\mathsf{m}\cup\mathsf{Q})$ the modified Selmer structure of (3.2.2).

5.3.12. Let $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}_{\mathsf{S}\cup\mathsf{m}}$, and fix representatives $(Q_n)_{n\in\mathbb{N}}$ and $(m_n)_{n\in\mathbb{N}}$; for $\mathfrak{F}$-many n, $Q_n m_n$ is a squarefree product of primes inert in K. Our patched Kolyvagin class is the element

$$\kappa(\mathsf{m}, \mathsf{Q}) \in \mathsf{H}^1(K^{\mathsf{S}\cup\mathsf{m}\cup\mathsf{Q}}/K, T_f)$$

whose image in $\mathsf{H}^1(K^{\mathsf{S}\cup\mathsf{m}\cup\mathsf{Q}}/K, T_j)$ is represented by the sequence of images of the classes $\overline{c}_j(m_n, Q_n)$, well-defined for $\mathfrak{F}$-many n.

If $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}_{\mathsf{S}\cup\mathsf{m}}$, then we similarly set

$$\lambda(\mathsf{m}, \mathsf{Q}) \in \mathcal{O} \simeq \varprojlim \mathcal{U}\left(\left\{\mathcal{O}/\pi^j\right\}\right)$$

to be the element whose image in $\mathcal{O}/\pi^j$ is represented by the sequence $\lambda_j(m_n, Q_n)$.

Proposition 5.3.13. *For any* $\mathsf{m} \in \mathsf{K}$ *and* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}_{\mathsf{S}\cup\mathsf{m}}$*,*

$$(\kappa(\mathsf{m}, \mathsf{Q})) \subset \mathrm{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f).$$

Moreover:

(1) For all $\mathsf{m} \in \mathsf{K}$ *and all Kolyvagin-admissible* $\mathsf{l} \notin \mathsf{m}$*, and all* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}_{\mathsf{S}\cup\mathsf{ml}}$*,*

$$(\mathrm{loc}_{\mathsf{l}}^{\pm}(\kappa(\mathsf{m}, \mathsf{Q}))) = (\partial_{\mathsf{l}}^{\mp}(\kappa(\mathsf{ml}, \mathsf{Q})))$$

as submodules of $\mathcal{O}$.

(2) For all $\mathsf{m} \in \mathsf{K}$ *and all* $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}^{\nu(N^-)}_{\mathsf{S}\cup\mathsf{m}}$*,*

$$(\partial_{\mathsf{q}}(\kappa(\mathsf{m}, \mathsf{Qq}))) = (\lambda(\mathsf{m}, \mathsf{Q}))$$

as submodules of $\mathcal{O}$.

(3) For all $\mathsf{m} \in \mathsf{K}$ *and all* $\{\mathsf{Qq}, \epsilon_{\mathsf{Qq}}\} \in \mathsf{N}_{\mathsf{S}\cup\mathsf{m}}^{\nu(N^-)-1}$,

$$(\mathrm{loc}_{\mathsf{q}}(\kappa(\mathsf{m},\mathsf{Q}))) = (\lambda(\mathsf{m},\mathsf{Qq}))$$

as submodules of $\mathcal{O}$.

In particular, for any fixed m, $(\kappa(\mathsf{m},\cdot), \lambda(\mathsf{m},\cdot))$ *forms a bipartite Euler system with parity* $\nu(N^-)$ *for the triple* $(T_f, \mathcal{F}(\mathsf{m}), \mathsf{S}\cup\mathsf{m})$.

Proof. To prove the first claim, we verify the local conditions for each $\mathsf{v} \in \mathsf{S}\cup\mathsf{m}\cup\mathsf{Q}$. If $\mathsf{v} = \underline{v}$ for a prime $v|N\infty$, then the local condition is all of $H^1(K_v, T_f)$, so there is nothing to show. (Indeed, $H^1(K_v, T_f)$ is torsion for any $v|N\infty$.) If $\mathsf{v} = \underline{v}$ for a prime $v|p$, then we have

$$\mathrm{Res}_{\mathsf{v}}\, \kappa(\mathsf{m},\mathsf{Q}) \in H^1_f(K_v, T_f) = \varprojlim_j H^1_f(K_v, T_j)$$

by Proposition 5.3.7.

If $\mathsf{v} = \mathsf{l} \in \mathsf{m}$, then, adopting as well the notation of (5.3.1), the class $c(m_n, Q_n)$ is zero when restricted to $K[m_n]_{\lambda_n}$ because $D_{\ell_n} = \ell_n(\ell_n+1)/2$ on $\mathbb{F}_{\lambda_n}$; it follows that $\mathrm{Res}_{\mathsf{v}}\, \kappa(\mathsf{m},\mathsf{Q}) \in \mathsf{H}^1_{\mathrm{tr}}(K_{\mathsf{l}}, T_f)$. The local conditions at $\mathsf{q} \in \mathsf{Q}$ are satisfied by Construction 5.1.2(1) because every factor of Q_n splits completely in $K[m_n]$; for the same reason, (2, 3) follow from Construction 5.1.2(2, 3). Moreover (1) is clear from Proposition 5.3.3(3). □

Remark 5.3.14. (1) The condition (disc) is only used to control the structure of the Galois group $\mathrm{Gal}(K[m]/K[1])$ in (5.3.1) above. In particular, one sees from the construction that the bipartite Euler system $(\kappa(1,\cdot), \lambda(1,\cdot))$ makes sense in the generality of (5.1.1). We will use this observation to prove the p-converse theorem (Corollary 8.1.3 below) without the assumption (disc).

(2) Note as well that, under (spl) and (ord), $(\kappa(1,\cdot), \lambda(1,\cdot))$ may be viewed as a specialization of $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$. Indeed, let $\mathbb{1} : \Lambda \to \mathcal{O}$ be the specialization at the trivial character. Then by the usual Heegner point norm relations [23, Proposition 3.10] (adapted to our context as in the proof of Proposition 5.2.2), $\mathbb{1}(\boldsymbol{\lambda}(\mathsf{Q})) = (\alpha_p - 1)^2(\lambda(1,\mathsf{Q}))$ and $\mathbb{1}(\boldsymbol{\kappa}(\mathsf{Q})) = (\alpha_p - 1)^2(\kappa(1,\mathsf{Q}))$ when $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$, respectively.

6. Deformation theory

In §4, we used geometric methods to produce level-raising congruences on the level of so-called "weak eigenforms" (i.e. ring maps from a Hecke algebra to $\mathcal{O}/\pi^j$), which typically do not lift to characteristic zero. To prove the main results, we also need to be able to π-adically approximate our fixed modular form f by genuine level-raised eigen-newforms. In this section, we provide this input via the relative deformation theory of Fakhruddin-Khare-Patrikis [28].

6.1. Review of relative deformation theory.

6.1.1. Let f, N, $\wp$, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5. However, for this section only we allow $p|N$ (since it does not change any of the statements of our results).

We will also consider the hypothesis:

(TW) If $p = 3$, then $\overline{T}_f$ is absolutely irreducible when restricted to $G_{\mathbb{Q}(\sqrt{-3})}$.

Consider the adjoint representation

$$L = \mathrm{ad}^0\, T_f$$

and its $\mathcal{O}$-dual, $L^\dagger \simeq L(1)$, and let $\overline{L}$ and $\overline{L}^* \simeq L^\dagger/\pi$ be the associated residual representations. Although we continue to assume $\overline{T}_f$ is absolutely irreducible, we do *not* assume $\overline{L}$ is absolutely irreducible.

6.1.2. We now recall the construction in [28, Proposition 4.7] of certain local conditions for the Galois cohomology of L. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete local Noetherian algebras R with a map $\mathcal{O} \to R$ inducing an isomorphism on residue fields. Fix a basis for T_f over $\mathcal{O}$, which identifies the Galois action on T_f with a homomorphism $\rho_f : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$, and let $\overline{\rho}_f$ be the reduction of ρ_f modulo π. For all primes $\ell \neq p$, let $\widetilde{R}_\ell \in \mathcal{C}_{\mathcal{O}}$ denote the framed universal deformation ring of $\overline{\rho}_f|_{G_{\mathbb{Q}_\ell}}$, of fixed determinant χ; that is,

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(\widetilde{R}_\ell, A) = \left\{\rho_A : G_{\mathbb{Q}_\ell} \to GL_2(A) \,:\, \rho_A \otimes_A A/\mathfrak{m}_A = \overline{\rho}_f|_{G_{\mathbb{Q}_\ell}},\ \det\rho_A = \chi\right\},\quad A \in \mathcal{C}_{\mathcal{O}}.$$

Similarly, for $\ell = p$, let $\widetilde{R}_p \in \mathcal{C}_{\mathcal{O}}$ denote the framed potentially semistable deformation ring of $\overline{T}_f|_{G_{\mathbb{Q}_p}}$ (constructed in [44]), with fixed Hodge-Tate weights 0 and 1, fixed determinant χ, and fixed Galois type agreeing with that of $\rho_f|_{G_{\mathbb{Q}_p}}$. For any ℓ, the generic fiber $\widetilde{R}_\ell[1/\pi]$ is generically formally smooth of pure dimension $3 + \delta_{\ell=p}$. The representation $\rho_f|_{G_{\mathbb{Q}_\ell}}$ defines a formally smooth point y_ℓ of $\operatorname{Spec} \widetilde{R}_\ell[1/\pi]$ by [1, Theorem D, Proposition 1.2.2]; let $\operatorname{Spec} R_\ell \subset \operatorname{Spec} \widetilde{R}_\ell$ be the Zariski closure of the irreducible component of $\operatorname{Spec} \widetilde{R}_\ell[1/\pi]$ containing y_ℓ.

Proposition 6.1.3. *There exists an open set $Y_\ell \subset \operatorname{Spec} R_\ell(\mathcal{O})$ containing y_ℓ, and a collection of submodules $Z_j = Z_j(y_\ell) \subset Z^1(G_{\mathbb{Q}_\ell}, L/\pi^j)$ which are free of rank $3+\delta_{\ell=p}$ over $\mathcal{O}/\pi^j$ for all $j \geq 0$, satisfying the following properties.*

1. *Let Y_n^ℓ be the image of Y_ℓ in $\operatorname{Spec} R_\ell(\mathcal{O}/\pi^n)$ and denote by $\pi_{n,j}^{Y_\ell} : Y_{n+j}^\ell \to Y_n^\ell$ the reduction maps for $n, j \geq 0$. Then given $j_0 > 0$, there exists $n_0 > 0$ such that, for all $n \geq n_0$ and all $0 \leq j \leq j_0$, the fibers of $\pi_{n,j}^{Y_\ell}$ are nonempty principal homogeneous spaces for Z_j.*
2. *The natural $\mathcal{O}$-module maps $\mathcal{O}/\pi^j \to \mathcal{O}/\pi^{j-1}$ and $\mathcal{O}/\pi^{j-1} \to \mathcal{O}/\pi^j$ induce surjections $Z_j \twoheadrightarrow Z_{j-1}$ and inclusions $Z_{j-1} \hookrightarrow Z_j$.*
3. *If $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is unramified, then Z_j is the subspace of unramified cocycles, and Y_ℓ can be chosen so that (1) holds with $n_0 = 1$ for all j_0.*

Remark 6.1.4. We note that, although the open subset $Y_\ell \subset \operatorname{Spec} R_\ell(\mathcal{O})$ is not uniquely determined by the property (1) in Proposition 6.1.3, the submodules $Z_j(y_\ell)$ depend only on y_ℓ. Indeed, if j is fixed, then for n sufficiently large the fiber of $Y_{n+j}^\ell \to Y_n^\ell$ over $y_\ell \pmod{\pi^n}$ is the fiber of $\operatorname{Spec} R_\ell(\mathcal{O}/\pi^{n+j}) \to \operatorname{Spec} R_\ell(\mathcal{O}/\pi^n)$ over $y_\ell \pmod{\pi^n}$, which depends only on y_ℓ, and this fiber determines $Z_j(y_\ell)$.

Proof. The existence of Y_ℓ and Z_j satisfying (1) and (2) is proved in [28, Proposition 4.7], so we consider (3). Note $\ell \neq p$ by the hypothesis that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is unramified. Let $\widetilde{R}_\ell \twoheadrightarrow R_\ell^{\mathrm{unr}}$ be the quotient parametrizing unramified deformations; then R_ℓ^{unr} is formally smooth over $\mathcal{O}$ of dimension 3, and it follows that $R_\ell = R_\ell^{\mathrm{unr}}$. The formal smoothness of R_ℓ^{unr} then immediately implies that properties (1) and (2) are satisfied with $Z_j = Z^1_{\mathrm{unr}}(G_{\mathbb{Q}_\ell}, L/\pi^j)$ the space of unramified cocycles, $Y_\ell = \operatorname{Spec} R_\ell(\mathcal{O})$, and $n_0 = 1$ for all j_0. □

6.1.5. For all primes ℓ and all $j \geq 0$, let $H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^j)$ be the image of the subset $Z_j(y_\ell) \subset Z^1(\mathbb{Q}_\ell, L/\pi^j)$ of Proposition 6.1.3. Also let

$$H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L) = \varprojlim_j H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^j).$$

Proposition 6.1.6. *For all primes ℓ, the quotient*

$$\frac{H^1(\mathbb{Q}_\ell, L)}{H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L)}$$

is π-torsion-free.

Proof. The π-torsion of $H^1(\mathbb{Q}_\ell, L)/H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L)$ is identified with the inverse limit of the kernels

$$K_j := \ker\left(\frac{H^1(\mathbb{Q}_\ell, L/\pi^j)}{H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^j)} \xrightarrow{\pi_*} \frac{H^1(\mathbb{Q}_\ell, L/\pi^{j+1})}{H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^{j+1})}\right),$$

where π_* is the map induced by $\pi : L/\pi^j \to L/\pi^{j+1}$. Since $Z_j(y_\ell)$ contains all coboundaries,

$$K_j = \ker\left(\frac{Z^1(\mathbb{Q}_\ell, L/\pi^j)}{Z_j(y_\ell)} \xrightarrow{\pi_*} \frac{Z^1(\mathbb{Q}_\ell, L/\pi^{j+1})}{Z_{j+1}(y_\ell)}\right).$$

So it suffices to show

$$\pi_* Z^1(\mathbb{Q}_\ell, L/\pi^j) \cap Z_{j+1}(y_\ell) = Z_j(y_\ell). \tag{73}$$

Now $\pi_* Z^1(\mathbb{Q}_\ell, L/\pi^j) \cap Z_{j+1}(y_\ell)$ is contained in $Z_{j+1}(y_\ell)[\pi^j] \approx (\mathcal{O}/\pi^j)^{3+\delta_{\ell=p}}$, and also contains $Z_j(y_\ell)$ by Proposition 6.1.3(2). By counting ranks, we see that (73) holds, as desired. □

6.1.7. Now suppose q is a j-admissible prime with sign ϵ_q, and let $\underline{\epsilon}_q$ denote the unramified character of $G_{\mathbb{Q}_q}$ sending Frob_q to ϵ_q.

Recall from Definition 4.6.2 the uniquely determined subspace $\mathrm{Fil}^+_{q,\epsilon_q} T_j \subset T_j$ free of rank one over $\mathcal{O}/\pi^j$, on which $G_{\mathbb{Q}_q}$ acts by $\chi\underline{\epsilon}_q$. We define

$$\mathrm{Fil}^+_{q,\epsilon_q} L/\pi^j = \mathrm{Hom}\left(T_j/\mathrm{Fil}^+_{q,\epsilon_q} T_j, \mathrm{Fil}^+_{q,\epsilon_q} T_j\right) \subset L/\pi^j,$$

a free $\mathcal{O}/\pi^j$-submodule of rank one, and

$$H^1_{\mathrm{ord},\epsilon_q}(\mathbb{Q}_q, L/\pi^j) = \mathrm{im}\left(H^1(\mathbb{Q}_q, \mathrm{Fil}^+_{q,\epsilon_q} L/\pi^j) \to H^1(\mathbb{Q}_q, L/\pi^j)\right).$$

As always the subscripts ϵ_q will usually be omitted when they are clear from context.

A representation $\rho : G_{\mathbb{Q}_q} \to GL_2(A)$, for any $\mathbb{Z}_p$-algebra A, will be called *Steinberg* if it is conjugate to a representation of the form $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ with $*$ a ramified cocycle in $H^1(\mathbb{Q}_q, A(1))$. We denote by $\mathrm{Spec}\, R^{\mathrm{ord}}_{q,\epsilon_q} \subset \mathrm{Spec}\, \widetilde{R}_q$ the Zariski closure of the union of the irreducible components of $\mathrm{Spec}\, \widetilde{R}_q[1/\pi]$ that contain a point corresponding to a Steinberg representation twisted by $\underline{\epsilon}_q$.

Proposition 6.1.8. *Suppose q is j-admissible with sign ϵ_q and $\rho_n \in \mathrm{Spec}\, R^{\mathrm{ord}}_{q,\epsilon_q}(\mathcal{O}/\pi^n)$ is a lift of $T_j|_{G_{\mathbb{Q}_q}}$ (with any framing), for some $n \geq j$. Then for all $r \leq j$, the fiber of the reduction map*

$$\mathrm{Spec}\, R^{\mathrm{ord}}_{q,\epsilon_q}(\mathcal{O}/\pi^{n+r}) \to \mathrm{Spec}\, R^{\mathrm{ord}}_{q,\epsilon_q}(\mathcal{O}/\pi^n)$$

over ρ_n is a nonempty principal homogeneous space under

$$Z^1_{\mathrm{ord},\epsilon_q}(\mathbb{Q}_q, L/\pi^r) := \ker\left(Z^1(\mathbb{Q}_q, L/\pi^r) \to \frac{H^1(\mathbb{Q}_q, L/\pi^r)}{H^1_{\mathrm{ord},\epsilon_q}(\mathbb{Q}_q, L/\pi^r)}\right),$$

which is free of rank 3 over $\mathcal{O}/\pi^r$.

Proof. Without loss of generality, we may assume that Frob_q acts on T_j via the diagonal matrix $\begin{pmatrix} q\epsilon_q & 0 \\ 0 & \epsilon_q \end{pmatrix}$.

By the explicit calculations in [68, Propositions 5.5, 5.6], $R^{\mathrm{ord}}_{q,\epsilon_q}$ is a power series ring $\mathcal{O}[\![X, Y, B]\!]$ with universal deformation

$$\rho^{\mathrm{ord}}_q(\sigma) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$$
$$\rho^{\mathrm{ord}}_q(\phi) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} q\epsilon_q & 0 \\ 0 & \epsilon_q \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix},$$

where σ is a generator of tame inertia and ϕ is a lift of arithmetic Frobenius. (We note this calculation crucially uses $q \not\equiv 1 \pmod p$.) In particular, $\mathrm{Spec}\, R^{\mathrm{ord}}_{q,\epsilon_q}$ is formally smooth, and by the discussion in [28, §4.1, Lemma 4.5], the fiber of the reduction map in the proposition over ρ_n is a principal homogeneous space under a certain submodule Z'_r of $Z^1(\mathbb{Q}_q, L/\pi^r)$ which is free of rank three over $\mathcal{O}/\pi^r$ and contains all coboundaries. It is also clear that the cocycles of the form $\phi \mapsto 0, \sigma \mapsto \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ are contained in Z'_r, and these generate $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L/\pi^r)$. By comparing ranks, we find $Z'_r = Z^1_{\mathrm{ord},\epsilon_q}(\mathbb{Q}_q, L/\pi^r)$. □

6.1.9. Let N_j denote the set of weakly admissible pairs $\{Q, \epsilon_Q\}$ such that each $q|Q$ is j-admissible with sign $\epsilon_Q(q)$ (notation slightly different from (5.1.1)). If $\{Q, \epsilon_Q\} \in N_j$ is a weakly admissible pair, we will consider the (non-patched) Selmer groups

$$\mathrm{Sel}_Q(\mathbb{Q}, L/\pi^j) := \ker\left(H^1(\mathbb{Q}^{S\cup Q}/\mathbb{Q}, L/\pi^j) \to \prod_{\ell|Np} \frac{H^1(\mathbb{Q}_\ell, L/\pi^j)}{H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^j)} \times \prod_{q|Q} \frac{H^1(\mathbb{Q}_q, L/\pi^j)}{H^1_{\mathrm{ord},\epsilon_Q(q)}(\mathbb{Q}_q, L/\pi^j)}\right),$$

where S is the set of places dividing $Np\infty$. We also have the dual Selmer group $\mathrm{Sel}_Q(\mathbb{Q}, L^*[\pi^j])$ defined using orthogonal complement local conditions.

Finally, define, for any finite set of places Σ containing all $v|Np\infty$:

$$Ш^1_\Sigma = \ker\left(H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \overline{L}^*) \to \prod_{v\in\Sigma} H^1(\mathbb{Q}_v, \overline{L}^*)\right). \tag{74}$$

Proposition 6.1.10. *There exists a finite set of places Σ, containing all $v|Np\infty$, such that*

$$Ш^1_\Sigma = 0.$$

Proof. We claim it suffices to show that

$$H^1(\mathbb{Q}(\overline{T}_f)/\mathbb{Q}, \overline{L}^*) = 0, \tag{75}$$

where this cohomology group makes sense because $G_{\mathbb{Q}(\overline{T}_f)}$ acts trivially on $\overline{L}^*$ as $\det \rho_f = \chi$. Indeed, suppose Σ is a finite set of places containing all $v|Np\infty$, and $c \in Ш^1_\Sigma$ is nonzero. Then by (75), the restriction map

$$H^1(\mathbb{Q}, \overline{L}^*) \to H^1(\mathbb{Q}(\overline{T}_f), \overline{L}^*)$$

is injective, so c restricts to a nonzero homomorphism $c' : G_{\mathbb{Q}(\overline{T}_f)} \to \overline{L}^*$. Let $\ell \notin \Sigma$ be a prime which is totally split in $\mathbb{Q}(\overline{T}_f)$ but not in the extension cut out by c' (which is possible by the Chebotarev Density Theorem). Then $\mathrm{Res}_\ell\, c \neq 0$, hence the inclusion $Ш^1_{\Sigma\cup\{\ell\}} \subset Ш^1_\Sigma$ is strict. Since $Ш^1_\Sigma$ is finite, iterating this process produces a set Σ such that $Ш^1_\Sigma = 0$.

We now show (75). If $\mu_p \not\subset \mathbb{Q}(\overline{L})$, then the center of $\mathrm{Gal}(\mathbb{Q}(\overline{T}_f)/\mathbb{Q})$ contains elements that act by nontrivial scalars on $\overline{L}^*$, and (75) follows from inflation-restriction. So suppose that $\mu_p \subset \mathbb{Q}(\overline{L})$. In particular, the projective image $\overline{G} = \mathrm{Gal}(\mathbb{Q}(\overline{L})/\mathbb{Q})$ of $\overline{\rho}_f$ has a cyclic quotient of order $p-1$.

Since $\overline{T}_f$ is absolutely irreducible, and since every subgroup of $PGL_2(\mathcal{O}/\pi)$ is naturally a subgroup of $PSL_2(\overline{\mathbb{F}}_p)$, a classical result of Dickson [73, Chapter 3, Theorem 6.25] implies that $\overline{G}$ is isomorphic to a dihedral group, or A_4, S_4, or A_5, or either $PSL_2(\mathbb{F}_{p^n})$ or $PGL_2(\mathbb{F}_{p^n})$ for some n. Because $PSL_2(\mathbb{F}_q)$ is simple for $q > 3$, for $\overline{G}$ to have a cyclic quotient of order $p-1$ requires that $p = 3$ and $\overline{G}$ is isomorphic to either a dihedral group, or $PGL_2(\mathbb{F}_{p^n})$ for some n. (Recall here that S_4 is isomorphic to $PGL_2(\mathbb{F}_3)$.) If the order of $\overline{G}$ is prime to p, then (75) will hold automatically, so we may assume without loss of generality that $\overline{G}$ is isomorphic to $PGL_2(\mathbb{F}_{p^n})$ for some n.

