DECOMPOSING FLAG VARIETIES

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Introduction

Let G be a split reductive algebraic group over a field k of characteristic zero. If $P \subset G$ is a parabolic subgroup, then the quotient G/P has the structure of a projective variety called the (generalized) **flag variety**. When $B = P$ is a Borel subgroup, G/B is called a complete flag variety; the terminology comes from the most classical case $G = GL_n/\mathbb{C}$, where G/B parameterizes complete flags in $\mathbb{C}^n$ and G/P parameterizes partial flags. Flag varieties were first studied in the 19th century [16] for their rich applications to combinatorial algebraic geometry, and are now viewed as fundamental objects in their own right.

This expository paper will primarily focus on the decomposition of the flag variety into algebraic components, indexed by combinatorial parameters: first the Schubert cells, and then a refinement known as the Deodhar components. The Schubert cells, which are highly classical, form a so-called affine paving of G/P indexed by cosets in the Weyl group of G (or just elements if $P = B$ is a Borel subgroup). Their closures, the Schubert varieties, form a basis of the Chow ring, where the intersection theory is then governed by the combinatorics of the Weyl group. We will also highlight the connection between Schubert varieties on G/P and the representation theory of G. Deodhar components are a more recent innovation, dating to Deodhar's 1985 paper [9] and involving finer combinatorial data than the Schubert cells. In the last section, we will discuss an influential result [14] of Marsh and Rietsch, from 2004, where an explicit section from each Deodhar component to G is constructed. This construction is then used to characterize the so-called totally positive part of G/B over $\mathbb{R}$. In particular, Marsh and Rietsch elegantly resolve a conjecture by Lusztig, that the totally positive parts of certain components of G/B are isomorphic to affine half-space $\mathbb{R}^n_{>0}$.

Sources and references. Flag varieties have an extensive literature, not all of which could be included here. The exposition is most directly inspired by the surveys [18, 4]. We have departed from these in two main respects; first, we have included a review of the features of the Weyl group that will be reflected in the geometry of G/P, in an effort to make the exposition as self-contained and accessible as possible. Second, we have endeavored to include the case $P \neq B$ in all statements (until the final section on Deodhar components); this allows us to explicitly translate between the general theory and the concrete example $\mathbb{G}_{m,n}$.

In addition to the two survey articles, we also have relied on Manivel's book [13] for the section on the Grassmanian, and on papers of Chevalley [7, 6] and Serre [17] for some points of the classical theory that we were unable to find elsewhere. For general reference, we have used [3, 10, 11]. (These are also excellent resources for the basic facts about reductive groups and their Weyl groups which are used freely in this paper.) In the final section, we of course rely heavily on [14].

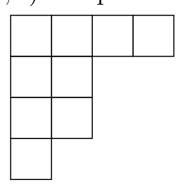
Convention. To avoid technicalities, we assume throughout that G is connected, semisimple, and simply connected. This excludes GL_n, but the flag varieties GL_n/P are isomorphic to flag varieties for SL_n.

1. The complex Grassmanian

The exposition in this section generally follows [13].

1.1. **Definitions.** Let $G = SL_{n+m}/\mathbb{C}$, and let $\{e_1, \ldots, e_{n+m}\}$ be the standard basis of $\mathbb{C}^{n+m}$. If P is the parabolic subgroup of G stabilizing the subspace generated by $\{e_1, \ldots, e_m\}$, then the homogeneous space G/P is the usual Grassmanian $\mathbb{G}_{m,n}$, a complex projective variety parameterizing the m-dimensional subspaces of $\mathbb{C}^{n+m}$.

1.2. **Partitions.** In combinatorics, a partition λ of length m is a decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. A partition is often represented graphically by its Ferrers diagram, in which the ith row has λ_i boxes. For example, the partition $(4, 2, 2, 1)$ is represented by the diagram:



Note that additional zeros do not change the diagram, but if we work with partitions of a fixed length (or simply identify partitions that differ only by the number of trailing zeros) there is no ambiguity. We say a partition μ is contained in λ, written $\mu \subset \lambda$, if the Ferrers diagram of μ is a subset of the Ferrers diagram of λ. This is of course equivalent to the condition that $\mu_j \leq \lambda_j$ for all j.