Let $G^{(1)} = \mathrm{Gal}(\mathbb{Q}(\overline{T}_f)/\mathbb{Q}(\mu_p)) \subset SL_2(\mathcal{O}/\pi)$. The natural map $G^{(1)} \to \overline{G}$ has kernel and cokernel of cardinality at most 2, so, comparing with the classification in [73, Chapter 3, Theorem 6.17, Case II], we conclude that $G^{(1)}$ contains a subgroup H isomorphic to $SL_2(\mathbb{F}_{p^n})$ with index at most 2. Moreover, $\mathbb{F}_{p^n}$ is a subfield of $\mathcal{O}/\pi$ and the embedding $H \hookrightarrow SL_2(\mathcal{O}/\pi)$ is $GL_2(\mathcal{O}/\pi)$-conjugate to the standard one by [73, Chapter 3, Lemma 6.18]. In particular, $H^1(H, \overline{L}^*) = H^1(H, \overline{L}) = 0$ by [24, Lemma 2.48], and because H contains a p-Sylow subgroup of $G^{(1)}$, $H^1(G^{(1)}, \overline{L}^*) = 0$ as well. This shows (75) because of the inflation-restriction exact sequence:

$$0 = H^1(\mathbb{Q}(\mu_p)/\mathbb{Q}, (\overline{L}^*)^{G_{\mathbb{Q}(\mu_p)}}) \to H^1(\mathbb{Q}(\overline{T}_f)/\mathbb{Q}, \overline{L}^*) \to H^1(\mathbb{Q}(\overline{T}_f)/\mathbb{Q}(\mu_p), \overline{L}^*) = H^1(G^{(1)}, \overline{L}^*).$$

□

Theorem 6.1.11. *Let f be as above, satisfying (TW). Suppose given a weakly admissible pair $\{Q, \epsilon_Q\} \in N_j$ and an integer $k \leq j$ satisfying the following conditions:*

(1) The map

$$\mathrm{Sel}_Q(L^*[\pi^k]) \to \mathrm{Sel}_Q(\overline{L}^*)$$

is identically zero.

(2) For each $\ell|Np$, let $n_0(\ell, k)$ be a number satisfying the conclusion of Proposition 6.1.3(1) for $j_0 = k$, and let $N_0(k) = \max_{\ell|Np}\{n_0(\ell, k)\}$. Then $j - k + 1 \geq N_0(k)$.

Then there is an eigen-newform g of weight 2, level NQ, and trivial character, with a prime $\wp_g$ of the ring of integers of its coefficient field $\mathcal{O}_g$, such that:

- *The completion $\mathcal{O}_{g,\wp_g}$ is a subring of $\mathcal{O}$.*
- *There is a congruence of Galois representations (in some basis)*

$$\rho_f \equiv \rho_{g,\wp_g} \pmod{\pi^{j-k+1}}.$$

- *The inertial types of $\rho_{g,\wp_g}|_{G_{\mathbb{Q}_\ell}}$ and $\rho_f|_{G_{\mathbb{Q}_\ell}}$ agree for all $\ell|N$ with $\ell \neq p$.*

- *$\rho_{g,\wp_g}|_{G_{\mathbb{Q}_p}}$ has the same Galois type as $\rho_f|_{G_{\mathbb{Q}_p}}$ and is potentially crystalline if and only if $\rho_f|_{G_{\mathbb{Q}_p}}$ is.*
- *For all $q|Q$, $\rho_{g,\wp_g}|_{G_{\mathbb{Q}_q}}$ is a Steinberg representation twisted by the unramified character* $\mathrm{Frob}_q \mapsto \epsilon_Q(q)$.

Proof. We will construct a Galois representation

$$\tau : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$$

satisfying the following properties:

- $\tau \equiv \rho_f \pmod{\pi^{j-k+1}}$.
- $\det \tau = \chi$.
- For all $\ell \nmid Q$, $\tau|_{G_{\mathbb{Q}_\ell}}$ defines a point of $\operatorname{Spec} R_\ell$.
- For all $q|Q$, $\tau|_{G_{\mathbb{Q}_q}}$ defines a point of $\operatorname{Spec} R^{\mathrm{ord}}_{q,\epsilon_Q(q)}$.

Let us first show that the existence of the representation τ is sufficient for the theorem. Since τ is odd and potentially semistable with distinct Hodge-Tate weights 0 and 1, we may apply the modularity lifting theorem of [56, Theorem 1.0.4] (and see the main result of [75] for the case $p = 3$, using (TW)) to conclude that τ arises from a modular newform g, which is automatically of weight two and trivial character. Now by modularity and [1, Theorem D, Proposition 1.2.2], $\tau|_{G_{\mathbb{Q}_\ell}}$ defines not only a point of $\operatorname{Spec} R_\ell$ but a smooth point of $\operatorname{Spec} \widetilde{R}_\ell$ for all ℓ. Since the potentially crystalline locus of $\operatorname{Spec} \widetilde{R}_p$ is a union of irreducible components (cf. [44, Theorems 3.3.4, 3.3.8]), τ is potentially crystalline if and only if ρ_f is. By construction, $\tau|_{G_{\mathbb{Q}_p}}$ has the same Galois type as $\rho_f|_{G_{\mathbb{Q}_p}}$. Hence g and f have the same conductor at p by [66, Theorem 2.2]. For $\ell \neq p$, the inertial type is constant on components of $\operatorname{Spec} \widetilde{R}_\ell[1/\pi]$, except possibly at the nonsmooth points (count dimensions using [2, Theorems 3.3.2, 3.3.7], or see the calculations in [68, §5]); it follows that τ has the same inertial type as ρ_f at all $\ell \nmid Q$, and is Steinberg twisted by $\mathrm{Frob}_q \mapsto \epsilon_Q(q)$ for all $q|Q$. Since for all $\ell \neq p$, the ℓ-part of the conductor of g is the conductor of the Weil-Deligne representation associated to $\tau|_{G_{\mathbb{Q}_\ell}}$ by [13, Théorème A], we see that g has level NQ.

We now turn to the construction of τ. For each $\ell \nmid Q$, let Y_ℓ be a set satisfying Proposition 6.1.3 for $j_0 = k$ with $n_0 = N_0(k)$ for $\ell|Np$, and with $n_0 = 1$ if $\ell \nmid Np$. Also let Y^ℓ_m be as in Proposition 6.1.3 for all $m \geq 1$. We will construct τ as the inverse limit of representations

$$\tau_m : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/\pi^m),$$

compatible under reduction maps, with the following key property: for all m, $\tau_m|_{G_{\mathbb{Q}_\ell}}$ lies in Y^ℓ_m if $\ell \nmid Q$, and $\tau_m|_{G_{\mathbb{Q}_q}}$ defines a point of $\operatorname{Spec} R^{\mathrm{ord}}_q(\mathcal{O}/\pi^m)$ if $q|Q$. The representations τ_m are constructed inductively, *but* when constructing τ_{m+1}, we will allow ourselves to modify the representations $\tau_{m-k+2}, \ldots, \tau_m$. (This is the "relative" aspect of the construction.) Before we begin the construction, let us fix once and for all a set Σ of places containing all $v|NpQ\infty$ such that $\text{Ш}^1_\Sigma = 0$ (possible by Proposition 6.1.10). Our base case is $\tau_j = \rho_f \pmod{\pi^j}$. Suppose we have defined τ_m for some $m \geq j$. For each $\ell \in \Sigma$, we may fix a local lift $\rho_{m+1,\ell}$ of $\tau_m|_{G_{\mathbb{Q}_\ell}}$ with the following property: if $\ell \nmid Q$, then $\rho_{m+1,\ell}$ lies in Y^ℓ_{m+1}, and if $\ell = q|Q$, then $\rho_{m+1,q}$ lies in $\operatorname{Spec} R^{\mathrm{ord}}_q(\mathcal{O}/\pi^{m+1})$. This is possible by Propositions 6.1.3 and 6.1.8 and by the key property of τ_m (using $m \geq j \geq N_0(k)$). In particular, the obstruction to lifting τ_m modulo π^{m+1} vanishes locally. By global Poitou-Tate duality, the vanishing of Ш^1_Σ implies the vanishing of

$$\ker\left(H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, \overline{L}) \to \prod_{v\in\Sigma} H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, \overline{L})\right),$$

so there exists a representation $\rho_{m+1} : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/\pi^{m+1})$ which is unramified outside Σ and lifts τ_m. Comparing $\rho_{m+1,\ell}$ to ρ_{m+1} as lifts of $\tau_{m-k+1}|_{G_{\mathbb{Q}_\ell}}$ for all $\ell \in \Sigma$, their difference defines a collection of local cocycles

$$(c_\ell) \in \bigoplus_{\ell\in\Sigma} \frac{H^1(\mathbb{Q}_\ell, L/\pi^k)}{H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^k)}.$$

Now, since ρ_{m+1} lifts τ_m, Propositions 6.1.3 and 6.1.8 imply that (c_ℓ) has trivial image in

$$\bigoplus_{\ell\in\Sigma} \frac{H^1(\mathbb{Q}_\ell, L/\pi^{k-1})}{H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L/\pi^{k-1})}.$$

By the argument in [28, p. 3578], it then follows from the hypothesis (1) of the theorem that there is a global cocycle $c \in H^1(\mathbb{Q}, L/\pi^k)$ whose localizations at $\ell \in \Sigma$ agree with (c_ℓ). Adjusting ρ_{m+1} by the cocycle c, we obtain a representation τ_{m+1} with the desired key property. (Note that we are repeatedly using $m+1-k \geq j+1-k \geq N_0(k)$ to apply Proposition 6.1.3.) We now *redefine* $\tau_{m-k+2}, \ldots, \tau_m$ to be the reductions of τ_{m+1}; since τ_{m+1} is a lift of τ_{m-k+1} by construction, the representations τ_i with $j \leq i \leq m-k+1$ do not need to be redefined. This completes the inductive step of the construction, hence the proof of the theorem. □

6.2. **Patching adjoint Selmer groups.**

6.2.1. Patched Selmer groups provide a convenient framework to produce weakly admissible pairs $\{Q, \epsilon_Q\}$ satisfying the conditions of Theorem 6.1.11. *For this subsection only*, we drop the condition from Definition 3.1.2 that $\mathrm{Frob}_{\mathsf{q}}$ has nontrivial image in $\mathrm{Gal}(K/\mathbb{Q})$, as no quadratic imaginary field is needed for the discussion.

Suppose that q is an admissible ultraprime with sign ϵ_{q} (in this modified sense). In the notation of (3.1.3), we have the exact sequence of $\mathcal{O}[\mathrm{Frob}_{\mathsf{q}}]$-modules

$$0 \to \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f \to T_f \to T_f / \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f \to 0,$$

and we define

$$\mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} L = \mathrm{Hom}(T_f / \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f, \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} T_f) \subset L$$

and

$$\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L) = \mathrm{im}\left(\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, \mathrm{Fil}^+_{\mathsf{q},\epsilon_{\mathsf{q}}} L) \to \mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)\right).$$

It is clear that, if q is represented by a sequence $(q_n)_{n\in\mathbb{N}}$, then

$$\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L) = \varprojlim_j \mathcal{U}\left(\left\{H^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{q_n}, L/\pi^j)\right\}_{n\in\mathbb{N}}\right),$$

where $H^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{q_n}, L/\pi^j)$ is well-defined for all n such that q_n is j-admissible with sign ϵ_{q}. We also define $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)$ as the orthogonal complement of $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L)$ under the local Tate pairing. Note that, since $\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)$ is torsion-free by a direct calculation, $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)$ and $\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L)$ are exact annihilators. We will require the restriction maps

$$\begin{aligned} \mathrm{loc}_{\mathsf{q},\epsilon_{\mathsf{q}}} &: \mathsf{H}^1(\mathbb{Q}, L) \to \frac{\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}, \\ \mathrm{loc}^\dagger_{\mathsf{q},\epsilon_{\mathsf{q}}} &: \mathsf{H}^1(\mathbb{Q}, L^\dagger) \to \frac{\mathsf{H}^1(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}{\mathsf{H}^1_{\mathrm{ord},\epsilon_{\mathsf{q}}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^\dagger)}. \end{aligned} \tag{76}$$

As usual, we drop the subscript ϵ_{q} when it is clear from context.

6.2.2. Let $\mathsf{S} \subset \mathsf{M}_{\mathbb{Q}}$ be the set of constant ultraprimes $\underline{v}$ for $v|Np\infty$. For any $\{\mathsf{PQR}, \epsilon_{\mathsf{PQR}}\} \in \mathsf{N}$, we define the Selmer structure $(\mathcal{S}^{\mathsf{P}}_{\mathsf{R}}(\mathsf{Q}), \mathsf{S} \cup \mathsf{PQR})$ for L:

$$\mathsf{H}^1_{\mathcal{S}^{\mathsf{P}}_{\mathsf{R}}(\mathsf{Q})}(\mathbb{Q}_{\mathsf{v}}, L) = \begin{cases} H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L), & \mathsf{v} = \underline{\ell} \in \mathsf{S}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{Q}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) + \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{P}, \\ \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L), & \mathsf{v} = \mathsf{q} \in \mathsf{R}, \\ \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{v}}, L), & \mathsf{v} \notin \mathsf{S} \cup \mathsf{PQR}. \end{cases} \tag{77}$$

When P, Q, or R is empty, it is omitted from the notation.

Proposition 6.2.3. *For all* $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$,

$$d_{\mathsf{Q}} := \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})}(L) = \mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})^\dagger}(L^\dagger).$$

Proof. For all primes ℓ, Proposition 6.1.3 implies that

$$\mathrm{rk}_{\mathcal{O}}\, H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L) = 3 + \delta_{\ell=p} - \mathrm{rk}_{\mathcal{O}} \left(\varprojlim_j B^1(\mathbb{Q}_\ell, L/\pi^j) \right),$$

where $B^1(\mathbb{Q}_\ell, L/\pi^j)$ denotes the coboundaries; because L has rank three over $\mathcal{O}$, we conclude

$$\mathrm{rk}_{\mathcal{O}}\, H^1_{\mathcal{S}}(\mathbb{Q}_\ell, L) = \mathrm{rk}_{\mathcal{O}}\, H^0(\mathbb{Q}_\ell, L) + \delta_{\ell=p}.$$

Since

$$\mathrm{rk}_{\mathcal{O}}\, \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) = \mathrm{rk}_{\mathcal{O}}\, \mathsf{H}^0(\mathbb{Q}_{\mathsf{q}}, L) = 1$$

for all $\mathsf{q} \in \mathsf{Q}$, the claim results from Proposition 2.6.13. □

In the language of patching, we can reformulate Theorem 6.1.11 as follows.

Theorem 6.2.4. *Assume f satisfies (TW), and suppose given a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$ such that $d_{\mathsf{Q}} = 0$. Fix a sequence $\{Q_n, \epsilon_{Q_n}\}$ of weakly admissible pairs representing $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$. Then there is a sequence (defined for $\mathfrak{F}$-many n) of eigen-newforms g_n of weight 2, level NQ_n, and trivial character, with a prime $\wp_{g_n}$ of the ring of integers $\mathcal{O}_{g_n}$ of the coefficient field, such that:*

- *The completion $\mathcal{O}_{g_n,\wp_{g_n}}$ is isomorphic to $\mathcal{O}$.*
- *The inertial types of $\rho_{g_n,\wp_{g_n}}|_{G_{\mathbb{Q}_\ell}}$ and $\rho_f|_{G_{\mathbb{Q}_\ell}}$ agree for all $\ell|N$ with $\ell \neq p$.*
- *$\rho_{g_n,\wp_{g_n}}|_{G_{\mathbb{Q}_p}}$ has the same Galois type as ρ_f and is potentially crystalline if and only if ρ_f is.*
- *For all $q_n|Q_n$, $\rho_{g_n,\wp_{g_n}}|_{G_{\mathbb{Q}_{q_n}}}$ is a Steinberg representation twisted by the unramified character $\mathrm{Frob}_{q_n} \mapsto \epsilon_{Q_n}(q_n)$.*
- *For any fixed j, there is a congruence of Galois representations (in some basis)*

  $$\rho_f \equiv \rho_{g_n,\wp_{g_n}} \pmod{\pi^j}$$

  for $\mathfrak{F}$-many n.

In particular, for fixed $j \geq 1$ the maps

$$\mathbb{T}_{Q_n} = \mathbb{T}_{N^+,N^-Q_n} \to \mathcal{O}/\pi^j$$

of Remark 4.6.8 admit $\mathcal{O}$-valued lifts for $\mathfrak{F}$-many n.

Proof. First observe that it suffices to construct such a sequence with $\mathcal{O}_{g_n,\wp_{g_n}}$ a subring of $\mathcal{O}$, since both $\mathcal{O}_{g_n,\wp_{g_n}}$ and $\mathcal{O}$ are integrally closed and E is generated over $\mathbb{Q}_p$ by $\mathrm{tr}\,\rho(g)$ for $g \in G_{\mathbb{Q}}$.

Since $d_{\mathsf{Q}} = 0$, there exists some $k \geq 0$ such that the natural map

$$\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})^\dagger}(L^\dagger/\pi^k) \to \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})^\dagger}(\overline{L}^*) \tag{78}$$

is identically zero. (To see this, use Proposition 2.5.6 to conclude that the inverse limit over k of the image of the map in (78) vanishes.) Also, Proposition 6.1.6 implies $\mathcal{S}$ is saturated as a Selmer structure for L, and $\mathcal{S}(\mathsf{Q})$ is as well by a direct local calculation for $\mathsf{q} \in \mathsf{Q}$. Proposition 2.6.12 therefore shows that $\mathrm{Sel}_{\mathcal{S}(\mathsf{Q})^\dagger}(L^\dagger/\pi^k) = \mathrm{Sel}_{\mathcal{S}(\mathsf{Q})^*}(L^*[\pi^k])$. Hence for $\mathfrak{F}$-many n, the maps

$$\mathrm{Sel}_{Q_n}(L^*[\pi^k]) \to \mathrm{Sel}_{Q_n}(\overline{L}^*)$$

are also identically zero. Since k is fixed independently of n, the theorem is now immediate from Theorem 6.1.11. □

6.3. **Annihilating two Selmer groups.**

6.3.1. Now fix an imaginary quadratic field K of discriminant prime to Np and a self-dual Selmer structure $(\mathcal{F}, \mathsf{S}_K)$ for the G_K-module T_f over $\mathcal{O}$, where S_K contains all constant ultraprimes $\underline{s}$ with $s|Np\infty$. For our application of Theorem 6.2.4, we will want to choose $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}_K}$ such that $d_{\mathsf{Q}} = 0$, and $r_{\mathsf{Q}} = \mathrm{rk}_{\mathcal{O}}\, \mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f) = 0$. In this subsection, we show that such a choice is possible (Proposition 6.3.6 below). The proof is inspired by [15, §3.3], and begins with a series of lemmas. We note that the results of this subsection crucially require the assumption from §1.5 that f not have CM.

Lemma 6.3.2. *There exists an integer j that, for all $n \geq 0$,*

$$\pi^j H^1(K(T_f)/\mathbb{Q}, L/\pi^n) = \pi^j H^1(K(T_f)/\mathbb{Q}, L^\dagger/\pi^n) = 0.$$

Proof. Let $F = \mathbb{Q}(\mu_{p^\infty}) \subset K(T_f)$, and note that L and $L^\dagger$ are isomorphic G_F-modules. Since f is non-CM, $(L \otimes \mathbb{Q}_p)^{G_F} = 0$ by [61, Theorem 5.5] or [54, Theorem B.5.2], and so $(L/\pi^n)^{G_F} = (L^\dagger/\pi^n)^{G_F}$ is uniformly bounded in n.

The pro-p-Sylow subgroup of $\mathrm{Gal}(K(T_f)/F)$ is a compact p-adic Lie group with semisimple Lie algebra; hence, by [28, Lemma B.1], the cohomology $H^1(K(T_f)/F, L/\pi^n) = H^1(K(T_f)/F, L^\dagger/\pi^n)$ is uniformly bounded in n.

Now, by inflation-restriction, we have exact sequences

$$\begin{aligned} &0 \to H^1(F/\mathbb{Q}, (L/\pi^n)^{G_F}) \to H^1(K(T_f)/\mathbb{Q}, L/\pi^n) \to H^1(K(T_f)/F, L/\pi^n), \\ &0 \to H^1(F/\mathbb{Q}, (L^\dagger/\pi^n)^{G_F}) \to H^1(K(T_f)/\mathbb{Q}, L^\dagger/\pi^n) \to H^1(K(T_f)/F, L^\dagger/\pi^n), \end{aligned} \tag{79}$$

where the outer terms are uniformly bounded in n; the lemma follows. ☐

For the next lemma, we abbreviate $L_m := L/\pi^m$, $L_m^\dagger := L^\dagger/\pi^m$, and (as usual) $T_m := T_f/\pi^m$. Also, for any torsion $\mathcal{O}$-module M and element $y \in M$, let $\mathrm{ord}(y)$ be the smallest integer $t \geq 0$ such that $\pi^t y = 0$.

Lemma 6.3.3. *There is a constant C, depending only on T_f, with the following property. Given cocycles $\phi \in H^1(\mathbb{Q}, L_m)$, $\psi \in H^1(\mathbb{Q}, L_m^\dagger)$, and $c_1, c_2 \in H^1(K, T_m)^\delta$ for some $\delta \in \{\pm 1\}$, and given an integer $n \geq m$, there exist infinitely many primes $q \nmid Np$ such that all the cocycles are unramified at q and:*

- *q is n-admissible with sign δ.*
- $\mathrm{ord}\,\mathrm{loc}_{q,\delta}\,\phi \geq \mathrm{ord}\,\phi - C$.
- $\mathrm{ord}\,\mathrm{loc}_{q,\delta}^\dagger\,\psi \geq \mathrm{ord}\,\psi - C$.
- $\mathrm{ord}\,\mathrm{loc}_{q,\delta}\,c_i \geq \mathrm{ord}\,c_i - C$ *for* $i = 1, 2$.

In the second and third bullet points, $\mathrm{loc}_{q,\delta}$ and $\mathrm{loc}_{q,\delta}^\dagger$ are defined analogously to (76). Also $H^1(K, T_m)^\delta$ refers to the δ-eigenspace for complex conjugation.

Proof. Let us first fix a complex conjugation $c \in G_\mathbb{Q}$ and choose a basis for T_f in which c acts as $\begin{pmatrix} -\delta & 0 \\ 0 & \delta \end{pmatrix}$.

The restriction of the cocycles ϕ, ψ, c_i to $G_{K(T_n)}$ may be considered as a homomorphism

$$h : G_{K(T_n)} \to L_m \oplus L_m^\dagger \oplus (T_m)^2$$

compatible with the action of G_K; let H be the image of this homomorphism. Let $g_z \in G_K$ be an element that acts by a scalar $z \neq \pm 1$ on T_f, which exists by [61, Theorem 5.5] or [54, Theoerem B.5.2]; then we have:

$$\begin{aligned} H \supset &(g_z - z)(g_z - z^2)H + (g_z - z)(g_z - 1)H + (g_z - z^2)(g_z - 1)H \\ &\supset (z-1)(z^2-1)(z^2-z)\left(\pi_{L_m}(H) \oplus \pi_{L_m^\dagger}(H) \oplus \pi_{T_m^2}(H)\right), \end{aligned}$$

where $\pi_\bullet$ are the projection operators. Now, since $L \otimes \mathbb{Q}_p$ and $L^\dagger \otimes \mathbb{Q}_p$ are absolutely irreducible by *loc. cit.*, the natural maps $E[G_K] \to \mathrm{End}_E(L \otimes \mathbb{Q}_p)$ and $E[G_K] \to \mathrm{End}_E(L^\dagger \otimes \mathbb{Q}_p)$ are surjective. Combining these observations with Lemma 6.3.2 and inflation-restriction, we see that, for some constant C depending only on T_f, there exists $\gamma \in G_{K(T_f)} \subset G_{K(T_n)}$ satisfying:

- The $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ component of $\phi(\gamma)$ has order at least $\mathrm{ord}\,\phi - C$.
- The $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ component of $\psi(\gamma)$ has order at least $\mathrm{ord}\,\psi - C$.
- The component of $c_i(\gamma)$ in the δ eigenspace has order at least $\mathrm{ord}\,c_i - C$, where $i = 1, 2$.

For the final item, we are using the elementary fact that a group cannot be the union of two nontrivial subgroups, as well as the irreducibility of T_f.

Since $\phi(c^2) = c\phi(c) + \phi(c) = 0$, $\phi(c)$ lies in the -1 eigenspace for complex conjugation, whereas $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalue 1; hence the $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ component of $\phi(c\gamma)$ has order at least $\mathrm{ord}\,\phi - C$. Similarly, the $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ component of $\psi(c\gamma)$ has order at least $\mathrm{ord}\,\psi - C$.

Let $F \supset K(T_n)$ be the fixed field of the kernel of h. Then any prime with Frobenius $c\gamma$ in $\mathrm{Gal}(F/\mathbb{Q})$ satisfies the conclusion of the lemma; cf. the proof of Lemma 3.3.11 for the assertions about c_i. ☐

Corollary 6.3.4. *Suppose given a finite set of ultraprimes $\mathsf{S}' \subset \mathsf{M}_{\mathbb{Q}}$ containing the image of S_K, and non-torsion cocycles:*

- $\phi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L)$;
- $\psi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L^\dagger)$;
- $c_1, c_2 \in \mathsf{H}^1(K^{\mathsf{S}_K}/K, T_f)^\delta$ *for some* $\delta \in \{\pm 1\}$.

Then there exist infinitely many admissible ultraprimes $\mathsf{q} \notin \mathsf{S}'$ with sign δ such that:

- $\operatorname{loc}_{\mathsf{q},\delta} \phi \neq 0$.
- $\operatorname{loc}^\dagger_{\mathsf{q},\delta} \psi \neq 0$.
- $\operatorname{loc}_{\mathsf{q},\delta} c_i \neq 0$ *for* $i = 1, 2$.

Proof. Fix an integer C satisfying the conclusion of Lemma 6.3.3. Since ϕ, ψ, c_1, and c_2 are all non-torsion, we may choose an integer $m \geq 1$ such that ϕ, ψ, c_1, and c_2 have non-π^C-torsion images in $\mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L/\pi^m)$, $\mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L^\dagger/\pi^m)$, and $\mathsf{H}^1(K^{\mathsf{S}_K}/K, T_m)$, respectively. The rest of the proof is the same as that of Theorem 3.3.9, replacing Lemma 3.3.11 with Lemma 6.3.3. □

We also require the following variant.

Lemma 6.3.5. *Let $\phi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L)$ and $c \in \mathsf{H}^1(K^{\mathsf{S}_K}/K, T_f)^\delta$ be non-torsion, where $\mathsf{S}' \subset \mathsf{M}_{\mathbb{Q}}$ is a finite set containing the image of S_K. Then for any cocycle $\psi \in \mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L^\dagger)$, there exist infinitely many admissible ultraprimes $\mathsf{q} \notin \mathsf{S}'$ with sign δ such that $\operatorname{loc}_{\mathsf{q},\delta} \phi \neq 0$, $\operatorname{loc}_{\mathsf{q},\delta} c \neq 0$, and $\operatorname{loc}^\dagger_{\mathsf{q},\delta} \psi = 0$.*

Proof. The proof of Lemma 6.3.3 also shows the following:

Claim. There is a constant C, depending only on T_f, with the following property. Given integers $n \geq m$ and cocycles $\phi \in H^1(\mathbb{Q}, L/\pi^m)$, $c \in H^1(K, T_m)^\delta$, and $\psi_j \in H^1(\mathbb{Q}, L^\dagger/\pi^j)$ for $j = 1, \ldots, n$, there exist infinitely many n-admissible primes q with sign δ such that all the cocycles are unramified at q and:

- $\operatorname{ord} \operatorname{loc}_{q,\delta} \phi \geq \operatorname{ord} \phi - C$.
- $\operatorname{loc}^\dagger_{q,\delta} \psi_j = 0$ for all j.
- $\operatorname{ord} \operatorname{loc}_{q,\delta} c \geq \operatorname{ord} c - C$.

Fix a prime C satisfying the conclusion of the claim. Since ϕ and c are non-torsion, we may choose an integer $m \geq 1$ such that ϕ and c have non-π^C-torsion images $\overline{\phi}$ and $\overline{c}$ in $\mathsf{H}^1(\mathbb{Q}^{\mathsf{S}'}/\mathbb{Q}, L/\pi^m)$ and $\mathsf{H}^1(K^{\mathsf{S}_K}/K, T_m)$, respectively. Recall that $\overline{\phi}$ and $\overline{c}$ are the equivalence classes of sequences $(\overline{\phi}_n)_{n\in\mathbb{N}}$, $(\overline{c}_n)_{n\in\mathbb{N}}$ with $\overline{\phi}_n \in H^1(\mathbb{Q}, L/\pi^m)$ and $\overline{c}_n \in H^1(K, T_m)$, while ψ is an equivalence class of sequences

$$(\psi_{n,j})_{n,j\in\mathbb{N}} \in \varprojlim_j \mathcal{U}\left(\left\{H^1(\mathbb{Q}^{S_n}/\mathbb{Q}, L^\dagger/\pi^j)\right\}_{n\in\mathbb{N}}\right),$$

where $(S_n)_{n\in\mathbb{N}}$ represents S'. For each n, choose $q_n \notin S_n$ as in the claim with $\phi = \overline{\phi}_n$, $\psi_j = \psi_{n,j}$ for $j = 1, \cdots, n$, and $c = \overline{c}_n$. Let $\mathsf{q} \in \mathsf{M}_{\mathbb{Q}}$ be the ultraprime represented by $(q_n)_{n\in\mathbb{N}}$, which is admissible with sign δ. For fixed j, $\operatorname{loc}_{q_n,\delta} \psi_{n,j} = 0$ for all $n \geq j$, i.e. for $\mathfrak{F}$-many n, so $\operatorname{loc}_{\mathsf{q}} \psi \equiv 0 \pmod{\pi^j}$. Since this holds for all j, we conclude $\operatorname{loc}_{\mathsf{q}} \psi = 0$. We also have $\operatorname{loc}_{\mathsf{q}} \phi \neq 0$ and $\operatorname{loc}_{\mathsf{q}} c \neq 0$ by construction, so q satisfies the conclusion of the lemma. There are infinitely many choices of each q_n, hence of q. □

Proposition 6.3.6. *Assume f satisfies (non-CM), and suppose given a self-dual Selmer structure $(\mathcal{F}, \mathsf{S}_K)$ for T_f. Then there exists $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}_K}$ such that*

$$r_{\mathsf{Q}} = d_{\mathsf{Q}} = 0.$$

(Recall that $r_{\mathsf{Q}} = \operatorname{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f)$.)

Proof. Without loss of generality, by Corollary 3.3.13 we may assume that $r_1 = 0$; for if not, choose any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}_K}$ with $r_{\mathsf{Q}} = 0$, and then relabel $\mathcal{F}(\mathsf{Q})$ as $\mathcal{F}$.

We will show that, if $d_1 > 0$, we may find $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ such that $r_{\mathsf{Q}} = 0$ and $d_{\mathsf{Q}} < d_1$; this clearly suffices by induction. By Proposition 6.2.3, there exist non-torsion elements $\phi \in \operatorname{Sel}_{\mathcal{S}}(L)$, $\psi \in \operatorname{Sel}_{\mathcal{S}^\dagger}(L^\dagger)$. Using Corollary 6.3.4, choose any admissible $\mathsf{q} \notin \mathsf{S}_K$ with sign ϵ_{q} such that $\operatorname{loc}_{\mathsf{q}} \phi \neq 0$, $\operatorname{loc}^\dagger_{\mathsf{q}} \psi \neq 0$. Then by Proposition 2.6.13,

$$\operatorname{rk}_{\mathcal{O}} \operatorname{Sel}_{(\mathcal{S}_{\mathsf{q}})^\dagger}(L^\dagger) + \operatorname{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{S}^{\mathsf{q}}}(L) = 2 + \operatorname{rk}_{\mathcal{O}} \operatorname{Sel}_{(\mathcal{S}^{\mathsf{q}})^\dagger}(L^\dagger) + \operatorname{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{S}_{\mathsf{q}}}(L).$$

In particular, the images of the localization maps

$$\mathrm{loc}_{\mathsf{q}} : \frac{\mathrm{Sel}_{\mathcal{S}^{\mathsf{q}}}(L)}{\mathrm{Sel}_{\mathcal{S}_{\mathsf{q}}}(L)} \hookrightarrow \frac{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)} \oplus \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L)}$$

and

$$\mathrm{loc}^{\dagger}_{\mathsf{q}} : \frac{\mathrm{Sel}_{(\mathcal{S}_{\mathsf{q}})^{\dagger}}(L^{\dagger})}{\mathrm{Sel}_{(\mathcal{S}^{\mathsf{q}})^{\dagger}}(L^{\dagger})} \hookrightarrow \frac{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^{\dagger})}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^{\dagger}) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^{\dagger})} \oplus \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^{\dagger})}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{q}}, L^{\dagger}) \cap \mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{q}}, L^{\dagger})}$$

have total rank two. Hence the image in the ordinary part is zero for both maps, and $d_{\mathsf{q}} < d_1$. Now, Proposition 3.3.8 shows that $r_{\mathsf{q}} = 1$, and a generator $c \in \mathrm{Sel}_{\mathcal{F}(\mathsf{q})}(T_f)$ has $\partial_{\mathsf{q}} c \neq 0$. In particular, c has nonzero component in the ϵ_{q} eigenspace for τ, because it is easy to check that $\partial_{\mathsf{q}} \tau c = \epsilon_{\mathsf{q}} \partial_{\mathsf{q}} c$.

Now consider the set P of admissible ultraprimes $\mathsf{s} \notin \mathsf{S}_K$ with sign $\epsilon_{\mathsf{s}} = \epsilon_{\mathsf{q}}$ such that $\mathrm{loc}_{\mathsf{s}}\, c \neq 0$, which is nonempty by Theorem 3.3.9. If, for any $\mathsf{s} \in \mathsf{P}$, $d_{\mathsf{qs}} \leq d_{\mathsf{q}}$, then we may take $\mathsf{Q} = \mathsf{qs}$ and complete our induction step. For example, this will occur provided $d_{\mathsf{q}} > 0$, by Corollary 6.3.4 and the argument above; so without loss of generality, $d_{\mathsf{q}} = 0$ and $d_{\mathsf{qs}} = 1$ for all $\mathsf{s} \in \mathsf{P}$. By definition, we therefore have non-torsion elements $\phi(\mathsf{s}) \in \mathrm{Sel}_{\mathcal{S}(\mathsf{qs})}(L)$ and $\psi(\mathsf{s}) \in \mathrm{Sel}_{\mathcal{S}(\mathsf{qs})^{\dagger}}(L^{\dagger})$ such that $\mathrm{loc}_{\mathsf{s}}\, \phi(s)$ and $\mathrm{loc}_{\mathsf{s}}\, \psi(s)$ are nontrivial.

Choose any $\mathsf{s}_1 \in \mathsf{P}$, and then, using Lemma 6.3.5, choose $\mathsf{s}_2 \in \mathsf{P}$ such that $\mathrm{loc}_{\mathsf{s}_2} \phi(\mathsf{s}_1) \neq 0$ but $\mathrm{loc}_{\mathsf{s}_2} \psi(\mathsf{s}_1) = 0$. By another application of Proposition 3.3.8, $r_{\mathsf{qs}_1\mathsf{s}_2} = 1$, and a generator c' of $\mathrm{Sel}_{\mathcal{F}(\mathsf{qs}_1\mathsf{s}_2)}(T_f)$ again has nonzero component in the ϵ_{q} eigenspace. We now use Corollary 6.3.4 to choose $\mathsf{s}_3 \in \mathsf{P}$ such that $\mathrm{loc}_{\mathsf{s}_3} c' \neq 0$, $\mathrm{loc}_{\mathsf{s}_3} \phi(\mathsf{s}_2) \neq 0$, and $\mathrm{loc}_{\mathsf{s}_3} \psi(\mathsf{s}_1) \neq 0$. Note that $\mathrm{rk}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{S}^{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}(\mathsf{q})}(L) = 3$ by another application of Proposition 2.6.13; up to torsion, $\phi(\mathsf{s}_i)$ are generators. So to show that $d_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = d_{\mathsf{q}}$, it suffices to show that the images of $\phi(\mathsf{s}_i)$ form a rank-three subspace of

$$S := \bigoplus_{i=1}^{3} \frac{\mathsf{H}^1_{\mathrm{unr}}(\mathbb{Q}_{\mathsf{s}_i}, L) + \mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{s}_i}, L)}{\mathsf{H}^1_{\mathrm{ord}}(\mathbb{Q}_{\mathsf{s}_i}, L)}$$

under the localization

$$\mathrm{loc} : \frac{\mathrm{Sel}_{\mathcal{S}^{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}(\mathsf{q})}(L)}{\mathrm{Sel}_{\mathcal{S}_{\mathsf{s}_1\mathsf{s}_2\mathsf{s}_3}(\mathsf{q})}(L)} \hookrightarrow S.$$

By pairing $\phi(\mathsf{s}_i)$ and $\psi(\mathsf{s}_j)$ for $i \neq j$ and applying Proposition 2.6.10, we see that $\mathrm{loc}_{\mathsf{s}_i} \phi(\mathsf{s}_j) \neq 0$ if and only if $\mathrm{loc}_{\mathsf{s}_j} \psi(\mathsf{s}_i) \neq 0$. Hence, the images of $\phi(\mathsf{s}_i)$ in S are of the form:

$$\begin{aligned} \mathrm{loc}(\phi(\mathsf{s}_1)) &= (0, *, \cdot) \\ \mathrm{loc}(\phi(\mathsf{s}_2)) &= (0, 0, *), \\ \mathrm{loc}(\phi(\mathsf{s}_3)) &= (*, \cdot, 0), \end{aligned}$$

where $*$ is nonzero and $\cdot$ may or may not be zero. This completes the proof since it shows $d_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = d_{\mathsf{q}} = 0$ and we have $r_{\mathsf{qs}_1\mathsf{s}_2\mathsf{s}_3} = 0$ by Proposition 3.3.8. ☐

7. Proof of main results: anticyclotomic main conjectures

7.1. **Notation.**

7.1.1. Let f, N, $\wp$, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5, and assume $\wp$ is ordinary. We also fix a quadratic imaginary field K in which p splits and suppose that N admits a factorization $N = N^+N^-$ as in (5.1.1).

For the next two sections, we depart from our earlier notation of (3.4.1), (5.2.1) and denote by K_∞^{ac} the anticyclotomic $\mathbb{Z}_p$-extension of K. Also let $K_\infty^{\mathrm{cyc}} = K\mathbb{Q}_\infty$, where $\mathbb{Q}_\infty/\mathbb{Q}$ is the cyclotomic $\mathbb{Z}_p$-extension, and $K_\infty = K_\infty^{\mathrm{cyc}} K_\infty^{\mathrm{ac}}$. For $? = \emptyset, \mathrm{cyc}, \mathrm{ac}$, let $\Gamma_K^? = \mathrm{Gal}(K_\infty^?/K)$, so that $\Gamma_K \cong \Gamma_K^{\mathrm{cyc}} \times \Gamma_K^{\mathrm{ac}}$. We continue to reserve the notation Λ for the anticyclotomic Iwasawa algebra $\mathcal{O}[\![\Gamma_K^{\mathrm{ac}}]\!]$. Recall from (3.4.2) the free rank-one direct summand $\mathrm{Fil}_v^+ T_f$ for each $v|p$, and let $\mathrm{gr}_v T_f = T_f/\mathrm{Fil}_v^+ T_f$. Also, for a topological $\mathcal{O}$-module M, let $M^\vee = \mathrm{Hom}_{\mathcal{O}}(M, E/\mathcal{O})$. Here and for the remainder of the text, the notation $\mathrm{Hom}_{\mathcal{O}}$ refers to continuous $\mathcal{O}$-module homomorphisms.

For a finite set Σ of places of K, for $? = \emptyset, \mathrm{cyc}$, or ac, and for A the ring of integers of a finite extension of $\mathbb{Q}_p$ containing $\mathcal{O}$, we will consider the Selmer group

$$\mathrm{Sel}^{\Sigma}_{K^?_\infty,A}(f) := \ker\Big(H^1(K, T_f \otimes_{\mathcal{O}} A[\![\Gamma^?_K]\!]^\vee) \to \prod_{\substack{v\notin\Sigma\\ v\nmid p}} H^1(I_v, T_f\otimes_{\mathcal{O}} A[\![\Gamma^?_K]\!]^\vee) \times \prod_{v|p} H^1(I_v, \mathrm{gr}_v\, T_f \otimes_{\mathcal{O}} A[\![\Gamma^?_K]\!]^\vee)\Big). \tag{80}$$

Note that the canonical isomorphism $\mathcal{O}[\![\Gamma^?_K]\!]^\vee \otimes_{\mathcal{O}} \mathrm{Hom}(A,\mathcal{O}) \xrightarrow{\sim} A[\![\Gamma^?_K]\!]^\vee$ induces an isomorphism

$$\mathrm{Sel}^{\Sigma}_{K^?_\infty,\mathcal{O}}(f)\otimes_{\mathcal{O}}\mathrm{Hom}(A,\mathcal{O}) \cong \mathrm{Sel}^{\Sigma}_{K^?_\infty,A}(f). \tag{81}$$

If Σ is finite set of places of $\mathbb{Q}$, let Σ_K be the set of places of K lying above an element of Σ; we shall abbreviate $\mathrm{Sel}^{\Sigma}_{K^?_\infty,A}(f) := \mathrm{Sel}^{\Sigma_K}_{K^?_\infty,A}(f)$. In the anticyclotomic case, we also have the Selmer group $\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)$, where the Selmer structure is determined by (43) via the duality of (3.4.1). This Selmer group is related to (80) by the following lemma.

Lemma 7.1.2. *There is an exact sequence of Λ-modules:*

$$0 \to \mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\iota \to \mathrm{Sel}^{\emptyset}_{K^{\mathrm{ac}}_\infty,\mathcal{O}} \to \prod_{\ell|N^-} H^1_{\mathrm{unr}}(K_\ell, W_f)\otimes_{\mathcal{O}}\mathrm{Hom}_{\mathcal{O}}(\Lambda,\mathcal{O}).$$

Here ι denotes twisting the Λ-action by the canonical involution defined by the inversion map on Γ^{ac}_K.

Proof. Recall that $\mathbf{W}_f = (\mathbf{T}^*_f)^\iota$. We may therefore identify $\mathbf{W}_f$ with $(T_f\otimes_{\mathcal{O}}\Lambda^\vee)^\iota$ via (20).

Then by local Poitou-Tate duality, we have

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\iota = \ker\Big(H^1(K, T_f\otimes\Lambda^\vee) \to \prod_{v\nmid N^- p} H^1(I_v, T_f\otimes\Lambda^\vee)\times\prod_{v|N^-} H^1(K_v, T_f\otimes\Lambda^\vee) \times \prod_{v|p} H^1(K_v, \mathrm{Fil}^+_v\, T_f\otimes\Lambda)^\vee\Big).$$

Since the pairing on T_f identifies $\mathrm{gr}_v\, T_f$ with $\mathrm{Hom}_{\mathcal{O}}(\mathrm{Fil}^+_v\, T_f, \mathcal{O}(1))$, $H^1(K_v, \mathrm{Fil}^+_v\, T_f\otimes\Lambda)^\vee$ is canonically identified with $H^1(K_v, \mathrm{gr}_v\, T_f\otimes\Lambda^\vee)$ under local duality, so we conclude

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\iota = \ker\Big(\mathrm{Sel}^{\emptyset}_{K^{\mathrm{ac}}_\infty,\mathcal{O}}(f) \to \prod_{v|N^-} H^1_{\mathrm{unr}}(K_v, T_f\otimes\Lambda^\vee)\times\prod_{v|p} H^1_{\mathrm{unr}}(K_v, \mathrm{gr}_v\, T_f\otimes\Lambda^\vee)\Big).$$

If $v|p$, then $H^1_{\mathrm{unr}}(K_v, \mathrm{gr}_v\, T_f\otimes\Lambda^\vee) = H^1_{\mathrm{unr}}(K_v, \mathrm{gr}_v\, T_f\otimes_{\mathcal{O}}\mathcal{O}^\vee) = 0$; recall here that $\mathrm{gr}_v\, T_f$ is unramified with Frobenius eigenvalue $\alpha_p \neq 1$ in the notation of (5.2.3). Then the lemma follows from the identity $H^1_{\mathrm{unr}}(K_\ell, T_f\otimes\Lambda^\vee) = H^1_{\mathrm{unr}}(K_\ell, W_f)\otimes_{\mathcal{O}}\mathrm{Hom}_{\mathcal{O}}(\Lambda,\mathcal{O})$ for all $\ell|N^-$, which holds because G_{K_ℓ} acts trivially on Λ. ☐

Remark 7.1.3. By [58, Proposition A.2], the sequence in Lemma 7.1.2 is also exact on the right, but this fact will not be needed for our results.

For the proof of Theorem 7.2.1, we will also need:

Lemma 7.1.4. *For $? = \emptyset, \mathrm{cyc}$, or ac, $\overline{T}_f$ is absolutely irreducible as a representation of $G_{K^?_\infty}$.*

Proof. By Lemma 3.3.4, $\overline{T}_f$ is absolutely irreducible as a representation of G_K. In particular, for any finite field k containing $\mathcal{O}/\pi$, the pro-p-group $\mathrm{Gal}(K^?_\infty/K)$ acts without fixed points on the set of $G_{K^?_\infty}$-stable k-lines of $\overline{T}_f\otimes_{\mathcal{O}/\pi} k$, so this set has cardinality a multiple of p. However, if $G_{K^?_\infty}$ stabilizes more than two lines in $\overline{T}_f\otimes_{\mathcal{O}/\pi} k$, then it acts by scalars, in which case it stabilizes a prime-to-p number of lines. So there are no $G_{K^?_\infty}$-stable lines in $\overline{T}_f\otimes_{\mathcal{O}/\pi} k$ for any k, which proves the lemma. ☐

7.2. **A result of Skinner-Urban.** The following result is deduced from the proof the Iwasawa main conjecture for modular forms [71].

Theorem 7.2.1 (Skinner-Urban)**.** *Let K be an imaginary quadratic field of discriminant prime to Np in which p splits. Assume that $p \nmid 2N$, that $\wp$ is ordinary for f, and that:*

- *the mod $\wp$ representation $\overline{T}_f$ is absolutely irreducible as an $\mathcal{O}[G_{\mathbb{Q}}]$-module;*
- *$N = N^+N^-$, where every factor of N^+ is split in K and N^- is the squarefree product of an odd number of primes inert in K.*

Then

$$\mathrm{char}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(1))^2$$

as ideals of Λ, where $\boldsymbol{\lambda}(1) \in \Lambda$ is as in (5.2.5).

Proof. Let A be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing $\mathcal{O}$, that is large enough to satisfy the hypotheses of [71, Theorem 12.7]. As in the proof of *loc. cit.*, there exists a Hida family $\mathbf{f}$ of ordinary eigenforms of tame level N and trivial nebentypus $\chi_{\mathbf{f}}$, parametrized by an integral domain $\mathbb{I}$ that is faithfully flat over A, such that $\mathbf{f}_\phi = f$ for a certain specialization $\phi : \mathbb{I} \to A$. Let Σ be the set of rational primes dividing $Np\,\mathrm{disc}(K)$, and let $\mathscr{L}^\Sigma_{\mathbf{f},K} \in \mathbb{I}[\![\Gamma_K]\!]$ be the three-variable p-adic L-function constructed in [71, Theorem 12.6]. (In the notation of *loc. cit.*, we are taking $\xi = 1$, and $(\mathbf{dist})_{\mathbf{f}}$ is satisfied because $(\overline{T}_f|_{I_p})^{ss} = \overline{\chi} \oplus \mathbb{1}$, where $\overline{\chi}$ is the mod p cyclotomic character.)

Let $\mathbb{1}_{\mathrm{cyc}} : A[\![\Gamma_K^{\mathrm{cyc}}]\!] \to A$ be the specialization at the trivial character, which we also view as a map $A[\![\Gamma_K]\!] \to A[\![\Gamma_K^{\mathrm{ac}}]\!]$. Comparing the interpolation properties of $\mathscr{L}^\Sigma_{\mathbf{f},K}$ and $\boldsymbol{\lambda}(1)$ from [71, Theorem 12.6] and [19, Theorem A], we conclude

$$\mathbb{1}_{\mathrm{cyc}} \circ \phi(\mathscr{L}^\Sigma_{\mathbf{f},K}) \doteq \boldsymbol{\lambda}(1)^2 \cdot \prod_{\substack{v \in \Sigma_K \\ v \nmid p}} \det\left(1 - \mathrm{Frob}_v^{-1} | \mathbf{T}_f^{I_v}\right) \cdot \frac{\Omega_{f,N^-}}{\Omega^+_{f_{\alpha_p}} \Omega^-_{f_{\alpha_p}}}, \tag{82}$$

where $\Omega^\pm_{f_{\alpha_p}}$ and Ω_{f,N^-} are the periods appearing in [71, Theorem 12.7] and [19, Theorem A], respectively, and $\doteq$ denotes equality up to a unit in $A[\![\Gamma_K^{\mathrm{ac}}]\!]$. (We write $\Omega^\pm_{f_{\alpha_p}}$ instead of $\Omega^\pm_f$, as in [71], because these periods are canonically associated to the p-stabilization of f, with U_p-eigenvalue the number α_p from (5.2.3).) The ratio $\Omega_{f,N^-}/(\Omega^+_{f_{\alpha_p}} \Omega^-_{f_{\alpha_p}})$ is a nonzero element of $\mathcal{O}$, and it is described more explicity in §8.3 below.

Let us expand

$$\mathscr{L}^\Sigma_{\mathbf{f},K} = \alpha_0 + \alpha_1(\gamma^{\mathrm{ac}} - 1) + \alpha_2(\gamma^{\mathrm{ac}} - 1)^2 + \cdots, \ \alpha_i \in \mathbb{I}[\![\Gamma_K^{\mathrm{cyc}}]\!], \tag{83}$$

where γ^{ac} is a topological generator. For all height-one primes $P \subset \mathbb{I}[\![\Gamma_K]\!]$ which are pullbacks of primes $P_{\mathrm{cyc}} \subset \mathbb{I}[\![\Gamma_K^{\mathrm{cyc}}]\!]$, and for any $i \geq 0$, (83) shows

$$\mathrm{ord}_P \mathscr{L}^\Sigma_{\mathbf{f},K} \leq \mathrm{ord}_P \alpha_i. \tag{84}$$

Now consider the three-variable Selmer group $\mathrm{Sel}^\Sigma_{K_\infty,A}(\mathbf{f})$ defined in [71, §3.3.10];[3] by the discussion in [71, §3.3.11], using Lemma 7.1.4 and the fact that p splits in K to verify the hypotheses of [71, Proposition 3.7], we have

$$\mathrm{char}_{A[\![\Gamma_K]\!]} \mathrm{Sel}^\Sigma_{K_\infty,A}(f)^\vee \subset \phi\left(\mathrm{char}_{\mathbb{I}[\![\Gamma_K]\!]} \mathrm{Sel}^\Sigma_{K_\infty,A}(\mathbf{f})^\vee\right). \tag{85}$$

The same argument as [71, Proposition 3.9] (replacing [71, Proposition 3.7] with its analogue for $F = K_\infty^{\mathrm{ac}}$) shows that

$$\mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]} \mathrm{Sel}^\Sigma_{K_\infty^{\mathrm{ac}},A}(f)^\vee \subset \mathbb{1}_{\mathrm{cyc}}\left(\mathrm{char}_{A[\![\Gamma_K]\!]} \mathrm{Sel}^\Sigma_{K_\infty,A}(f)^\vee\right).$$

Combining this with (85) shows that

$$\mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]} \mathrm{Sel}^\Sigma_{K_\infty^{\mathrm{ac}},A}(f)^\vee \subset \mathbb{1}_{\mathrm{cyc}} \circ \phi\left(\mathrm{char}_{A[\![\Gamma_K]\!]} \mathrm{Sel}^\Sigma_{K_\infty,A}(\mathbf{f})^\vee\right). \tag{86}$$

[3] The subscript A does not appear in *loc. cit.*, because a choice of coefficient ring has already been made implicit. We include it here to be consistent with (80).

If $P \subset \mathbb{I}[\![\Gamma_K]\!]$ is a height-one prime *not* pulled back from a prime $P_{\mathrm{cyc}} \subset \mathbb{I}[\![\Gamma_K^{\mathrm{cyc}}]\!]$, then by [71, Proposition 13.6] and the discussion in §7.4 of *op. cit.*, [71, Theorem 7.7] applies to show

$$\mathrm{ord}_P\, \mathscr{L}^{\Sigma}_{\mathbf{f},K} \leq \mathrm{ord}_P\, \mathrm{char}_{\mathbb{I}[\![\Gamma_K]\!]}\, \mathrm{Sel}^{\Sigma}_{K,A}(\mathbf{f})^{\vee}. \tag{87}$$

Combining (84) and (87) shows that

$$(\alpha_i)\cdot \mathrm{char}_{\mathbb{I}[\![\Gamma_K]\!]}\, \mathrm{Sel}^{\Sigma}_{K_\infty,A}(\mathbf{f})^{\vee} \subset \left(\mathscr{L}^{\Sigma}_{\mathbf{f},K}\right) \tag{88}$$

for any of the α_i in (83). Now we specialize both sides by $\mathbb{1}_{\mathrm{cyc}}\circ\phi$, using (86) and (82), to obtain

$$\mathbb{1}_{\mathrm{cyc}}\circ\phi(\alpha_i)\cdot \mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]}\, \mathrm{Sel}^{\Sigma}_{K_\infty^{\mathrm{ac}},A}(f)^{\vee} \subset \boldsymbol{\lambda}(1)^2\cdot \prod_{\substack{v\in\Sigma_K\\ v\nmid p}} \det\left(1-\mathrm{Frob}_v^{-1}\,|\mathbf{T}_f^{I_v}\right)\cdot \frac{\Omega_{f,N^-}}{\Omega^+_{f_{\alpha_p}}\Omega^-_{f_{\alpha_p}}}\cdot A[\![\Gamma_K^{\mathrm{ac}}]\!]. \tag{89}$$

Note that $\mathrm{Sel}^{\Sigma}_{K_\infty^{\mathrm{ac}},A}(f)$ is already known to be $A[\![\Gamma_K^{\mathrm{ac}}]\!]$-cotorsion by Proposition 5.2.8, Theorem 3.4.9, and Lemma 7.1.2. Hence by [58, Proposition A.2], we have

$$\mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]}\, \mathrm{Sel}^{\Sigma}_{K_\infty^{\mathrm{ac}},A}(f)^{\vee} = \mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]}\, \mathrm{Sel}^{\emptyset}_{K_\infty^{\mathrm{ac}},A}(f)^{\vee}\cdot \prod_{\substack{v\in\Sigma_K\\ v\nmid p}} \mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]}\left(\frac{H^1(K_v, T_f\otimes_{\mathcal{O}} A[\![\Gamma_K^{\mathrm{ac}}]\!]^{\vee})}{H^1_{\mathrm{unr}}(K_v, T_f\otimes_{\mathcal{O}} A[\![\Gamma_K^{\mathrm{ac}}]\!]^{\vee})}\right)^{\vee}.$$

By local Poitou-Tate duality, the local terms appearing above for $v\in\Sigma_K$ agree up to units with the ones in (82), and it also easy to check that they are not identically zero. Hence from (89), we can cancel the local factors to deduce

$$\mathbb{1}_{\mathrm{cyc}}\circ\phi(\alpha_i)\cdot \mathrm{char}_{A[\![\Gamma_K^{\mathrm{ac}}]\!]}\, \mathrm{Sel}^{\emptyset}_{K_\infty^{\mathrm{ac}},A}(f)^{\vee} \subset \boldsymbol{\lambda}(1)^2\cdot \frac{\Omega_{f,N^-}}{\Omega^+_{f_{\alpha_p}}\Omega^-_{f_{\alpha_p}}}\cdot A[\![\Gamma_K^{\mathrm{ac}}]\!] \tag{90}$$

in $A[\![\Gamma_K^{\mathrm{ac}}]\!]$. Finally, note that by (82) and the nonvanishing of $\boldsymbol{\lambda}(1)$ (see Proposition 5.2.8), we may choose α_i so that $\mathbb{1}_{\mathrm{cyc}}\circ\phi(\alpha_i)\neq 0$. Inverting p, descending coefficients to $\mathcal{O}$ using (81), and applying Lemma 7.1.2, we have

$$\left(\mathrm{char}_\Lambda\, \mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^{\vee}\right)^{\iota} \subset (\boldsymbol{\lambda}(1))^2 \tag{91}$$

in $\Lambda\otimes\mathbb{Q}_p$. The action of ι can be removed by [19, Theorem B]. To upgrade (91) to a divisibility in Λ, it suffices to note that $\boldsymbol{\lambda}(1)\not\equiv 0 \pmod{\wp}$ by Remark 5.2.9.

□

Remark 7.2.2. (1) In §8 below, we will need the more refined observation that α_i in (90) can be chosen so that

$$\mathrm{ord}_\pi\, \mathbb{1}_{\mathrm{cyc}}\circ\phi(\alpha_i) = \sum_{\ell|N^-}\mathrm{ord}_\pi(1-\ell^2) + \sum_{\ell|\,\mathrm{disc}(K)}\mathrm{ord}_\pi(1+\ell-a_\ell) + \mathrm{ord}_\pi\frac{\Omega_{f,N^-}}{\Omega^+_{f_{\alpha_p}}\Omega^-_{f_{\alpha_p}}}.$$

(Here we extend ord_π to a valuation on A.) This follows from comparing the μ-invariants in (82): the μ-invariant of $\boldsymbol{\lambda}(1)$ vanishes, as noted in Remark 5.2.9; the local Euler factor at $v|N^+$ is nonzero modulo π by a simple calculation, using that such primes are not infinitely split in K_∞^{ac}; the local Euler factor at $\ell|N^-$ is $(1-\ell^{-2}) = -\ell^{-2}(1-\ell^2)$ because $T_f|_{G_{K_\ell}}$ is a ramified extension of the form $\begin{pmatrix}\chi & * \\ 0 & 1\end{pmatrix}$; and the local Euler factor at $\ell|\,\mathrm{disc}(K)$ is $\det(1-\mathrm{Frob}_\ell^{-1}\,|T_f) = \ell^{-1}(1+\ell-a_\ell)$ because T_f is unramified at ℓ and, if λ is the unique prime of K above ℓ, then Frob_λ is trivial in Γ_K^{ac}.

(2) In [71, Theorem 3.26], the divisibility (88) is established without the factor of α_i, but under the additional assumption that $\overline{T}_f$ is ramified at all $\ell|N^-$. The presence of the extra factor α_i means that, in the generality of Theorem 7.2.1, (88) carries no information about specializations that do not factor through $\mathbb{1}_{\mathrm{cyc}}\circ\phi$.

7.3. **The Heegner point main conjecture.** In this subsection, we prove the following main theorem.

Theorem 7.3.1. *Let f be a non-CM cuspidal eigenform of weight two and trivial character, new of level N, with ring of integers $\mathcal{O}_f$ of its coefficient field. Let $\wp \subset \mathcal{O}_f$ be an ordinary prime of residue characteristic p, and let K be an imaginary quadratic field. Suppose:*

- *$N = N^+N^-$, where every factor of N^+ is split in K, and N^- is a squarefree product of primes inert in K.*
- *$p \nmid 2N$ is split in K.*
- *The modulo $\wp$ representation $\overline{T}_f$ associated to f is absolutely irreducible; if $p = 3$, assume that $\overline{T}_f$ is not induced from a character of $G_{\mathbb{Q}(\sqrt{-3})}$.*

Then, for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)}$ such that $\boldsymbol{\kappa}(\mathsf{Q}) \neq 0$, we have

$$\mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = \mathrm{crk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f) = 1$$

and

$$\mathrm{char}_\Lambda \left(\left(\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)^\vee \right)_{\mathrm{tors}} \right) = \mathrm{char}_\Lambda \left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f)}{(\boldsymbol{\kappa}(\mathsf{Q}))} \right)^2 \text{ in } \Lambda \otimes \mathbb{Q}_p.$$

For all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$ such that $\boldsymbol{\lambda}(\mathsf{Q}) \neq 0$,

$$\mathrm{rk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = \mathrm{crk}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f) = 0$$

and

$$\mathrm{char}_\Lambda \left(\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)^\vee \right) = (\boldsymbol{\lambda}(\mathsf{Q}))^2 \text{ in } \Lambda \otimes \mathbb{Q}_p.$$

Under condition (sclr), the equalities hold in Λ.

Proof. Let $(\mathcal{F}, \mathsf{S})$ be the Selmer structure on T_f defined by (44) for $\mathfrak{P} = (T)$.[4] Now apply Proposition 6.3.6 to $(\mathcal{F}, \mathsf{S})$ to obtain a pair $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, represented by a sequence $\{Q_n, \epsilon_{Q_n}\}$ of weakly admissible pairs. Let g_n be the resulting sequence of newforms of level NQ_n obtained from Theorem 6.2.4; g_n may only be defined for $\mathfrak{F}$-many n. Without loss of generality, we assume that all $q_n | Q_n$ are inert in K.

Step 1. $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}^{\nu(N^-)+1}$.

Proof. By Proposition 3.4.5 and Nakayama's Lemma, $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f) = 0$ implies $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})}(\mathbf{T}_f) = 0$, which by Theorem 3.4.9 and the nontriviality of $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$ implies the claim. □

Step 2. *For any fixed j,*

$$(\boldsymbol{\lambda}(\mathsf{Q})) \equiv (\boldsymbol{\lambda}_{g_n}(1)) \pmod{\pi^j, T^j}$$

for $\mathfrak{F}$-many n.

Proof. We use the notations of §4.6 and §5.2. By definition, the image of $\boldsymbol{\lambda}(\mathsf{Q})$ modulo (π^j, T^j) is a map $\mathrm{Gal}(K_j/K) \to \mathcal{O}$ obtained, for $\mathfrak{F}$-many n, by evaluating a map $F_n : M_{Q_n} \to \mathcal{O}(f)/\pi^j$ of $\mathbb{T}^{Q_n}$-modules at certain CM points, where $\mathcal{O}(f)$ is defined to be $\mathcal{O}$ with $\mathbb{T}^{Q_n}$-module structure determined by f as in Definition 4.6.2. Recall that the map F_n is chosen to be surjective after $\mathcal{O}$-linearization and to factor through $\mathcal{O}(f)/\pi^{j+C}$ for the constant C of Lemma 4.6.5, and that these properties uniquely determine F_n up to a scalar in $\mathcal{O}^\times$. When g_n has a sufficiently deep congruence to f, $\mathcal{O}(g_n)/\pi^{j+C} = \mathcal{O}(f)/\pi^{j+C}$ as $\mathbb{T}^{Q_n}$-modules, and the composite $G_n : M_{Q_n} \to \mathcal{O}(g_n) \to \mathcal{O}(g_n)/\pi^j$ therefore induces a unit multiple of F_n, where $G_n : M_{Q_n} \to \mathcal{O}(g_n)$ is the quaternionic modular form associated to g_n by the Jacquet-Langlands correspondence. (Here $\mathcal{O}(g_n)$ is $\mathcal{O}$ with $\mathbb{T}^{Q_n}$-module structure determined by the Hecke eigenvalues of g_n.) But by Remark 5.1.3, G_n is the very map whose evaluation at CM points is used to define $\boldsymbol{\lambda}_{g_n}(1)$, and the claim follows. □

Fix j, and restrict to those $\mathfrak{F}$-many n such that all $q_n | Q_n$ are j-admissible and inert in K. Then $H^1(K_{q_n}, \mathbf{W}_f[\pi^j]) = H^1(K_{q_n}, T_j) \otimes_{\mathcal{O}} \mathrm{Hom}(\Lambda, \mathcal{O})$ for all $q_n | Q_n$, and we let

$$H^1_{\mathrm{ord}}(K_{q_n}, \mathbf{W}_f[\pi^j]) \coloneqq H^1_{\mathrm{ord}}(K_{q_n}, T_j) \otimes_{\mathcal{O}} \mathrm{Hom}(\Lambda, \mathcal{O})$$

[4]It is easy to check that this is the same Selmer structure as defined in (72) with $\mathsf{m} = 1$, but we omit the proof since it is not needed for the main results.

in the notation of (60). Define

$$\begin{aligned}\mathrm{Sel}_{\mathcal{F}_\Lambda(Q_n)^*}(\mathbf{W}_f[\pi^j]) = \ker\Big(H^1(K,\mathbf{W}_f[\pi^j]) \to \prod_{v\nmid NpQ_n\infty} H^1(I_v,\mathbf{W}_f)\times\prod_{v|N} H^1(K_v,\mathbf{W}_f)\\ \times\prod_{v|Q_n}\frac{H^1(K_{q_n},\mathbf{W}_f[\pi^j])}{H^1_{\mathrm{ord}}(K_{q_n},\mathbf{W}_f[\pi^j])}\times\prod_{v|p} H^1(K_v,\mathrm{gr}_v\,\mathbf{W}_f)\Big).\end{aligned}\tag{92}$$

Here, for $v|p$, $\mathrm{gr}_v\,\mathbf{W}_f = (\mathrm{gr}_v\,T_f\otimes\Lambda^\vee)^\iota$, which is a direct summand of $\mathbf{W}_f$ as a $\Lambda[G_{K_v}]$-module. Also, by [65, Corollary B.3.4], (92) coincides with

$$\begin{aligned}\ker\Big(H^1(K,\mathbf{W}_f[\pi^j]) \to \prod_{v\nmid N^-pQ_n\infty} H^1(I_v,\mathbf{W}_f)\times\prod_{v|N^-} H^1(K_v,\mathbf{W}_f)\\ \times\prod_{v|Q_n}\frac{H^1(K_{q_n},\mathbf{W}_f[\pi^j])}{H^1_{\mathrm{ord}}(K_{q_n},\mathbf{W}_f[\pi^j])}\times\prod_{v|p} H^1(K_v,\mathrm{gr}_v\,\mathbf{W}_f)\Big).\end{aligned}\tag{93}$$

Step 3. *For any fixed j, $\mathrm{Sel}_{\mathcal{F}_\Lambda(Q_n)^*}(\mathbf{W}_f[\pi^j])$ and $\mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n}[\pi^j])$ are isomorphic as Λ-modules for $\mathfrak{F}$-many n.*

Here $\mathcal{F}_{g_n,\Lambda}$ refers to the Selmer structure for $\mathbf{T}_{g_n}$ defined the same way as (43).

Proof. For $\mathfrak{F}$-many n, $\mathbf{W}_f[\pi^j]$ is isomorphic to $\mathbf{W}_{g_n}[\pi^j]$ as a $\Lambda[G_K]$-module, so it suffices to compare the local conditions at all $v|NpQ_n$. For $v|N$, using the first definition (92), we must compare the kernels

$$\begin{aligned}&\ker\left(H^1(K_v,\mathbf{W}_f[\pi^j])\to H^1(K_v,\mathbf{W}_f)\right),\\ &\ker\left(H^1(K_v,\mathbf{W}_{g_n}[\pi^j])\to H^1(K_v,\mathbf{W}_{g_n})\right).\end{aligned}\tag{94}$$

Suppose first that $T_f^{I_v}\neq 0$; then since T_f and T_{g_n} have the same conductor at v, $T_{g_n}^{I_v}\neq 0$ as well, so both $T_f|_{G_{K_v}}$ and $T_{g_n}|_{G_{K_v}}$ are Steinberg representations twisted by the same unramified character. One readily checks that, for j sufficiently large depending on $T_f|_{G_{K_v}}$, $T_f\equiv T_{g_n} \pmod{\pi^j}$ implies $T_f|_{G_{K_v}}\cong T_{g_n}|_{G_{K_v}}$; then $\mathbf{W}_f|_{G_{K_v}}\cong\mathbf{W}_{g_n}|_{G_{K_v}}$, so the kernels (94) clearly coincide. On the other hand, if $T_f^{I_v}=0$, then $\mathbf{W}_f^{G_{K_v}}\subset\mathbf{W}_f^{I_v}$ is annihilated by π^M for some $M\geq 0$. The same is then true for $\mathbf{W}_{g_n}^{G_{K_v}}$ for $\mathfrak{F}$-many n. Since the kernels in (94) are identified with $\mathbf{W}_f^{G_{K_v}}/\pi^j\mathbf{W}_f^{G_{K_v}}$ and $\mathbf{W}_{g_n}^{G_{K_v}}/\pi^j\mathbf{W}_{g_n}^{G_{K_v}}$, respectively, these kernels will coincide provided $\mathbf{W}_f[\pi^{j+M}]\cong\mathbf{W}_{g_n}[\pi^{j+M}]$, which occurs for $\mathfrak{F}$-many n.

For places $v|p$, it suffices to show that the kernels

$$\ker\left(H^1(K_v,\mathrm{gr}_v\,\mathbf{W}_f[\pi^j])\to H^1(K_v,\mathrm{gr}_v\,\mathbf{W}_f)\right)\tag{95}$$

$$\ker\left(H^1(K_v,\mathrm{gr}_v\,\mathbf{W}_{g_n}[\pi^j])\to H^1(K_v,\mathrm{gr}_v\,\mathbf{W}_{g_n})\right)\tag{96}$$

coincide. Note that (95) is identified with $(\mathrm{gr}_v\,\mathbf{W}_f)^{G_{K_v}}/\pi^j(\mathrm{gr}_v\,\mathbf{W}_f)^{G_{K_v}}$. If we put $\mathrm{gr}_v\,W_f=\mathrm{gr}_v\,\mathbf{W}_f[T]\cong(\mathrm{gr}_v\,T_f)^\vee$, this is just $(\mathrm{gr}_v\,W_f)^{G_{K_v}}/\pi^jW_f^{G_{K_v}}$ since $(\mathrm{gr}_v\,\mathbf{W}_f)^{I_v}=\mathrm{gr}_v\,W_f$, and likewise for (96). Since Frob_v acts on $\mathrm{gr}_v\,W_f$ by $\alpha_p\neq 1$ in the notation of (5.2.3), $(\mathrm{gr}_v\,W_f)^{G_{K_v}}$ is finite and hence identified with $(\mathrm{gr}_v\,W_{g_n})^{G_{K_v}}$ for $\mathfrak{F}$-many n. So the same reasoning as for $v|N$ shows that (95), (96) coincide for $\mathfrak{F}$-many n.

Finally, for $q_n|Q_n$, it suffices to compare the local condition

$$\ker\left(H^1(K_{q_n},\mathbf{W}_{g_n}[\pi^j])\to H^1(K_{q_n},\mathbf{W}_{g_n})\right)\tag{97}$$

with $H^1_{\mathrm{ord}}(K_{q_n},\mathbf{W}_f[\pi^j])$ when q_n is j-admissible. Let $\{e_1,e_2\}$ be a basis for T_{g_n} with respect to which $T_{g_n}|_{G_{\mathbb{Q}_{q_n}}}$ has the form $\begin{pmatrix}\underline{\epsilon}\chi & *\\ 0 & \underline{\epsilon}\end{pmatrix}$, with $*$ ramified and $\underline{\epsilon}$ the unramified local character $\mathrm{Frob}_{q_n}\mapsto\epsilon_{Q_n}(q_n)$. Then a direct calculation shows that the kernel (97) is the image of $H^1(K_{q_n},e_1\otimes\Lambda^\vee[\pi^j])$, which is the same as the ordinary local condition for $\mathbf{W}_f$ under the isomorphism $\mathbf{W}_f[\pi^j]\cong\mathbf{W}_{g_n}[\pi^j]$. $\square$

Step 4. *For any fixed j,*

$$\mathrm{Fitt}_\Lambda\,\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)^\vee\equiv\mathrm{Fitt}_\Lambda\,\mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n})^\vee \pmod{\pi^j,T^j}$$

for $\mathfrak{F}$-many n.

Proof. Since Fitting ideals are stable under base change, it suffices to show

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)[\pi^j, T^j]^\vee = \mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n})[\pi^j, T^j]^\vee. \tag{98}$$

Recall that $\overline{T}_f$ has no G_K-fixed points by Lemma 3.3.4. Hence Lemma 2.4.12 applied to the short exact sequences $0 \to \mathbf{W}_f[\pi^j] \to \mathbf{W}_f \xrightarrow{\pi^j} \mathbf{W}_f \to 0$ and $0 \to \mathbf{W}_f[T^j, \pi^j] \to \mathbf{W}_f[\pi^j] \xrightarrow{T^j} \mathbf{W}_f[\pi^j] \to 0$ shows that

$$\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)[\pi^j, T^j] = \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f[\pi^j])[T^j] = \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f[\pi^j, T^j]),$$

and likewise for g_n. So it suffices to show that

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f[\pi^j, T^j])^\vee = \mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n}[\pi^j])[T^j]^\vee \tag{99}$$

for $\mathfrak{F}$-many n. Unraveling the definition of the patched Selmer group, and again using Lemma 2.4.12, we find that

$$\begin{aligned}\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f[\pi^j, T^j]) &= \mathcal{U}\left(\left\{\mathrm{Sel}_{\mathcal{F}_\Lambda(Q_n)^*}(\mathbf{W}_f[\pi^j, T^j])\right\}_{n\in\mathbb{N}}\right)\\ &= \mathcal{U}\left(\left\{\mathrm{Sel}_{\mathcal{F}_\Lambda(Q_n)^*}(\mathbf{W}_f[\pi^j])[T^j])\right\}_{n\in\mathbb{N}}\right),\end{aligned}$$

where the Selmer structure $\mathcal{F}_{\Lambda(Q_n)^*}$ for $\mathbf{W}_f[\pi^j, T^j]$ is induced by the local conditions in (93), so Step 3 implies (99).

□

Step 5. *Conclusion of the proof.*

For $\mathfrak{F}$-many n, N^-Q_n is the squarefree product of an odd number of primes inert in K. By Theorem 7.2.1, for such n we have:

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}^*_{g_n,\Lambda}}(\mathbf{W}_{g_n})^\vee \subset (\boldsymbol{\lambda}_{g_n}(1))^2 \subset \Lambda. \tag{100}$$

By Steps 2 and 4, (100) implies that

$$\mathrm{Fitt}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(\mathsf{Q}))^2 \subset \Lambda. \tag{101}$$

Since the characteristic ideal of any Λ-module is the smallest divisorial ideal containing the Fitting ideal, (101) implies

$$\mathrm{char}_\Lambda \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)^\vee \subset (\boldsymbol{\lambda}(\mathsf{Q}))^2 \subset \Lambda. \tag{102}$$

Combining with Theorem 3.4.9 completes the proof.

□

Corollary 7.3.2. *Under the hypotheses of Theorem 7.3.1, if additionally $\nu(N^-)$ is even, then the Heegner point main conjecture holds for f in $\Lambda \otimes \mathbb{Q}_p$; that is, there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \approx \Lambda \oplus M \oplus M$$

for some torsion Λ-module M, and

$$\mathrm{char}_\Lambda\left(\frac{\mathrm{Sel}_{\mathcal{F}_\Lambda}(\mathbf{T}_f)}{(\boldsymbol{\kappa}(1))}\right) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. Under condition (sclr), the equality holds in Λ.

Corollary 7.3.3. *Under the hypotheses of Theorem 7.3.1, if additionally $\nu(N^-)$ is odd, then the anticyclotomic main conjecture holds for f in $\Lambda \otimes \mathbb{Q}_p$; that is, there is a pseudo-isomorphism of Λ-modules:*

$$\mathrm{Sel}_{\mathcal{F}^*_\Lambda}(\mathbf{W}_f)^\vee \approx M \oplus M$$

for some torsion Λ-module M, and

$$(\boldsymbol{\lambda}(1)) = \mathrm{char}_\Lambda(M)$$

as ideals of $\Lambda \otimes \mathbb{Q}_p$. Under condition (sclr), the equality holds in Λ.

Corollary 7.3.4. *Under the hypotheses of Theorem 7.3.1, the bipartite Euler system*

$$(\kappa(1, \cdot), \lambda(1, \cdot))$$

of Remark 5.3.14(1) is nontrivial.

Proof. Keep the notation of the proof of Theorem 7.3.1. By Proposition 3.3.6, $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f)$ is finite. Since $\mathsf{H}^1(K^{\mathsf{Q}\cup\mathsf{S}}/K, \mathbf{W}_f)[T] = \mathsf{H}^1(K^{\mathsf{Q}\cup\mathsf{S}}/K, W_f)$ by Lemmas 2.4.12 and 3.3.4, the cokernel of $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f) \to \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)[T]$ injects into

(103)

$$\prod_{\mathsf{v}\in\mathsf{S}\cup\mathsf{Q}} \ker\left(\frac{\mathsf{H}^1(K_\mathsf{v}, W_f)}{\mathsf{H}^1_{\mathcal{F}(\mathsf{Q})}(K_\mathsf{v}, W_f)} \to \frac{\mathsf{H}^1(K_\mathsf{v}, \mathbf{W}_f)}{\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(K_\mathsf{v}, \mathbf{W}_f)}\right) = \prod_{\mathsf{v}\in\mathsf{S}\cup\mathsf{Q}} \ker\left(H^1_{\mathcal{F}(\mathsf{Q})}(K_\mathsf{v}, T_f)^\vee \to \mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_\mathsf{v}, \mathbf{T}_f)^\vee\right).$$

We claim (103) is finite: indeed, the natural map $\mathsf{H}^1_{\mathcal{F}_\Lambda(\mathsf{Q})}(K_\mathsf{v}, \mathbf{T}_f) \to \mathsf{H}^1_{\mathcal{F}(\mathsf{Q})}(K_\mathsf{v}, T_f)$ is surjective for $\mathsf{v} = \mathsf{q} \in \mathsf{Q}$ or for $\mathsf{v} = \underline{v}$ with $v|N^-$ by the discussion in the proof of Proposition 3.4.5, and has finite cokernel for all other $\mathsf{v} \in \mathsf{S}$ by [37, Lemma 2.2.7]. Hence $\mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f)[T]$ is finite as well, which implies

$$\mathrm{char}_\Lambda \, \mathrm{Sel}_{\mathcal{F}_\Lambda(\mathsf{Q})^*}(\mathbf{W}_f) \not\subset (T).$$

Theorem 7.3.1 then shows $(\boldsymbol{\lambda}(\mathsf{Q})) \not\subset (T)$, hence $\lambda(1, \mathsf{Q}) \neq 0$ by Remark 5.3.14(2). □

8. Proof of main results: Kolyvagin's conjecture and p-converse theorem

8.1. **Nonvanishing of $(\kappa(1,\cdot), \lambda(1,\cdot))$ and p-converse theorem.** In this section, we shall prove the following theorem:

Theorem 8.1.1. *Let f be a non-CM cuspidal eigenform of weight two and trivial character, new of level N, with ring of integers $\mathcal{O}_f$ of its coefficient field. Let $\wp \subset \mathcal{O}_f$ be a prime, and let K be an imaginary quadratic field. Assume:*

- *$N = N^+N^-$, where every factor of N^+ is split in K, and N^- is a squarefree product of an even number of primes inert in K.*
- *The residue characteristic p of $\wp$ does not divide $2N\,\mathrm{disc}(K)$.*
- *The modulo $\wp$ representation $\overline{T}_f$ associated to f is absolutely irreducible; and if $p = 3$, $\overline{T}_f$ is not induced from a character of $G_{\mathbb{Q}(\sqrt{-3})}$.*
- *If p is inert in K, then there exists some prime $\ell_0||N$.*
- *If a_p is not a $\wp$-adic unit, then there exist primes $\ell_i||N$ for $i = 1, 2$ (possibly with $\ell_1 = \ell_2$) such that $\overline{T}_f|_{G_{\mathbb{Q}_{\ell_i}}}$ is ramified for $i = 1, 2$ and $\overline{T}_f^{G_{\mathbb{Q}_{\ell_1}}} = (\overline{T}_f \otimes \chi_K)^{G_{\mathbb{Q}_{\ell_2}}} = 0$, where χ_K is the quadratic character of $G_\mathbb{Q}$ associated to K.*

Then $(\kappa(1,\cdot), \lambda(1,\cdot))$ is nontrivial.

8.1.2. If p is split in K and $\wp$ is ordinary, then Theorem 8.1.1 is simply Corollary 7.3.4. In the inert or non-ordinary cases, the anticyclotomic main conjecture is currently not known in full generality; however, since all we are interested in is specialization at the trivial character, we are able to nonetheless prove Theorem 8.1.1 by combining cyclotomic main conjectures for quadratic twists of f.

Corollary 8.1.3. *Let f, $\wp$, and K be as in Theorem 8.1.1. Then*

$$\mathrm{rk}_\mathcal{O} \, \mathrm{Sel}_\mathcal{F}(T_f) = 1 \iff L'(f/K, 1) \neq 0.$$

Here, $(\mathcal{F}, \mathsf{S})$ is the Selmer structure for the G_K-module T_f defined by (72) with $\mathsf{m} = 1$.

Proof. If $\nu(N^-)$ is odd then $\mathrm{rk}_\mathcal{O} \, \mathrm{Sel}_\mathcal{F}(T_f)$ is even by Theorem 3.3.14 and Theorem 8.1.1, and $L'(f/K, 1) = 0$ by root number considerations.

So without loss of generality, we may assume $\nu(N^-)$ is even. Then by Remark 5.1.3, $\kappa(1,1) \in \mathrm{Sel}_\mathcal{F}(T_f)$ is the Kummer image of the classical Heegner point $y_K \in A_f(K)$; in particular, by the Gross-Zagier Theorem of [79, Theorem 1.2.1], $\kappa(1,1) \neq 0$ if and only if $L'(f/K, 1) \neq 0$. On the other hand, $\kappa(1,1) \neq 0$ if and only if $\mathrm{rk}_\mathcal{O} \, \mathrm{Sel}_\mathcal{F}(T_f) = 1$ by Theorem 3.3.14 and Theorem 8.1.1, and this gives the corollary. □

Before completing the proof of Theorem 8.1.1 in §8.6 below, we first give another application, to the nonvanishing of Kolyvagin classes.

8.2. **Kolyvagin's conjecture.**

8.2.1. Assume the condition (disc) from (5.3.1).

For any $\mathsf{m} \in \mathsf{K}$ and any $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}\cup\mathsf{m}}$, define the m-transverse Selmer ranks

$$r_{\mathsf{m}}^{\pm}(\mathsf{Q}) = \mathrm{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f)^{\pm}, \tag{104}$$

where $\pm$ refers to the τ eigenvalue ± 1 for a generator $\tau \in \mathrm{Gal}(K/\mathbb{Q})$; note that this is well-defined because the local conditions defining $\mathcal{F}(\mathsf{m}, \mathsf{Q})$ are all τ-stable. When $\mathsf{Q} = 1$, we simply write $r_{\mathsf{m}}^{\pm}$. When $\mathsf{m} = 1$, the $r_1^{\pm}$ are the classical Selmer ranks of f.

Proposition 8.2.2. *Fix $\mathsf{m} \in \mathsf{K}$, and let $\mathsf{l} \notin \mathsf{m}$ be Kolyvagin-admissible. Then for all $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}\cup\mathsf{ml}}$, and for each $\delta \in \{\pm\}$, either:*

- $r_{\mathsf{ml}}^{\delta}(\mathsf{Q}) = r_{\mathsf{m}}^{\delta}(\mathsf{Q}) - 1$, $\mathrm{loc}_{\mathsf{l}}^{\delta}(\operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f))^{\delta} \neq 0$, *and* $\partial_{\mathsf{l}}^{\delta}(\operatorname{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f))^{\delta} = 0$.
- $r_{\mathsf{ml}}^{\delta}(\mathsf{Q}) = r_{\mathsf{m}}^{\delta}(\mathsf{Q}) + 1$, $\mathrm{loc}_{\mathsf{l}}^{\delta}(\operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f))^{\delta} = 0$, *and* $\partial_{\mathsf{l}}^{\delta}(\operatorname{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f))^{\delta} \neq 0$.

Proof. If $\mathcal{F}^{\mathsf{l}}(\mathsf{m}, \mathsf{Q}) = \mathcal{F}(\mathsf{ml}, \mathsf{Q}) + \mathcal{F}(\mathsf{m}, \mathsf{Q})$ and $\mathcal{F}_{\mathsf{l}}(\mathsf{m}, \mathsf{Q}) = \mathcal{F}(\mathsf{ml}, \mathsf{Q}) \cap \mathcal{F}(\mathsf{m}, \mathsf{Q})$, then we have a τ-equivariant exact sequence

$$0 \to \operatorname{Sel}_{\mathcal{F}_{\mathsf{l}}(\mathsf{m},\mathsf{Q})}(T_f) \to \operatorname{Sel}_{\mathcal{F}^{\mathsf{l}}(\mathsf{m},\mathsf{Q})}(T_f) \to \mathsf{H}^1(K_{\mathsf{l}}, T_f),$$

where the image of the final arrow has $\mathcal{O}$-rank two and is self-annihilating under the local Tate pairing by Propositions 2.6.10 and 2.6.13. Since the Tate pairing of two classes with opposite τ eigenvalues is necessarily zero, the proposition follows as in the proof of Proposition 3.3.8. □

Lemma 8.2.3. *Suppose given elements $c^{\pm} \in \mathsf{H}^1(K, T_f)^{\pm}$. Then there exists a Kolyvagin-admissible ultraprime l such that*

$$c^{\pm} \neq 0 \implies \mathrm{loc}_{\mathsf{l}}^{\pm} c^{\pm} \neq 0.$$

If (sclr) holds for T_f, then the same is true for elements $c^{\pm} \in \mathsf{H}^1(K, T_f/\pi^j)$.

Proof. The proof of Theorem 3.3.9 applies almost verbatim, except that in the proof of Lemma 3.3.11 we will have two homomorphisms

$$\phi^{\pm} \in \mathrm{Hom}_{G_K}(G_L, \overline{T}_f)^{\pm},$$

and we must choose $g \in G_L$ so that $\phi^{\epsilon}(g)$ has nonzero component in the τ eigenspace of sign ϵ for both choices ϵ (unless ϕ^{ϵ} is itself 0); for each ϵ, this condition is satisfied outside a proper subgroup of G_L, so indeed there exists $g \in G_L$ such that both conditions are satisfied. With this modification, the rest of the proof applies unchanged. □

Lemma 8.2.4. *Suppose that the bipartite Euler system $(\kappa(1, \cdot), \lambda(1, \cdot))$ of (5.3.12) is nontrivial. Then, for all $\mathsf{m} \in \mathsf{K}$, $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ is nontrivial.*

In particular, for all $\mathsf{m} \in \mathsf{K}$ and $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}\cup\mathsf{m}}$:

$$\mathrm{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) \equiv \nu(N^-) + 1 \pmod 2;$$

and

$$\mathrm{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) \leq 1 \iff \begin{cases} \kappa(\mathsf{m}, \mathsf{Q}) \neq 0, & \nu(N^-) + |\mathsf{Q}| \text{ even} \\ \lambda(\mathsf{m}, \mathsf{Q}) \neq 0, & \nu(N^-) + |\mathsf{Q}| \text{ odd.} \end{cases}$$

Proof. Recall that, for fixed m, the pair $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ forms a bipartite Euler system with parity $\nu(N^-)$ for the self-dual Selmer structure $(\mathcal{F}(\mathsf{m}), \mathsf{S} \cup \mathsf{m})$ on T_f. We will prove that, for any $\mathsf{ml} \in \mathsf{K}$, if $(\kappa(\mathsf{m}, \cdot), \lambda(\mathsf{m}, \cdot))$ is nontrivial then so is $(\kappa(\mathsf{ml}, \cdot), \lambda(\mathsf{ml}, \cdot))$; this suffices by Theorem 3.3.14.

Choose $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}_{\mathsf{S}\cup\mathsf{ml}}^{\nu(N^-)+1}$ such that $\operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Q})}(T_f) = 0$; this is possible by Corollary 3.3.13. By Proposition 8.2.2, we may choose a nonzero

$$d \in \operatorname{Sel}_{\mathcal{F}(\mathsf{ml},\mathsf{Q})}(T_f).$$

Applying Theorem 3.3.9 to d, let q be admissible with sign ϵ_{q} such that $\mathsf{q} \notin \mathsf{Q} \cup \mathsf{ml}$ and $\mathrm{loc}_{\mathsf{q}} d \neq 0$. By Proposition 3.3.8 for the Selmer structures $\mathcal{F}(\mathsf{m}, \mathsf{Qq})$ and $\mathcal{F}(\mathsf{m}, \mathsf{Q})$,

$$\mathrm{rk}_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Qq})}(T_f) = 1. \tag{105}$$

Hence, by hypothesis, $\kappa(\mathsf{m}, \mathsf{Qq})$ generates $\operatorname{Sel}_{\mathcal{F}(\mathsf{m},\mathsf{Qq})}(T_f)$ up to finite index, and in particular $\partial_{\mathsf{q}} \kappa(\mathsf{m}, \mathsf{Qq}) \neq 0$. Now, taking the sum of local pairings and using Proposition 2.6.10,

$$0 = \sum_{\mathsf{v}} \langle d, \kappa(\mathsf{m}, \mathsf{Qq}) \rangle_{\mathsf{v}} = \langle d, \kappa(\mathsf{m}, \mathsf{Qq}) \rangle_{\mathsf{l}} + \langle d, \kappa(\mathsf{m}, \mathsf{Qq}) \rangle_{\mathsf{q}}. \tag{106}$$

Since the latter pairing is nonzero by construction, the former is as well, and so $\mathrm{Res}_{\mathsf{l}}\,\kappa(\mathsf{m},\mathsf{Qq}) \neq 0$. By Proposition 5.3.13(1), $\kappa(\mathsf{ml},\mathsf{Qq}) \neq 0$ as well, which completes the proof. □

8.2.5. For any $\mathsf{m} \in \mathsf{K}$, define the vanishing order of the Kolyvagin system at m:

$$\nu_{\mathsf{m}} = \begin{cases} \min\{|\mathsf{n}| \,:\, \mathsf{mn} \in \mathsf{K},\, \lambda(\mathsf{mn},1) \neq 0\}, & \nu(N^-) \text{ odd}, \\ \min\{|\mathsf{n}| \,:\, \mathsf{mn} \in \mathsf{K},\, \kappa(\mathsf{mn},1) \neq 0\}, & \nu(N^-) \text{ even}. \end{cases} \tag{107}$$

Theorem 8.2.6. *Assume (disc). If $(\kappa(1,\cdot),\lambda(1,\cdot))$ is nontrivial, and in particular under the hypotheses of Theorem 8.1.1, we have for all $\mathsf{m} \in \mathsf{K}$:*

- *If $\nu(N^-)$ is odd, then $\nu_{\mathsf{m}} = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\}$ and $\epsilon_f = (-1)^{r_{\mathsf{m}}^\pm + |\mathsf{m}|}$.*
- *If $\nu(N^-)$ is even, then $\nu_{\mathsf{m}} = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - 1$ and $\epsilon_f \cdot (-1)^{|\mathsf{m}|+\nu_{\mathsf{m}}+1}$ is the τ-eigenvalue of the larger eigenspace.*

Proof. For ease of notation, let $\delta = 0$ if $\nu(N^-)$ is odd, and $\delta = 1$ if $\nu(N^-)$ is even. Suppose given $\mathsf{mn} \in \mathsf{K}$ such that $\lambda(\mathsf{mn},1)$ or $\kappa(\mathsf{mn},1)$ is nontrivial; then

$$\mathrm{rk}_{\mathcal{O}}\,\mathrm{Sel}_{\mathcal{F}(\mathsf{mn})}(T_f) = \delta$$

by Lemma 8.2.4. In particular, the kernel of the localization map

$$\mathrm{Sel}_{\mathcal{F}(\mathsf{m})}(T_f)^\pm \to \bigoplus_{\mathsf{l}\in\mathsf{n}} H^1_{\mathrm{unr}}(K_{\mathsf{l}}, T_f)^\pm$$

has rank at most δ. It follows that $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - \delta \leq \nu_{\mathsf{m}}$. We now show that equality holds by induction on $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\}$. If $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} \leq \delta$, then Lemma 8.2.4 implies $\nu_{\mathsf{m}} = 0 = \max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} - \delta$.

Now suppose that $\max\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\} > \delta$, and let ϵ be the sign of the larger value of $r_{\mathsf{m}}^\pm$ (choose either if they agree). If $r_{\mathsf{m}}^{-\epsilon} > 0$, then by Lemma 8.2.3 and Proposition 8.2.2, there exists $\mathsf{l} \in \mathsf{K}$ not in m such that $r_{\mathsf{ml}}^\pm = r_{\mathsf{m}}^\pm - 1$. In this case, $\max\left\{r_{\mathsf{ml}}^+, r_{\mathsf{ml}}^-\right\} = \max\{r_{\mathsf{m}}^-, r_{\mathsf{m}}^-\} - 1$. Hence (by the inductive hypothesis)

$$\nu_{\mathsf{m}} \leq \nu_{\mathsf{ml}} + 1 = \max\left\{r_{\mathsf{ml}}^+, r_{\mathsf{ml}}^-\right\} - \delta + 1 = \max\left\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\right\} - \delta.$$

Since we have already shown the opposite equality, this completes the inductive step under the assumption $r_{\mathsf{m}}^{-\epsilon} > 0$.

If on the other hand $r_{\mathsf{m}}^{-\epsilon} = 0$, then $r_{\mathsf{m}}^{\epsilon} \geq \delta + 2$, since $r_{\mathsf{m}}^+ + r_{\mathsf{m}}^- \equiv \delta \pmod 2$ and we have assumed $r_{\mathsf{m}}^\epsilon > \delta$. Then by Lemma 8.2.3 and Proposition 8.2.2 again, we may choose $\mathsf{l} \in \mathsf{K}$ such that $r_{\mathsf{ml}}^\epsilon = r_{\mathsf{m}}^\epsilon - 1$, while necessarily $r_{\mathsf{ml}}^{-\epsilon} = 1 \leq \delta + 1 \leq r_{\mathsf{ml}}^\epsilon$. Hence $\max\left\{r_{\mathsf{ml}}^+, r_{\mathsf{ml}}^-\right\} = r_{\mathsf{m}}^\epsilon - 1$, and the same argument as above again completes the inductive step.

Finally, we consider the parity assertions of the theorem. If $\nu(N^-)$ is even, i.e. if $\delta = 1$, then the Selmer ranks $r_{\mathsf{m}}^\pm$ are always distinct by Lemma 8.2.4. As we pass from m to ml in the inductive step above, the sign of the larger eigenspace is preserved, and $\nu_{\mathsf{m}} + |\mathsf{m}| = \nu_{\mathsf{ml}} + |\mathsf{ml}|$. It therefore suffices to show that $\epsilon_f \cdot (-1)^{|\mathsf{m}|+\nu_{\mathsf{m}}+1}$ is the eigenvalue of the larger τ eigenspace when $\nu_{\mathsf{m}} = 0$, i.e. when $\kappa(\mathsf{m},1) \neq 0$. In this case it follows from Proposition 5.3.9.

When $\nu(N^-)$ is odd, then whenever $\lambda(\mathsf{mn}) \neq 0$, Proposition 5.3.9 implies that $\epsilon_f = (-1)^{|\mathsf{mn}|}$. Hence

$$\epsilon_f = (-1)^{\nu_{\mathsf{m}}+|\mathsf{m}|} = (-1)^{\max\left\{r_{\mathsf{m}}^+, r_{\mathsf{m}}^-\right\}+|\mathsf{m}|},$$

which proves the claim since $r_{\mathsf{m}}^\pm$ have the same parity when $\nu(N^-)$ is odd.

□

8.2.7. It remains to relate the nonvanishing of the patched Kolyvagin classes to the nonvanishing of their unpatched analogues. For this, recall the notation of Definition 5.3.4, and let m be a squarefree product of primes inert in K. If $\nu(N^-)$ is even, let $c(m) \in H^1(K, T_f/I_m)$ be the class $\overline{c}_{v_\wp(I_m)}(m,1)$; if $\nu(N^-)$ is odd, let $\lambda(m) \in \mathcal{O}/I_m$ be the element $\lambda_{v_\wp(I_m)}(m,1)$.

The classical vanishing order, generalizing Kolyvagin's original definition, is defined as:

$$\nu_{\mathrm{classical}} \coloneqq \begin{cases} \min\{\nu(m) \,:\, \lambda(m) \neq 0\}, & \nu(N^-) \text{ odd}, \\ \min\{\nu(m) \,:\, c(m) \neq 0\}, & \nu(N^-) \text{ even}. \end{cases} \tag{108}$$

Corollary 8.2.8. *Assume (disc). If $(\kappa(1,\cdot),\lambda(1,\cdot))$ is nontrivial, and in particular under the hypotheses of Theorem 8.1.1, $\nu_{\mathrm{classical}}$ is finite. If (sclr) holds for f, then $\nu_{\mathrm{classical}} = \nu_1$, and in particular:*

- *If* $\nu(N^-)$ *is odd, then* $\nu_{\text{classical}} = \max\left\{r_1^+, r_1^-\right\}$ *and* $r_1^\pm \equiv \frac{\epsilon_f - 1}{2} \pmod 2$.
- *If* $\nu(N^-)$ *is even, then* $\nu_{\text{classical}} = \max\left\{r_1^+, r_1^-\right\} - 1$ *and* $\epsilon_f \cdot (-1)^{1+\nu_{\text{classical}}}$ *is the larger* τ *eigenspace.*

Proof. The finiteness of the classical vanishing order is clear by construction: if a patched Kolyvagin class or element is nontrivial, then infinitely many of the classical Kolyvagin classes or elements defining it are nontrivial. This also shows $\nu_{\text{classical}} \leq \nu_1$. We will check that equality holds under the condition (sclr). Suppose first $\nu(N^-)$ is even. Given some nonzero $\overline{c}_j(m,1)$, one may show as in [50, p. 309] that there exists a sequence of squarefree products m_n of primes inert in K with $v_\wp(I_{m_n}) \to \infty$, $\nu(m_n) = \nu(m)$, and $\overline{c}_j(m_n, 1) \neq 0$. (In [50], additional hypotheses are put on the image of the Galois action, but the argument goes through under (sclr) by arguing as in Lemma 8.2.3.) In particular, the sequence $(m_n)_{n\in\mathbb{N}}$ defines a nonzero $\kappa(\mathsf{m}, 1)$ witnessing $\nu_1 \leq \nu_{\text{classical}}$.

Now suppose that $\nu(N^-)$ is odd, and that $\lambda_j(m, 1) \neq 0$ where $\nu(m) = \nu_{\text{classical}}$. We choose an auxiliary ultraprime $\mathsf{q} \in \mathsf{M}_\mathbb{Q}$ (with either sign ϵ_q) such that $\mathrm{Frob}_\mathsf{q} \in G_\mathbb{Q}$ is a complex conjugation, represented by a sequence $(q_n)_{n\in\mathbb{N}}$ where without loss of generality each q_n is inert in K and j-admissible with sign ϵ_q. By the non-patched analogue of Proposition 5.3.13(2), we have $\overline{c}_j(m, q_n) \neq 0$ for $\mathfrak{F}$-many n. Then once again, the argument of [50, p. 309] shows that there exists a sequence of squarefree products m_n of primes inert in K with $q_n \nmid m_n$, $\nu(m_n) = \nu(m)$, $v_\wp(I_{m_n}) \to \infty$, and $\overline{c}_j(m_n, q_n) \neq 0$ for $\mathfrak{F}$-many n. We therefore obtain a nonzero patched class $\kappa(\mathsf{m}, \mathsf{q})$ with $|\mathsf{m}| = \nu_{\text{classical}}$. By Lemma 2.4.13, $\mathrm{loc}_\mathsf{q} \mathrm{Sel}_{\mathcal{F}(\mathsf{m},1)}(T_f) = 0$. Hence by Proposition 3.3.8, we have $\partial_\mathsf{q}\kappa(\mathsf{m}, \mathsf{q}) \neq 0$, so $\lambda(\mathsf{m}, 1) \neq 0$ by Proposition 5.3.13(2). This shows $\nu_1 \leq \nu_{\text{classical}}$ and completes the proof. □

The rest of §8 is dedicated to proving Theorem 8.1.1, culminating in §8.6 below.

8.3. **Comparing periods.**

8.3.1. Let f, $\wp$, N, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5. For any factorization $N = N_1N_2$ with N_2 squarefree and coprime to N_1, let $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ be the maximal ideal associated to f and $\wp$; recall from (4.2.1) that $\mathbb{T}_{N_1,N_2}$ is the N_2-new quotient of the full cuspidal Hecke algebra of level N.

8.3.2. There are two natural periods that appear when studying special values of L-functions for f, cf. the discussions in [58, §2] and [76, §2]. For any factorization $N = N_1N_2$, where N_1 and N_2 are coprime, let $\pi_f : \mathbb{T}_{N_1,N_2,\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathcal{O} \to \mathcal{O}$ be the map defined by the Hecke eigenvalues of f. The congruence ideal $\eta_f(N_1, N_2) \subset \mathcal{O}$ is defined as

$$\pi_f(\mathrm{Ann}_{\mathbb{T}_{N_1,N_2,\mathfrak{m}}\otimes_{\mathbb{Z}_p}\mathcal{O}}(\ker \pi_f)). \tag{109}$$

If $\eta_f(N_1, N_2)_0 \in \mathcal{O}_{f,(\wp)}$ generates $\eta_f(N_1, N_2)$, then Hida's canonical period, well-defined up to a $\wp$-adic unit, is given by:

$$\Omega_f^{can} = \frac{(f, f)}{\eta_f(N, 1)_0}, \tag{110}$$

where (f, f) is the Peterson inner product. Also define $\Omega_f^\pm$ to be the periods of [71, §3.3.3] for f, and recall the periods $\Omega_{f_{\alpha_p}}^\pm$ from the proof of Theorem 7.2.1 when $\wp$ is ordinary. Then Ω_f^{can} is related to $\Omega_{f_{\alpha_p}}^\pm$ and $\Omega_f^\pm$ by the following:

Proposition 8.3.3. *We have* $\Omega_f^{can} = \Omega_f^+\Omega_f^-$ *up to* $\wp$*-adic units. If* $\wp$ *is ordinary, then in addition* $\Omega_f^{can} = \Omega_{f_{\alpha_p}}^+\Omega_{f_{\alpha_p}}^-$ *up to* $\wp$*-adic units.*

Proof. The first claim follows from [76, Remark 2.7][5] or [72, Lemma 9.5], whose proof does not need the assumption in *op. cit.* that $p||N$. Now suppose $\wp$ is ordinary. By [71, Lemma 12.1] combined with [36, (4.7)], we have

$$\Omega_{f_{\alpha_p}}^+\Omega_{f_{\alpha_p}}^- = (1 - \alpha_p^2)\frac{(f, f)}{\eta_f(Np, 1)}$$

up to $\wp$-adic units, where $\eta_f(Np, 1)$ is the congruence ideal for f_{α_p} viewed as an eigenform of level $\Gamma_0(Np)$. The argument in [25, p. 388] shows that $\eta_f(Np, 1) = (a_p^2 - (p + 1)^2)\eta_f(N, 1)$. On the other hand, since

[5] To check that the two definitions of Ω_f^{can} in [76, Remark 2.7] coincide, one can argue using [36, Theorem 6.6].

$\alpha_p + p/\alpha_p = a_p$, we have $a_p^2 - (p+1)^2 = (\alpha_p^2 - 1)(1 - p^2/\alpha_p^2)$, where the second factor is a $\wp$-adic unit. Hence $\Omega^+_{f_{\alpha_p}}\Omega^-_{f_{\alpha_p}} = \Omega_f^{can}$ up to $\wp$-adic units, which completes the proof. $\square$

8.3.4. If $N = N_1N_2$ where N_2 is squarefree with an odd number of prime factors, then the f-isotypic part of the Hecke module $\mathcal{O}_f[X_{N_1,N_2}]_{(\wp)}$ is free of rank one over $\mathcal{O}_{f,(\wp)}$; let φ_{f,N_2} be a generator. For each $x \in X_{N_1,N_2}$, recall that x is an isomorphism class of oriented Eichler orders; let $\mathrm{Aut}(x)$ be the automorphism group of any representative (automorphisms in the sense of (4.4.1)) and set $e(x) := \#\,\mathrm{Aut}(x)$. Let $\langle\cdot,\cdot\rangle$ be the diagonal pairing on $\mathbb{Z}[X_{N_1,N_2}]$ with weights $e(x)$, and extend $\langle\cdot,\cdot\rangle$ $\mathcal{O}_f$-linearly to a pairing on $\mathcal{O}_f[X_{N_1,N_2}]$. Gross's period is defined (up to $\wp$-adic units) by:

$$\Omega_{f,N_2} = \frac{(f,f)}{\langle\varphi_{f,N_2},\varphi_{f,N_2}\rangle}. \tag{111}$$

This period is the same one from [19, Theorem A] that appeared in the proof of Theorem 7.2.1, and occurs naturally in anticyclotomic Iwasawa theory due to Gross's special value formula. In particular, for the central values, we have:

Proposition 8.3.5. *Let K be an imaginary quadratic field of discriminant prime to Np, and suppose that $N = N_1N_2$ where all factors of N_1 are split in K, and N_2 is a squarefree product of an odd number of primes inert in K. Then $L(f/K,1) \in \mathrm{Frac}(\mathcal{O}_f)\cdot\Omega_{f,N_2}$ and the element $\lambda(1) \in \mathcal{O}$ constructed in (5.3.12) satisfies:*

$$\frac{L(f/K,1)}{\Omega_{f,N_2}} = \lambda(1)^2 \tag{112}$$

up to $\wp$-adic units.

Proof. This is well known, but details can be found in [12, Theorems 1.2, 1.8]. $\square$

8.3.6. For any $\ell||N$, let $c_f(\ell)$ be the maximal exponent e such that T_f/π^e is unramified as a representation of $G_{\mathbb{Q}_\ell}$.

Theorem 8.3.7. *If $N = N_1N_2$ where N_2 is the squarefree product of an odd number of primes not dividing N_1, and if $\ell_0||N$ is any prime, then*

$$\sum_{\ell|N_2} c_f(\ell) - (\nu(N_2+2)c_f(\ell_0)) \le \mathrm{ord}_\wp \frac{\Omega_{f,N_2}}{\Omega_f^{can}} \le \sum_{\ell|N_2} c_f(\ell) + 2c_f(\ell_0).$$

Proof. By definition, we have

$$\frac{\Omega_{f,N_2}}{\Omega_f^{can}} = \frac{\eta_f(N,1)_0}{\langle\varphi_{f,N_2},\varphi_{f,N_2}\rangle},$$

so the theorem follows from Theorem A.3.6 of the appendix. $\square$

8.4. **Ordinary cyclotomic Iwasawa theory.**

8.4.1. In this subsection, we assume $\wp$ is a prime of good *ordinary* reduction for f. Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, and let $\Gamma_{\mathbb{Q}_\infty} = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Let $\mathrm{Fil}_p^+ T_f \subset T_f$ be the unique $G_{\mathbb{Q}_p}$-stable line on which I_p acts by the cyclotomic character, and let $\mathrm{gr}_p T_f = T_f/\mathrm{Fil}_p^+ T_f$. For the ring of integers A of a finite extension of $\mathbb{Q}_p$ containing $\mathcal{O}$ and a finite set Σ of finite places of $\mathbb{Q}$, we consider the cyclotomic Selmer group, analogous to (80):

$$\begin{aligned}\mathrm{Sel}^\Sigma_{\mathbb{Q}_\infty,A}(f) := \ker\Bigg(H^1(\mathbb{Q}, T_f\otimes_{\mathcal{O}} A[\![\Gamma_{\mathbb{Q}_\infty}]\!]^\vee) \to \prod_{\ell\notin\Sigma\cup\{p\}} H^1(I_\ell, T_f\otimes_{\mathcal{O}} A[\![\Gamma_{\mathbb{Q}_\infty}]\!]^\vee)\\ \times H^1(I_p, \mathrm{gr}_p T_f\otimes_{\mathcal{O}} A[\![\Gamma_{\mathbb{Q}_\infty}]\!]^\vee)\Bigg).\end{aligned} \tag{113}$$

We shall abbreviate $\Lambda_{\mathbb{Q}_\infty} := \mathcal{O}[\![\Gamma_{\mathbb{Q}_\infty}]\!]$. Let $L_p^\Sigma(\mathbb{Q}_\infty, f) \in \Lambda_{\mathbb{Q}_\infty}$ be the Σ-imprimitive cyclotomic p-adic L-function denoted $\mathcal{L}^\Sigma_{f,\psi}$ in the notation of [71, §3.4.4], with $\psi = 1$.

Kato has proven one direction of the main conjecture in this setting [40, Theorem 17.4]:

Theorem 8.4.2 (Kato). *Let f be a modular form of weight two, level N, and trivial character, and $\wp \subset \mathcal{O}_f$ a prime of good ordinary reduction with odd residue characteristic. Then $\mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)$ is $\Lambda_{\mathbb{Q}_\infty}$-cotorsion and*

$$L_p(\mathbb{Q}_\infty, f) \subset \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee$$

in $\Lambda_{\mathbb{Q}_\infty} \otimes \mathbb{Q}_p$.

Under stronger conditions, Kato additionally proved that the inclusion holds in $\Lambda_{\mathbb{Q}_\infty}$. For the opposite direction of the main conjecture, we deduce the following result from the work of Skinner-Urban [71]. In the proof, we shall freely use the notations from the proof of Theorem 7.2.1. In particular, for any A, let $\mathbb{1}_{\mathrm{cyc}} : A[\![\Gamma_{\mathbb{Q}_\infty}]\!] \to A$ be the specialization at the trivial character, which is consistent with our earlier notation since $\Gamma_{\mathbb{Q}_\infty}$ is identified with Γ_K^{cyc} for any quadratic imaginary field K.

Theorem 8.4.3. *Let K be an imaginary quadratic field of discriminant prime to Np in which p splits. Assume that $\wp$ is ordinary for f, that $p \nmid 2N$, and that:*

- *the mod $\wp$ representation $\overline{T}_f$ is absolutely irreducible as an $\mathcal{O}[G_{\mathbb{Q}}]$-module;*
- *$N = N_1 N_2$, where every factor of N_1 is split in K and N_2 is the squarefree product of an odd number of primes inert in K.*

Then there exists a finite extension of $\mathbb{Q}_p$ containing $\mathcal{O}$, with ring of integers A, and an element $\alpha \in A[\![\Gamma_{\mathbb{Q}_\infty}]\!]$ such that

$$\mathrm{ord}_\pi \mathbb{1}_{\mathrm{cyc}}(\alpha) = \mathrm{ord}_\pi \frac{\Omega_{f,N_2}}{\Omega_f^{can}} + \sum_{\ell | N_2} \mathrm{ord}_\pi(1-\ell^2) + \sum_{\ell | \operatorname{disc}(K)} \mathrm{ord}_\pi(1+\ell-a_\ell)$$

and

$$(\alpha) \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f\otimes\chi_K)^\vee \subset (L_p(\mathbb{Q}_\infty, f))(L_p(\mathbb{Q}_\infty, f\otimes\chi_K))$$

in $A[\![\Gamma_{\mathbb{Q}_\infty}]\!]$.

Proof. Let Σ be the set of rational primes dividing $Np\operatorname{disc}(K)$, and recall the divisibility established in the course of the proof of Theorem 7.2.1 for the characteristic ideal of the 3-variable Selmer group:

$$(\alpha_i) \cdot \mathrm{char}_{\mathbb{I}[\![\Gamma_K]\!]} \mathrm{Sel}^{\Sigma}_{K_\infty,A}(\mathbf{f})^\vee \subset \left(\mathscr{L}^{\Sigma}_{\mathbf{f},K}\right) \tag{114}$$

in $\mathbb{I}[\![\Gamma_K]\!]$, where, by Remark 7.2.2(1) and Proposition 8.3.3, $\alpha_i \in \mathbb{I}[\Gamma_K^{\mathrm{cyc}}]$ may be chosen such that

$$\mathrm{ord}_\pi \mathbb{1}_{\mathrm{cyc}} \circ \phi(\alpha_i) = \mathrm{ord}_\pi\left(\Omega_{f,N_2}/\Omega_f^{can}\right) + \sum_{\ell|N_2} \mathrm{ord}_\pi(1-\ell^2) + \sum_{\ell|\operatorname{disc}(K)} \mathrm{ord}_\pi(1+\ell-a_\ell).$$

Let $\alpha := \phi(\alpha_i)$ for such a choice of α_i. Combining (114) with (85) we have a divisibility

$$(\alpha) \cdot \mathrm{char}_{A[\![\Gamma_{K_\infty}]\!]} \mathrm{Sel}^{\Sigma}_{K_\infty,A}(f)^\vee \subset \phi\left(\mathscr{L}^{\Sigma}_{\mathbf{f},K}\right).$$

in $A[\![\Gamma_K]\!]$. Now, by Lemma 3.6, Proposition 3.9, and §3.4.6 of [71] – using Lemma 7.1.4 and that p splits in K to check the hypotheses of Proposition 3.9 in *loc. cit.* – the preceding divisibility yields

$$(\alpha) \cdot \mathrm{char}_{A[\![\Gamma_{\mathbb{Q}_\infty}]\!]} \mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,A}(f)^\vee \cdot \mathrm{char}_{A[\![\Gamma_{\mathbb{Q}_\infty}]\!]} \mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,A}(f\otimes\chi_K)^\vee \subset (L_p^{\Sigma}(\mathbb{Q}_\infty, f))(L_p^{\Sigma}(\mathbb{Q}_\infty, f\otimes\chi_K)). \tag{115}$$

Then by [71, Lemma 3.13(ii), Proposition 3.14], we obtain the imprimitive divisibility

$$(\alpha) \cdot \mathrm{char}_{A[\![\Gamma_{\mathbb{Q}_\infty}]\!]} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,A}(f)^\vee \cdot \mathrm{char}_{A[\![\Gamma_{\mathbb{Q}_\infty}]\!]} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,A}(f\otimes\chi_K)^\vee \subset (L_p(\mathbb{Q}_\infty, f))(L_p(\mathbb{Q}_\infty, f\otimes\chi_K)). \tag{116}$$

By the cyclotomic analogue of (81), we can replace the characteristic ideals on the left hand side with their analogues over $\mathcal{O}$, which gives the theorem. $\square$

8.4.4. Let $V_f = T_f \otimes \mathbb{Q}_p$ and let $H^1_f(\mathbb{Q}_p, W_f) \subset H^1(\mathbb{Q}_p, W_f)$ be the image of the Bloch-Kato local condition $H^1_f(\mathbb{Q}_p, V_f)$ under the natural map $H^1(\mathbb{Q}_p, V_f) \to H^1(\mathbb{Q}_p, W_f)$. Then we consider the $\mathbb{Q}$-Selmer group

$$\mathrm{Sel}(\mathbb{Q}, W_f) := \ker\left(H^1(\mathbb{Q}, W_f) \to \prod_{\ell\neq p} H^1(\mathbb{Q}_\ell, W_f) \times \frac{H^1(\mathbb{Q}_p, W_f)}{H^1_f(\mathbb{Q}_p, W_f)}\right). \tag{117}$$

This definition also makes sense without the assumption that $\wp$ be ordinary. However, in the ordinary case, we have:

Proposition 8.4.5. *Let α_p be as in (5.2.3). Then*

$$\operatorname{ord}_\pi \mathbb{1}_{\mathrm{cyc}}\left(\operatorname{char}_{\Lambda_{\mathbb{Q}_\infty}} \operatorname{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee\right) = \lg_{\mathcal{O}} \operatorname{Sel}(\mathbb{Q}, W_f) + \sum_{\ell \mid N} \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_f) + 2\operatorname{ord}_\pi(1-\alpha_p).$$

Proof. By Theorem 8.4.2 combined with [71, Proposition 3.13(i)], $\operatorname{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(f)$ is $\Lambda_{\mathbb{Q}_\infty}$-cotorsion for all finite sets of primes Σ. Then by [71, Proposition 3.20], $\operatorname{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee$ has no pseudo-null submodules for a sufficiently large finite set of primes Σ. By the same argument as [30, Proposition 4.14], this implies that $\operatorname{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee$ contains no pseudo-null submodules as well. Hence

$$\operatorname{ord}_\pi \mathbb{1}_{\mathrm{cyc}}(\operatorname{char}_{\Lambda_{\mathbb{Q}_\infty}} \operatorname{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee) = \lg_{\mathcal{O}} \operatorname{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)[T], \tag{118}$$

where $T = \gamma - 1 \in \mathcal{O}[\![\Gamma_{\mathbb{Q}_\infty}]\!]$ for a topological generator $\gamma \in \Gamma_{\mathbb{Q}_\infty}$.

Now note that, by the same reasoning as in the proof of [40, Lemma 17.9],

$$H^1_f(\mathbb{Q}_p, W_f) = \operatorname{im}\left(H^1_f(\mathbb{Q}_p, \operatorname{Fil}^+_p T_f \otimes \mathbb{Q}_p) \to H^1(\mathbb{Q}_p, W_f)\right). \tag{119}$$

In particular, $H^1_f(\mathbb{Q}_p, W_f)$ lies in the kernel of the natural map

$$H^1(\mathbb{Q}_p, W_f) \to H^1(\mathbb{Q}_p, \operatorname{gr}_p W_f),$$

and hence also in the kernel of the map $H^1(\mathbb{Q}_p, W_f) \to H^1(I_p, \operatorname{gr}_p T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty}.)$

Then by the arguments of [30, Lemma 4.3, Lemma 4.6], we have

$$\begin{aligned} \lg_{\mathcal{O}} \operatorname{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)[T] &= \lg_{\mathcal{O}} \operatorname{Sel}(\mathbb{Q}, W_f) + \sum_{\ell \neq p} \lg_{\mathcal{O}} \ker\left(H^1(\mathbb{Q}_\ell, W_f) \to H^1(\mathbb{Q}_\ell, T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})\right) \\ &\quad + \lg_{\mathcal{O}} \ker\left(\frac{H^1(\mathbb{Q}_p, W_f)}{H^1_f(\mathbb{Q}_p, W_f)} \to H^1(I_p, \operatorname{gr}_p T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})\right). \end{aligned} \tag{120}$$

For the kernels at $\ell \neq p$, note that the map $H^1(I_\ell, W_f) \to H^1(I_\ell, T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})$ is injective. Indeed, its kernel is identified with

$$(T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})^{I_\ell} / T(T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})^{I_\ell}$$

by the long exact sequence associated to multiplication by T on $T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty}$, and $(T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})^{I_\ell}$ is T-divisible because ℓ is unramified in $\mathbb{Q}_\infty$.

On the other hand, $H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty}) = 0$ by [58, Remark 3.1]. So the terms for $\ell \neq p$ in (120) coincide with

$$\lg_{\mathcal{O}} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_f),$$

which vanishes for $\ell \nmid Np$.

For the term at p, note first that, if A_f is the associated GL_2-type abelian variety to f as in Remark 5.1.3, then $H^1_f(\mathbb{Q}_p, W_f)$ coincides with the image of the local Kummer map $A_f(\mathbb{Q}_p) \to H^1_f(\mathbb{Q}_p, W_f)$ by [4, Examples 3.10.1, 3.11]. Then the argument of [30, Lemma 3.4], which is readily adapted to the case of GL_2-type abelian varieties, shows that the the term at p is $2\lg_{\mathcal{O}} H^0(\mathbb{Q}_p, \operatorname{gr}_p W_f) = 2\operatorname{ord}_\pi(1-\alpha_p)$, which completes the proof.

□

8.4.6. Denote by $\mu(f)$ the μ-invariant

$$\mu(f) \coloneqq \operatorname{ord}_{(\wp)}\left(\operatorname{char}_{\Lambda_{\mathbb{Q}_\infty}} \operatorname{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee\right) < \infty. \tag{121}$$

From Theorem 8.4.3, we obtain the following weak form of the BSD formula for f.

Corollary 8.4.7. *Let f and K be as in Theorem 8.4.3. Then we have*

$$\begin{aligned} \operatorname{ord}_\pi \frac{L(f,1)}{\Omega^+_f} &\leq \lg_{\mathcal{O}} \operatorname{Sel}(\mathbb{Q}, W_f) + \mu(f \otimes \chi_K) + \operatorname{ord}_\pi \frac{\Omega_{f,N_2}}{\Omega^{can}_f} \\ &\quad + \sum_{\ell \mid N_2} \operatorname{ord}_\pi(1-\ell^2) + \sum_{\ell \mid \operatorname{disc}(K)} \operatorname{ord}_\pi(1+\ell-a_\ell) \\ &\quad + \sum_{\ell \mid N} \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_f). \end{aligned}$$

Proof. The divisibility in $\Lambda_{\mathbb{Q}_\infty} \otimes \mathbb{Q}_p$ from Theorem 8.4.2 applied to the quadratic twist $f \otimes \chi_K$ can trivially be upgraded to a divisibility in $\Lambda_{\mathbb{Q}_\infty}$ by inserting a μ-invariant:

$$(\wp)^{\mu(f\otimes\chi_K)} \cdot (L_p(\mathbb{Q}_\infty, f \otimes \chi_K)) \subset \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f \otimes \chi_K)^\vee. \tag{122}$$

By Theorem 8.4.3, we also have

$$(\alpha) \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f \otimes \chi_K)^\vee \subset (L_p(\mathbb{Q}_\infty, f)) \cdot (L_p(\mathbb{Q}_\infty, f \otimes \chi_K)). \tag{123}$$

Combining these two equations, we have

$$\begin{aligned}(\alpha) \cdot (\wp)^{\mu(f\otimes\chi_K)} \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f \otimes \chi_K)^\vee \\ \subset (L_p(\mathbb{Q}_\infty, f)) \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f \otimes \chi_K)^\vee.\end{aligned} \tag{124}$$

Since characteristic ideals are divisorial and $\mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f \otimes \chi_K)^\vee \neq 0$ by Theorem 8.4.2, we obtain

$$(\alpha) \cdot (\wp)^{\mu(f\otimes\chi_K)} \cdot \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee \subset (L_p(\mathbb{Q}_\infty, f)). \tag{125}$$

We now specialize both sides at the cyclotomic character and obtain

$$\mathrm{ord}_\pi\, \mathbb{1}_{\mathrm{cyc}}(\alpha) + \mu(f \otimes \chi_K) + \mathrm{ord}_\pi\, \mathbb{1}_{\mathrm{cyc}}(\mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\emptyset}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee) \geq \mathrm{ord}_\pi\, \mathbb{1}_{\mathrm{cyc}}(L_p(\mathbb{Q}_\infty, f)). \tag{126}$$

Combining Proposition 8.4.5 with the interpolation formula for $L_p(\mathbb{Q}_\infty, f)$ in [71, §3.4.4] and the formula for $\mathrm{ord}_\pi\, \mathbb{1}_{\mathrm{cyc}}(\alpha)$ in Theorem 8.4.3, (126) gives the corollary.

□

The following lemma will be needed to control μ-invariants in our application of Corollary 8.4.7.

Lemma 8.4.8. *Continue to fix f as above, and let g be another Hecke eigenform of weight two and trivial character, new of level M, with $\mathcal{O}_g$ the ring of integers of the number field generated by the Hecke eigenvalues of g. Suppose given a good ordinary prime $\wp_g \subset \mathcal{O}_g$ such that $\mathcal{O}_{g,\wp_g}$ is isomorphic to $\mathcal{O}$, and let T_g be the corresponding $\mathcal{O}[G_\mathbb{Q}]$-module. If $T_g/\pi^j \cong T_f/\pi^j$ for some integer $j > \mu(f)$, then $\mu(f) = \mu(g)$.*

Proof. A direct calculation shows that the μ-invariant of $H^1(\mathbb{Q}_\ell, T_f \otimes \Lambda^\vee_{\mathbb{Q}_\infty})$ vanishes for all ℓ. Hence $\mu(f)$ is also the μ-invariant of $\mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(f)^\vee$ for any finite set of primes Σ, and likewise for g.

If Σ contains all primes dividing NM, then it is not difficult to check using Lemma 2.4.12 that

$$\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(f)[\pi^j] \simeq \mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(g)[\pi^j] \tag{127}$$

as $\Lambda_{\mathbb{Q}_\infty}$-modules. Let M_f and M_g be the Pontryagin duals of $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(f)$ and $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,\mathcal{O}}(g)$, respectively, and let $\mathfrak{P} = (\wp) \subset \Lambda_{\mathbb{Q}_\infty}$. Then we have an isomorphism of $\Lambda_{\mathbb{Q}_\infty}$-modules

$$M_f/\mathfrak{P}^j \simeq M_g/\mathfrak{P}^j.$$

Since $\mu(f) = \lg M_{f,(\mathfrak{P})} < j$, where $(\mathfrak{P})$ denotes the localization,

$$M_{f,(\mathfrak{P})}/\mathfrak{P}^j = M_{f,(\mathfrak{P})}/\mathfrak{P}^{j-1} \tag{128}$$

as $\Lambda_{\mathbb{Q}_\infty,(\mathfrak{P})}$-modules, which implies the same for g. Therefore $M_{g,(\mathfrak{P})}/\mathfrak{P}^j = M_{g,(\mathfrak{P})}$, so

$$\lg M_{g,(\mathfrak{P})} = \lg M_{g,(\mathfrak{P})}/\mathfrak{P}^j = \lg M_{f,(\mathfrak{P})}/\mathfrak{P}^j = \lg M_{f,(\mathfrak{P})},$$

as desired. □

8.5. **Non-ordinary Iwasawa theory.** In this section, we continue the notation of (8.3.1), and no longer assume that $\wp$ is ordinary for f. However, still let $\mathbb{Q}_\infty$ and $\Lambda_{\mathbb{Q}_\infty}$ be as in (8.4.1).

8.5.1. For an integer $i \geq 0$ and $M = W_f$ or T_f, write

$$H^i(\mathbb{Z}[1/p], M) := \ker\left(H^i(\mathbb{Q}, M) \to \prod_{\ell \neq p} H^i(I_\ell, M)\right) \tag{129}$$

and

$$H^1_f(\mathbb{Z}[1/p], M) := \ker\left(H^1(\mathbb{Z}[1/p], M) \to \frac{H^1(\mathbb{Q}_p, M)}{H^1_f(\mathbb{Q}_p, M)}\right), \tag{130}$$

where $H^1_f(\mathbb{Q}_p, W_f)$ is as in (8.4.4), and $H^1_f(\mathbb{Q}_p, T_f)$ is the kernel of the map $H^1(\mathbb{Q}_p, T_f) \to H^1(\mathbb{Q}_p, V_f)/H^1_f(\mathbb{Q}_p, V_f)$. The next proposition is a corollary to the motivic form of the cyclotomic main conjecture proved in [29].

Proposition 8.5.2. *Suppose that $\wp$ is* not *an ordinary prime for f, and that there exists $\ell || N$ such that $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$ is ramified and $\overline{T}_f^{G_{\mathbb{Q}_\ell}} = 0$. Then*

$$\mathrm{ord}_\pi \frac{L(f,1)}{\Omega_f^+} = \lg_{\mathcal{O}} H^1_f(\mathbb{Z}[1/p], W_f).$$

Proof. Let $\mathbf{z}(f) \subset H^1(\mathbb{Z}[1/p], T_f \otimes \Lambda_{\mathbb{Q}_\infty}) \otimes \mathbb{Q}_p$ be the subspace defined by Kato in [40, Theorem 12.5]. In fact, $\mathbf{z}(f)$ lies in $H^1(\mathbb{Z}[1/p], T_f \otimes \Lambda_{\mathbb{Q}_\infty})$. In [40, Theorem 12.5(4)] this is asserted under a stronger condition, but the proof only uses that any two $G_{\mathbb{Q}}$-stable $\mathcal{O}$-submodules of $T_f \otimes \mathbb{Q}_p$ differ by a scalar; this holds under the assumption that $\overline{T}_f$ is absolutely irreducible, cf. [69, Lemma 2.1.1].

Let $z(f) \subset H^1(\mathbb{Z}[1/p], T_f)$ be the image of $\mathbf{z}(f)$ under the specialization map

$$H^1(\mathbb{Z}[1/p], T_f \otimes \Lambda_{\mathbb{Q}_\infty}) \to H^1(\mathbb{Z}[1/p], T_f)$$

induced by $\mathbb{1}_{\mathrm{cyc}}$. By [29, Theorem 1.7, Proposition 3.20],[6] we have

$$\lg_{\mathcal{O}} \frac{H^1(\mathbb{Z}[1/p], T_f)}{z(f)} = \lg_{\mathcal{O}} H^2(\mathbb{Z}[1/p], T_f). \tag{131}$$

Note that it is to apply [29, Theorem 1.7] that we have assumed the existence of a prime ℓ as in the proposition.

Now by [40, Proposition 14.21(2)], (131) is equivalent to the desired formula, which completes the proof. □

Corollary 8.5.3. *Under the assumptions of Proposition 8.5.2, we have*

$$\mathrm{ord}_\pi \frac{L(f,1)}{\Omega_f^+} \leq \lg_{\mathcal{O}} \mathrm{Sel}(\mathbb{Q}, W_f) + \sum_{\ell | N} \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_f).$$

Proof. By definition, we have an exact sequence

$$0 \to \mathrm{Sel}(\mathbb{Q}, W_f) \to H^1_f(\mathbb{Z}[1/p], W_f) \to \prod_{\ell | N} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_f), \tag{132}$$

so the inequality is immediate from Proposition 8.5.2. □

Remark 8.5.4. In fact, the inequality in Corollary 8.5.3 is sharp: without loss of generality, we may assume $\mathrm{Sel}(\mathbb{Q}, W_f)$ is finite, and then [30, Proposition 4.13] implies that (132) is exact on the right as well.

[6] When consulting the preprint [29], the reader may find it helpful to note that the divisibility

$$\mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} \frac{H^1(\mathbb{Z}[1/p], T_f \otimes \Lambda_{\mathbb{Q}_\infty})}{\mathbf{z}(f)} \subset \mathrm{char}_{\Lambda_{\mathbb{Q}_\infty}} H^2(\mathbb{Z}[1/p], T_f \otimes \Lambda_{\mathbb{Q}_\infty})$$

of [40, Theorem 12.5(4)] holds – even without inverting p – under the assumptions of Proposition 8.5.2. The reasons for this are explained in detail in [69, p. 188]. Although the discussion in [69] is in the ordinary context, the same remarks apply here.

8.6. **Proof of Theorem 8.1.1.** By Corollary 7.3.4, we may assume without loss of generality that we are not in the split ordinary case; so by the assumptions in Theorem 8.1.1 we can fix once and for all a prime $\ell_0 || N$. In the ordinary case, fix as well an auxiliary quadratic imaginary field F, not contained in $K(T_f)$, such that ℓ_0 is inert in F and every other factor of $Np\operatorname{disc}(K)$ is split in F. Let $(\mathcal{F}, \mathsf{S})$ be the Selmer structure of (72) with $\mathsf{m} = 1$.

As in the proof of Theorem 7.3.1, apply Proposition 6.3.6 and Theorem 6.2.4 to obtain some $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\} \in \mathsf{N}$, represented by a sequence of weakly admissible pairs $\{Q_n, \epsilon_{Q_n}\}$ as in Remark 4.6.8, and a resulting sequence of newforms g_n of NQ_n; in the ordinary case, we make sure to choose each $\mathsf{q} \in \mathsf{Q}$ such that $\mathrm{Frob}_{\mathsf{q}}$ has trivial image in $\mathrm{Gal}(F/\mathbb{Q})$, which is clearly possible. The choice of $\{\mathsf{Q}, \epsilon_{\mathsf{Q}}\}$ from Proposition 6.3.6 includes the condition that $\mathrm{Sel}_{\mathcal{F}(\mathsf{Q})}(T_f) = 0$. We claim that $\lambda(\mathsf{Q}) \neq 0$, for which – by the same reasoning as Step 2 of Theorem 7.3.1 – it suffices to show $\mathrm{ord}_\pi \lambda_{g_n}(1)$ is uniformly bounded for $\mathfrak{F}$-many n. Combined with Proposition 8.3.5, we wish to show

$$\mathrm{ord}_\pi \frac{L(g_n/K, 1)}{\Omega_{g_n, N^- Q_n}} \leq C_1 \tag{133}$$

for some constant C_1 depending only on f, $\wp$, ℓ_0, F, and K, and for $\mathfrak{F}$-many n.

We first claim that, for a constant C_2 depending only on f, $\wp$, ℓ_0, and F, we have the inequality

$$\mathrm{ord}_\pi \frac{L(g_n, 1)}{\Omega^+_{g_n}} \leq \lg_{\mathcal{O}} \mathrm{Sel}(\mathbb{Q}, W_{g_n}) + \sum_{\ell | NQ_n} \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_{g_n}) + C_2. \tag{134}$$

In the non-ordinary case, this is immediate from Corollary 8.5.3. In the ordinary case, we have to bound the extra terms appearing in Corollary 8.4.7 for g_n and F, i.e.

$$\mu(g_n \otimes \chi_F) + \mathrm{ord}_\pi \frac{\Omega_{g_n, \ell_0}}{\Omega^{can}_{g_n}} + \mathrm{ord}_\pi(1 - \ell_0^2) + \sum_{\ell | \operatorname{disc}(F)} \mathrm{ord}_\pi(1 + \ell - a_\ell(g_n)). \tag{135}$$

The μ-invariant term is equal to $\mu(f \otimes \chi_F)$ for $\mathfrak{F}$-many n by Lemma 8.4.8; the term $\mathrm{ord}_\pi \frac{\Omega_{g_n, \ell_0}}{\Omega^{can}_{g_n}}$ is bounded by $3c_{g_n}(\ell_0)$ by Theorem 8.3.7, which coincides with $3c_f(\ell_0)$ for $\mathfrak{F}$-many n. Finally, for any $\ell | \operatorname{disc}(F)$, $1 + \ell - a_\ell(f) \neq 0$ by the Weil bound, so for $\mathfrak{F}$-many n $\mathrm{ord}_\pi(1 + \ell - a_\ell(f)) = \mathrm{ord}_\pi(1 + \ell - a_\ell(g_n))$. This proves (134). The exact same reasoning applied to $f \otimes \chi_K$ and $g_n \otimes \chi_K$ shows there exists a constant C_3, now depending on f, $\wp$, ℓ_0, F, and K, such that

$$\mathrm{ord}_\pi \frac{L(g_n \otimes \chi_K, 1)}{\Omega^+_{g_n \otimes \chi_K}} \leq \lg_{\mathcal{O}} \mathrm{Sel}(\mathbb{Q}, W_{g_n} \otimes \chi_K) + \sum_{\ell | NQ_n} \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, W_{g_n} \otimes \chi_K) + C_3 \tag{136}$$

holds for $\mathfrak{F}$-many n. By [72, Lemma 9.6] (whose proof does not require the assumption in *op. cit.* that $p||N$), $\Omega^+_{g_n \otimes \chi_K}$ coincides with $\Omega^-_{g_n}$ up to a $\wp$-adic unit; in combination with Proposition 8.3.3, we have

$$\Omega^+_{g_n \otimes \chi_K} \Omega^+_{g_n} = \Omega^{can}_{g_n}. \tag{137}$$

Let $\mathcal{F}_{g_n}$ be the Selmer structure for the G_K-module W_{g_n} which is dual to the analogue for g_n of (72) with $\mathsf{m} = 1$. By Shapiro's lemma (and comparing the definitions (72) and (117)), we also have

$$\mathrm{Sel}_{\mathcal{F}_{g_n}}(W_{g_n}) \cong \mathrm{Sel}(\mathbb{Q}, W_{g_n}) \oplus \mathrm{Sel}(\mathbb{Q}, W_{g_n} \otimes \chi_K). \tag{138}$$

Then combining (134), (136), (137), and (138), we have

$$\mathrm{ord}_\pi \frac{L(g_n/K, 1)}{\Omega^{can}_{g_n}} \leq \lg_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}}(W_{g_n}) + \sum_{v | NQ_n} \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(K_v, W_{g_n}) + C_2 + C_3 \tag{139}$$

for $\mathfrak{F}$-many n. Now note that, for all $v | N$, $\lg_{\mathcal{O}} H^1_{\mathrm{unr}}(K_v, W_{g_n}) = \lg_{\mathcal{O}} H^1_{\mathrm{unr}}(K_v, W_f) < \infty$ for $\mathfrak{F}$-many n; on the other hand, using the explicit form of $T_{g_n}|_{G_{\mathbb{Q}_{q_n}}}$ from the end of Step 3 of Theorem 7.3.1, we have $\lg_{\mathcal{O}} H^1_{\mathrm{unr}}(K_{q_n}, W_{g_n}) = c_{g_n}(q_n)$ for all $q_n | Q_n$. So (139) becomes

$$\mathrm{ord}_\pi \frac{L(g_n/K, 1)}{\Omega^{can}_{g_n}} \leq \lg_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}_{g_n}}(W_{g_n}) + \sum_{q_n | Q_n} c_{q_n}(g_n) + C_4 \tag{140}$$

for $\mathfrak{F}$-many n and for a constant C_4 depending on f, $\wp$, ℓ_0, F, and K.

Now by Theorem 8.3.7, (140) becomes

$$\operatorname{ord}_\pi \frac{L(g_n/K,1)}{\Omega_{g_n,N^-}} \leq \lg_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}_{g_n}}(W_{g_n}) - \sum_{\ell|N^-} c_{g_n}(\ell) + (\nu(N^-Q_n)+2)c_{g_n}(\ell_0) + C_4. \tag{141}$$

Again using that $c_{g_n}(\ell) = c_f(\ell)$ for $\mathfrak{F}$-many n and for all $\ell||N$, (141) becomes

$$\operatorname{ord}_\pi \frac{L(g_n/K,1)}{\Omega_{g_n,N^-}} \leq \lg_{\mathcal{O}} \operatorname{Sel}_{\mathcal{F}_{g_n}}(W_{g_n}) + C_5 \tag{142}$$

for $\mathfrak{F}$-many n and for a constant C_5 depending on f, $\wp$, ℓ_0, F, and K.

Now arguing as in in Steps 3 and 4 of the proof of Theorem 7.3.1, but replacing the local arguments at $v|p$ by [32, Lemma 7] combined with [60, Théorème 3.3.3], we find

$$\# \operatorname{Sel}_{\mathcal{F}_{g_n}}(W_{g_n})[\pi^j] = \# \operatorname{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f)[\pi^j] \tag{143}$$

for any j and for $\mathfrak{F}$-many n. Since $\operatorname{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f)$ is finite by Proposition 3.3.6(3) and the choice of Q, it follows from (143) that $\# \operatorname{Sel}_{\mathcal{F}_{g_n}}(W_{g_n}) = \# \operatorname{Sel}_{\mathcal{F}(\mathsf{Q})}(W_f)$ for $\mathfrak{F}$-many n.

In conjunction with (142), this shows (133) and completes the proof.

Appendix A. Degrees of modular parametrizations and congruence numbers

A.1. **Overview.** Let f, $\wp$, N, $\mathcal{O}$, E, π, V_f, T_f, and W_f be as in §1.5, and keep the notation of §8.3. In this appendix, we extend the results of [41, §3.2] and [64] on the degrees of modular parametrizations associated to f. There are two directions in which we must generalize their work: first to allow general coefficient rings $\mathcal{O}$, and second to not require that N be squarefree, at the cost of an error term in the final result.

In [64], it is assumed that $\mathcal{O}_f = \mathbb{Z}$; in [41], this hypothesis is relaxed to the assumption that $\mathcal{O}$ is absolutely unramified and generated over $\mathbb{Z}_p$ by the Hecke eigenvalues of f (whereas in general the Hecke eigenvalues only generate a finite-index subring of $\mathcal{O}$). In [58, §6], the results are stated for general coefficients with N squarefree, but some of the proofs are incomplete as written.

The ultimate goal of this appendix is Theorem A.3.6 below, which is crucially used in §8. If N is squarefree, then there exists a prime $\ell||N$ such that $\overline{T}_f|_{G_{\mathbb{Q}_\ell}}$ is ramified, in which case the error terms in Theorem A.3.6 can be chosen to vanish and we recover the statement of [58, Theorem 6.8].

A.2. Constructions and notations.

A.2.1. Recall the abelian variety A_f from Remark 5.1.3, and fix an isomorphism $\operatorname{End}(A_f) \cong \mathcal{O}_f$. Since the $G_{\mathbb{Q}}$-module $A_f[\wp]$ is absolutely irreducible, we can choose a polarization $\lambda : A_f \to A_f^\vee$ such that the induced map on $\wp$-adic Tate modules is an $\mathcal{O}$-linear isomorphism

$$\lambda_* : T_\wp A_f \xrightarrow{\sim} T_\wp A_f^\vee.$$

(Note that the Rosati involution associated to λ is trivial since, as f has trivial central character, $\mathcal{O}_f$ is the ring of integers of a totally real field.)

For a factorization $N = N_1 N_2$, where N_2 is the squarefree product of an even number of primes not dividing N_1, we choose a Hecke-equivariant map $\xi_{N_1,N_2} : J^{N_1,N_2} \to A_f$ such that the image of the induced map $T_p J^{N_1,N_2} \to T_p A_f \twoheadrightarrow T_\wp A_f$ is not contained in $\wp T_\wp A_f$. (Recall from (4.3.1) that J^{N_1,N_2} denotes the Jacobian of the Shimura curve X_{N_1,N_2}.)

A.2.2. For the rest of this subsection, abbreviate $J := J^{N_1,N_2}$ and $\xi := \xi_{N_1,N_2}$ for some factorization $N = N_1 N_2$ as above. If $\mathfrak{m} \subset \mathbb{T}_{N_1,N_2}$ is the maximal ideal associated to f and $\wp$, let

$$\xi_* : T_\mathfrak{m} J \otimes_{\mathbb{Z}_p} \mathcal{O} \to T_\wp A_f \tag{144}$$

be the natural map, with $T_\mathfrak{m} J$ as in (4.2.2). Let $d(\mathcal{O})$ be the different of $\mathcal{O}/\mathbb{Z}_p$, so that the modified trace pairing

$$\operatorname{tr}'(x,y) := \operatorname{tr}_{\mathcal{O}/\mathbb{Z}_p}(\pi^{-d(\mathcal{O})}xy) \tag{145}$$

defines an isomorphism

$$\mathcal{O} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}_p}(\mathcal{O}, \mathbb{Z}_p).$$

We then define the $\mathcal{O}$-linear pullback map

$$\xi^* : T_\wp A_f^\vee \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}, T_\mathfrak{m} J) \xrightarrow[\mathrm{tr}'^{-1}]{\sim} T_\mathfrak{m} J \otimes_{\mathbb{Z}_p} \mathcal{O} \tag{146}$$
$$x_A \mapsto (\alpha \mapsto \xi^*(\alpha x_A)).$$

The composite

$$T_\wp A_f \xrightarrow{\lambda_*} T_\wp A_f^\vee \xrightarrow{\xi^*} T_\mathfrak{m} J \otimes_{\mathbb{Z}_p} \mathcal{O} \xrightarrow{\xi_*} T_\wp A_f \tag{147}$$

is $\mathcal{O}$-linear and $G_\mathbb{Q}$-equivariant, hence is given by multiplication by some element of $\mathcal{O}$; let

$$\delta(N_1, N_2) \subset \mathcal{O}$$

be the ideal generated by this element. Since tr', λ, and ξ are all well-defined up to $\wp$-adic units, $\delta(N_1, N_2)$ is independent of the choices made to define it.

A.2.3. For any $\ell || N$, set $\mathcal{X}_\ell(A_f)_\mathcal{O} \coloneqq \mathcal{X}_\ell(A_f) \otimes_{\mathcal{O}_f} \mathcal{O}$, $\mathcal{X}_\ell(A_f^\vee)_\mathcal{O} \coloneqq \mathcal{X}_\ell(A_f) \otimes_{\mathcal{O}_f} \mathcal{O}$, and $\mathcal{X}_\ell(J)_\mathcal{O} \coloneqq \mathcal{X}_\ell(J)_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}$, with notation as in (4.1.1) and where $\mathfrak{m}$ denotes the $\mathfrak{m}$-adic completion. Let $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_J$ be the monodromy pairings on $\mathcal{X}_\ell(A_f^\vee) \times \mathcal{X}_\ell(A)$ and $\mathcal{X}_\ell(J) \times \mathcal{X}_\ell(J)$, respectively, using that J is canonically principally polarized.

Then we define $\langle \cdot, \cdot \rangle_{J,\mathcal{O}}$ to be the $\mathcal{O}$-valued pairing on $\mathcal{X}_\ell(J)_\mathcal{O}$ linearly extending $\langle \cdot, \cdot \rangle_J$, and define

$$\langle \cdot, \cdot \rangle_{A,\mathcal{O}} : \mathcal{X}_\ell(A^\vee)_\mathcal{O} \times \mathcal{X}_\ell(A)_\mathcal{O} \to \mathcal{O}$$

by

$$\mathrm{tr}'^{-1} \langle x_A, y_A \rangle_{A,\mathcal{O}}(\alpha) = \langle \alpha x_A, y_A \rangle_A = \langle x_A, \alpha y_A \rangle_A.$$

Repeating the construction of (A.2.2) on the level of character groups, we obtain $\mathcal{O}$-linear maps

$$\xi_* : \mathcal{X}_\ell(J)_\mathcal{O} \to \mathcal{X}_\ell(A_f^\vee)_\mathcal{O}$$

and

$$\xi^* : \mathcal{X}_\ell(A_f)_\mathcal{O} \to \mathcal{X}_\ell(J)_\mathcal{O}.$$

These maps can also be obtained functorially from (144), (146) by identifying the $\mathcal{O}$-modules $\mathcal{X}_\ell(A_f^\vee)_\mathcal{O}$, $\mathcal{X}_\ell(A_f)_\mathcal{O}$, and $\mathcal{X}_\ell(J)_\mathcal{O}$ as the maximal $\mathcal{O}$-submodules of $T_\wp A_f$, $T_\wp A_f^\vee$, and $T_\mathfrak{m} J \otimes_{\mathbb{Z}_p} \mathcal{O}$, respectively, on which the $G_{\mathbb{Q}_\ell}$-action is unramified and Frob_ℓ acts by ℓ times a root of unity, cf. the diagram on [20, p. 318]. In particular, for each ℓ the composite

$$\mathcal{X}_\ell(A_f^\vee)_\mathcal{O} \xrightarrow{\lambda^*} \mathcal{X}_\ell(A_f)_\mathcal{O} \xrightarrow{\xi^*} \mathcal{X}_\ell(J)_\mathcal{O} \xrightarrow{\xi_*} \mathcal{X}_\ell(A_f^\vee)_\mathcal{O} \tag{148}$$

is multiplication by a generator of $\delta(N_1, N_2)$.

Lemma A.2.4. *For any $\ell || N$, the map $\xi_* : \mathcal{X}_\ell(J)_\mathcal{O} \to \mathcal{X}_\ell(A_f^\vee)_\mathcal{O}$ is surjective.*

Proof. Let $B \subset J$ be the image of $A_f^\vee$ under the dual map

$$\xi^\vee : A_f^\vee \to J,$$

so that $\xi^\vee$ factors as

$$A_f^\vee \xrightarrow{\varphi} B \hookrightarrow J$$

with φ an isogeny. By [21, Theorem 8.2], the natural map $\mathcal{X}_\ell(J) \to \mathcal{X}_\ell(B)$ is surjective, so the image of ξ_* is identified with the $\wp$-adic completion of

$$X \coloneqq \sum_{\alpha \in \mathcal{O}_f} (\varphi \circ \alpha)^* \mathcal{X}_\ell(B) \subset \mathcal{X}_\ell(A_f^\vee).$$

View X, $\mathcal{X}_\ell(A_f^\vee)$, and $\mathcal{X}_\ell(B)$ as constant group schemes over $\overline{\mathbb{F}}_\ell$. By the discussion preceding [21, Theorem 8.6], there is a canonical inclusion

$$\ker \left(\mathrm{Hom}(\mathcal{X}_\ell(A_f^\vee), \mathbb{G}_m) \xrightarrow{(\varphi \circ \alpha)_*} \mathrm{Hom}(\mathcal{X}_\ell(B), \mathbb{G}_m) \right) \subset \ker \varphi \circ \alpha \tag{149}$$

for all $\alpha \neq 0$ in $\mathcal{O}_f$. On the other hand, by duality, we have

$$\mathrm{Hom}(\mathcal{X}_\ell(A_f^\vee)/X, \mathbb{G}_m) = \cap_{\alpha \in \mathcal{O}_f} \ker \left(\mathrm{Hom}(\mathcal{X}_\ell(A_f^\vee), \mathbb{G}_m) \xrightarrow{(\varphi \circ \alpha)_*} \mathrm{Hom}(\mathcal{X}_\ell(B), \mathbb{G}_m) \right), \tag{150}$$

so by (149) it suffices to show

$$\left(\cap_{\alpha\in\mathcal{O}_f}\ker\varphi\circ\alpha\right)_\wp = 0.$$

This follows from the choice of ξ, so the lemma is proved. □

A.2.5. For any $\ell||N$, let $\mathcal{X}_\ell(J)_\mathcal{O}[f]\subset\mathcal{X}_\ell(J)_\mathcal{O}$ denote the maximal f-isotypic subspace for the action of the Hecke algebra $\mathbb{T}_{N_1,N_2,\mathfrak{m}}\otimes_{\mathbb{Z}_p}\mathcal{O}$. By construction, the map $\xi^*:\mathcal{X}_\ell(A_f)_\mathcal{O}\to\mathcal{X}_\ell(J)_\mathcal{O}$ has image contained in $\mathcal{X}_\ell(J)_\mathcal{O}[f]$. Then Lemma A.2.4 has the following corollary.

Corollary A.2.6. *There is an isomorphism of $\mathcal{O}$-modules*

$$\frac{\mathcal{X}_\ell(J)_\mathcal{O}[f]}{\xi^*\mathcal{X}_\ell(A_f)_\mathcal{O}}\simeq\operatorname{coker}\left(\Phi_\ell(J)_\mathfrak{m}\otimes_{\mathbb{Z}_p}\mathcal{O}\to\Phi_\ell(A_f)\otimes_{\mathcal{O}_f}\mathcal{O}\right).$$

Proof. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}_\ell(J)_\mathcal{O} & \xrightarrow{\langle\cdot,\cdot\rangle_{J,\mathcal{O}}} & \operatorname{Hom}_\mathcal{O}(\mathcal{X}_\ell(J)_\mathcal{O},\mathcal{O}) & \longrightarrow & \Phi_\ell(J)_\mathfrak{m}\otimes_{\mathbb{Z}_p}\mathcal{O} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\xi_*} & & \downarrow{\scriptstyle\xi^*} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{X}_\ell(A_f^\vee)_\mathcal{O} & \xrightarrow{\langle\cdot,\cdot\rangle_{A,\mathcal{O}}} & \operatorname{Hom}_\mathcal{O}(\mathcal{X}_\ell(A_f)_\mathcal{O},\mathcal{O}) & \longrightarrow & \Phi_\ell(A_f)\otimes_{\mathcal{O}_f}\mathcal{O} & \longrightarrow & 0. \end{array} \tag{151}$$

Here, the exactness of the rows is immediate from [34, Théorème 11.5], and the commutativity results from the definitions of ξ_* and ξ^* along with the functoriality of the monodromy pairing. By Lemma A.2.4, the first vertical map is surjective, so the snake lemma induces an isomorphism

$$\operatorname{coker}\xi^*\simeq\operatorname{coker}\left(\Phi_\ell(J)_\mathfrak{m}\otimes_{\mathbb{Z}_p}\mathcal{O}\to\Phi_\ell(A_f)\otimes_{\mathcal{O}_f}\mathcal{O}\right).$$

On the other hand, since $\mathcal{X}_\ell(J)_\mathcal{O}/\mathcal{X}_\ell(J)_\mathcal{O}[f]$ is $\mathcal{O}$-torsion-free,

$$\operatorname{coker}\xi^*=\operatorname{coker}\left(\operatorname{Hom}(\mathcal{X}_\ell(J)_\mathcal{O}[f],\mathcal{O})\to\operatorname{Hom}_\mathcal{O}(\mathcal{X}_\ell(A_f)_\mathcal{O},\mathcal{O})\right),$$

which in turn is isomorphic to $\mathcal{X}_\ell(J)_\mathcal{O}[f]/\xi^*\mathcal{X}_\ell(A_f)_\mathcal{O}$ since both $\mathcal{X}_\ell(J)_\mathcal{O}[f]$ and $\mathcal{X}_\ell(A_f)_\mathcal{O}$ are free of rank one over $\mathcal{O}$. □

A.3. **Tamagawa factors and the method of Ribet-Takahashi.**

A.3.1. For any $\ell||N$, let $c_f(\ell)$ be as in (8.3.6). By the same argument as [41, Proposition 3(1)], $\Phi_\ell(A_f)\otimes_{\mathcal{O}_f}\mathcal{O}$ is isomorphic to $\mathcal{O}/c_f(\ell)$ as an $\mathcal{O}$-module.

Proposition A.3.2. *Suppose $N=N_1N_2$ where N_2 is the squarefree product of an even number of primes not dividing N_1. Then:*

(1) For all primes $\ell|N_1$, we have

$$\operatorname{ord}_\wp\langle\varphi_{f,N_2\ell},\varphi_{f,N_2\ell}\rangle=\operatorname{ord}_\wp\delta(N_1,N_2)-c_f(\ell).$$

(2) There is a constant $c(N_1,N_2)$ such that $0\le c(N_1,N_2)\le\min_{\ell|N_2}c_f(\ell)$ and, for all primes $\ell|N_2$,

$$\operatorname{ord}_\wp\langle\varphi_{f,N_2/\ell},\varphi_{f,N_2/\ell}\rangle=\operatorname{ord}_\wp\delta(N_1,N_2)+c_f(\ell)-2c(N_1,N_2).$$

(3) For any pair of distinct primes ℓ_1, ℓ_2 dividing N_2, we have

$$\operatorname{ord}_\wp\frac{\delta(N_1\ell_1\ell_2,N_2/\ell_1\ell_2)}{\delta(N_1,N_2)}=c_f(\ell_1)+c_f(\ell_2)-2c(N_1,N_2).$$

Proof. For any $\ell||N$, we have the identity of ideals of $\mathcal{O}$:

$$\begin{aligned}\langle\xi^*_{N_1,N_2}\mathcal{X}_\ell(A_f)_\mathcal{O},\xi^*_{N_1,N_2}\mathcal{X}_\ell(A_f)_\mathcal{O}\rangle_{J^{N_1,N_2},\mathcal{O}}&=\langle\xi_{N_1,N_2,*}\xi^*_{N_1,N_2}\mathcal{X}_\ell(A_f)_\mathcal{O},\mathcal{X}_\ell(A_f)_\mathcal{O}\rangle_{A,\mathcal{O}}\\&=\delta(N_1,N_2)\langle\mathcal{X}_\ell(A_f^\vee)_\mathcal{O},\mathcal{X}_\ell(A_f)_\mathcal{O}\rangle_{A,\mathcal{O}},\end{aligned}\tag{152}$$

where the first identity comes from the commutativity of the diagram (151). Write

$$c(N_1,N_2,\ell):=\lg_\mathcal{O}\operatorname{coker}\left(\Phi_\ell(J^{N_1,N_2})_\mathfrak{m}\otimes_{\mathbb{Z}_p}\mathcal{O}\to\Phi_\ell(A)\otimes_{\mathcal{O}_f}\mathcal{O}\right).$$

Then using Corollary A.2.6 and the exactness of the bottom row of (151), we obtain from (152)

$$2c(N_1,N_2,\ell)+\operatorname{ord}_\wp\langle\mathcal{X}_\ell(J^{N_1,N_2})_\mathcal{O}[f],\mathcal{X}_\ell(J^{N_1,N_2})_\mathcal{O}[f]\rangle_{J,\mathcal{O}}=\operatorname{ord}_\wp\delta(N_1,N_2)+c_f(\ell)\tag{153}$$

Now suppose $\ell | N_1$. By the results of [63, §3] (which are stated for modular curves but apply to Shimura curves as well with the obvious modifications), $\mathcal{X}_\ell(J^{N_1,N_2})_\mathfrak{m}$ is Hecke-equivariantly isomorphic to $\mathbb{Z}[X_{N_1/\ell, N_2\ell}]_\mathfrak{m}$, and this identification is compatible with the natural pairings. Hence

$$\langle \mathcal{X}_\ell(J^{N_1,N_2})_\mathcal{O}[f], \mathcal{X}_\ell(J^{N_1,N_2})_\mathcal{O}[f]\rangle_{J,\mathcal{O}} = \langle \varphi_{f,N_2\ell}, \varphi_{f,N_2\ell}\rangle.$$

On the other hand, since $\mathfrak{m}$ is non-Eisenstein, $\Phi_\ell(J^{N_1,N_2})_\mathfrak{m} = 0$ by [63, Theorem 3.12], so $c(N_1, N_2, \ell) = c_f(\ell)$. So in this case (153) becomes (1) of the proposition.

For $\ell | N_2$, the results of [63, §4] show that $\mathcal{X}_\ell(J^{N_1,N_2})_\mathfrak{m}$ is Hecke-equivariantly isomorphic to a submodule of $\mathbb{Z}[X_{N_1\ell, N_2/\ell}]_\mathfrak{m}$ containing $\varphi_{f,N_2/\ell}$, again compatibly with the natural pairings, so we conclude

$$\operatorname{ord}_\wp \langle \varphi_{f,N_2/\ell}, \varphi_{f,N_2/\ell}\rangle = \operatorname{ord}_\wp \delta(N_1, N_2) + c_f(\ell) - 2c(N_1, N_2, \ell). \tag{154}$$

We have $c(N_1, N_2, \ell) \leq c_f(\ell)$ by definition, so to prove (2), it suffices to show that $c(N_1, N_2, \ell)$ is independent of $\ell | N_2$; the common value will be defined to be $c(N_1, N_2)$. Write $\ell_1 = \ell$; since N_2 has an even number of prime factors, we may choose a prime $\ell_2 | N_2/\ell_1$.

Applying part (1) of the proposition to $N_1\ell_1\ell_2$, $N_2/\ell_1\ell_2$, and the prime ℓ_2, we find

$$\operatorname{ord}_\wp \langle \varphi_{f,N_2/\ell_1}, \varphi_{f,N_2/\ell_1}\rangle = \operatorname{ord}_\wp \delta(N_1\ell_1\ell_2, N_2/\ell_1\ell_2) - c_f(\ell_2). \tag{155}$$

Then comparing (154) and (155), we have

$$\operatorname{ord}_\wp \delta(N_1\ell_1\ell_2, N_2/\ell_1\ell_2) - \operatorname{ord}_\wp \delta(N_1, N_2) = c_f(\ell_1) + c_f(\ell_2) - 2c(N_1, N_2, \ell_1).$$

This expression is symmetric in ℓ_1 and ℓ_2, so $c(N_1, N_2, \ell_1) = c(N_1, N_2, \ell_2)$ for all primes $\ell_1, \ell_2 | N_2$, which completes the proof of (2) and also shows (3). ☐

Corollary A.3.3. *Suppose $N = N_1N_2$ where N_2 is the squarefree product of an even number of primes not dividing N_1. Then for any $\ell_0 | N_2$,*

$$\sum_{\ell | N_2} c_f(\ell) - \nu(N_2)c_f(\ell_0) \leq \operatorname{ord}_\wp \frac{\delta(N,1)}{\delta(N_1,N_2)} \leq \sum_{\ell|N_2} c_f(\ell).$$

Proof. This is immediate from repeatedly applying Proposition A.3.2(3), not choosing ℓ_0 as one of the primes ℓ_1, ℓ_2 until the last step. ☐

A.3.4. Proposition A.3.2 and Corollary A.3.3 are two of the three ingredients we need for the final comparison of periods. The third is below.

Lemma A.3.5. *Let $\ell_0 || N$ be a prime. Then we have*

$$\operatorname{ord}_\wp \delta(N, 1) \geq \operatorname{ord}_\wp \eta_f(N, 1) \geq \operatorname{ord}_\wp \delta(N, 1) - c_f(\ell_0).$$

Proof. The composite

$$T_\mathfrak{m} J^{N,1} \otimes_{\mathbb{Z}_p} \mathcal{O} \xrightarrow{\xi_{N,1*}} T_\wp A_f \xrightarrow{\lambda_*} T_\wp A_f^\vee \xrightarrow{\xi_{N,1}^*} T_\mathfrak{m} J^{N,1} \otimes_{\mathbb{Z}_p} \mathcal{O} \xrightarrow{w_N^*} T_\mathfrak{m} J^{N,1} \otimes_{\mathbb{Z}_p} \mathcal{O} \tag{156}$$

is $\mathcal{O}$-linear and equivariant for the full Hecke algebra (because the Rosati involution on $\mathbb{T}_{N,1} \subset \operatorname{End}(J^{N,1})$ is conjugation by the Atkin-Lehner involution w_N, cf. [35, Lemma 5.5]). Since $T_\mathfrak{m} J^{N,1}$ is free of rank two over $\mathbb{T}_{N,1,\mathfrak{m}}$ by Proposition 4.2.3, and since the residual representation associated to $\mathfrak{m}$ is absolutely irreducible, we conclude $w_N^* \xi_{N,1}^* \lambda_* \xi_{N,1,*} = y$ for some $y \in \mathbb{T}_{N,1,\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathcal{O}$. Then

$$\xi_{N,1,*} w_N^* \xi_{N,1}^* \lambda_* \xi_{N,1,*} = \xi_{N,1*} y = \pm \xi_{N,1,*} \xi_{N,1}^* \lambda_* \xi_{N,1*},$$

so $\pi_f(y)$ generates $\delta(N, 1)$, cf. (A.2.2). On the other hand, y lies in the annihilator of $\ker \pi_f$ since $yz = \pi_f(z)y$ for all $z \in \mathbb{T}_{N,1,\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathcal{O}$. This shows $\delta(N, 1) \subset \eta_f(N, 1)$, so $\operatorname{ord}_\wp \delta(N, 1) \geq \operatorname{ord}_\wp \eta_f(N, 1)$.

For the other inequality, note $\operatorname{ord}_\wp \eta_f(N, 1) \geq \operatorname{ord}_\wp \eta_f(N/\ell_0, \ell_0)$ since $\mathbb{T}_{N/\ell_0,\ell_0,\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathcal{O}$ is a quotient of $\mathbb{T}_{N,1,\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathcal{O}$. By [24, Lemma 4.17], we also have

$$\operatorname{ord}_\wp \eta_f(N/\ell_0, \ell_0) \geq \operatorname{ord}_\wp \langle \varphi_{f,\ell_0}, \varphi_{f,\ell_0}\rangle,$$

which is equal to $\operatorname{ord}_\wp \delta(N, 1) - c_f(\ell_0)$ by Proposition A.3.2(1). So indeed $\operatorname{ord}_\wp \eta_f(N, 1) \geq \operatorname{ord}_\wp \delta(N, 1) - c_f(\ell_0)$. ☐

Theorem A.3.6. *For any factorization $N = N_1N_2$ where N_2 is the squarefree product of an odd number of primes not dividing N_1, and for any prime $\ell_0||N$, we have*

$$\sum_{\ell|N_2} c_f(\ell) - (\nu(N_2)+2)c_f(\ell_0) \leq \mathrm{ord}_\wp \frac{\eta_f(N,1)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle} \leq \sum_{\ell|N_2} c_f(\ell) + 2c_f(\ell_0).$$

Proof. By Lemma A.3.5, it suffices to show

$$\sum_{\ell|N_2} c_f(\ell) - (\nu(N_2)+1)c_f(\ell_0) \leq \mathrm{ord}_\wp \frac{\delta(N,1)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle} \leq \sum_{\ell|N_2} c_f(\ell) + 2c_f(\ell_0). \tag{157}$$

Suppose first that $\ell_0|N_2$. If $\ell_0 = N_2$, then Proposition A.3.2(1) proves (157). So assume without loss of generality that there exists a prime $q|N_2$ with $q \neq \ell_0$. Then by Proposition A.3.2(1), we have

$$\mathrm{ord}_\wp \frac{\delta(N_1q, N_2/q)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle} = c_f(q), \tag{158}$$

and since $\ell_0|N_2/q$ Corollary A.3.3 shows

$$\sum_{\ell|N_2/q} c_f(\ell) - \nu(N_2/q)c_f(\ell_0) \leq \mathrm{ord}_\wp \frac{\delta(N,1)}{\delta(N_1q, N_2/q)} \leq \sum_{\ell|N_2/q} c_f(\ell). \tag{159}$$

Combining (158) and (159) yields (157), in fact with stricter bounds.

Now suppose $\ell_0|N_1$. Then by Proposition A.3.2(2), we have

$$-c_f(\ell_0) \leq \mathrm{ord}_\wp \frac{\delta(N_1/\ell_0, N_2\ell_0)}{\langle \varphi_{f,N_2}, \varphi_{f,N_2}\rangle} \leq c_f(\ell_0). \tag{160}$$

Moreover, Corollary A.3.3 shows

$$\sum_{\ell|N_2\ell_0} c_f(\ell) - \nu(N_2\ell_0)c_f(\ell_0) \leq \mathrm{ord}_\wp \frac{\delta(N,1)}{\delta(N_1/\ell_0, N_2\ell_0)} \leq \sum_{\ell|N_2\ell_0} c_f(\ell). \tag{161}$$

Combining (160) and (161) completes the proof of (157). □

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Email address: naomiss@math.princeton.edu

Department of Mathematics, Princeton University