The partition of length m whose Ferrers diagram is a rectangle of width n is written $m \times n$. If λ is a partition of length m and $\lambda_1 \leq n$, then $\lambda \subset m \times n$. The complement of the Ferrers diagram of λ in the Ferrers diagram of $m \times n$ is (after rotating) the Ferrers diagram of a partition, which we call $m \times n - \lambda$. The corresponding sequence is given by $(m \times n - \lambda)_j = n - \lambda_{m-j+1}$.

1.3. **Schubert varieties.** The standard basis defines a complete flag

$$0 = V_0 \subset \cdots \subset V_{n+m} = \mathbb{C}^{n+m}, \;\; V_i = \langle e_1, \ldots, e_i \rangle,$$

which we will use to stratify the Grassmanian. In particular, if $W \subset \mathbb{C}^{n+m}$ is of dimension m, then we consider the sequence of subspaces

$$0 = W \cap V_0 \subset \cdots W \cap V_{n+m} = W.$$

These satisfy:

$$0 \leq \dim(W \cap V_i) - \dim(W \cap V_{i-1}) \leq 1, \tag{1}$$

and the indices i such that $W \cap V_i \neq W \cap V_{i-1}$ constitute a strictly increasing sequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n + m$. The sequence $\{i_j\}$ is often called the jump sequence of W. If we set $\lambda_j = n + j - i_j$, then

$$n \geq \lambda_1 \geq \cdots \geq \lambda_m.$$

We call this sequence λ the partition associated to W, and write $\lambda = \lambda(W)$.

Definition 1.3.1. Let λ be a decreasing sequence of nonnegative integers of length m. Then the Schubert cell $\Omega_\lambda \subset \mathbb{G}_{m,n}$ (with respect to the standard flag $V_0 \subset \cdots \subset V_{n+m}$) is the set of m-dimensional subspaces $W \subset \mathbb{C}^{n+m}$ such that $\lambda(W) = \lambda$. The Zariski closures $X_\lambda = \overline{\Omega}_\lambda$ are called **Schubert varieties**; they are normal and irreducible, but singular in general.

Proposition 1.3.2. *The Schubert cells define a decomposition*

$$\mathbb{G}_{m,n} = \bigsqcup_{\lambda \subset m \times n} \Omega_\lambda.$$

For each λ, there is a decomposition

$$X_\lambda = \bigsqcup_{\mu \supset \lambda} \Omega_\mu.$$

Proof. The first claim is immediate from the definition, since each W is associated to a unique $\lambda(W)$. For the second, we note that Ω_λ is defined by the incidence conditions

$$\dim(W \cap V_j) = d_\lambda(j),$$

where $d_\lambda(j)$ is the index such that j lies in the interval

$$[n + d_\lambda(j) - \lambda_{d_\lambda(j)}, n + d(j) + 1 - \lambda_{d_\lambda(j)+1}).$$

Equivalently,

$$\dim(W + V_j) = m + j - d_\lambda(j).$$

In local coordinates, this condition may be expressed by the vanishing (resp. non-vanishing) of the minors of order $m + j - d_\lambda(j) + 1$ (resp. $m + j - d_\lambda(j)$) of a $(m + j) \times (m + n)$ matrix; so the closure of Ω_λ is defined by the conditions

$$\dim(W + V_j) \leq m + j - d_\lambda(j),$$

or equivalently

$$\dim(W \cap V_j) \geq d_\lambda(j).$$

Hence

$$X_\lambda = \bigsqcup_{\mu} \Omega_\mu,$$

where μ ranges over partitions such that $d_\mu(j) \geq d_\lambda(j)$ for all $1 \leq j \leq m$. One may check directly that this condition is equivalent to $\mu_j \geq \lambda_j$ for all j. □

Remark 1.3.3. We highlight two special cases: if λ is the zero partition, the corresponding Schubert cell Ω_λ is dense in the Grassmanian. Indeed, a generic subspace W meets V_j in a subspace of the smallest possible dimension, leading to the jump sequence $i_j = n + j$. On the other hand, if $\lambda = (n, \ldots, n)$ is the largest possible partition, the Schubert cell is a single point corresponding to $W = V_m$, which has jump sequence $i_j = j$. This point is contained in every Schubert variety.

Proposition 1.3.4. *Each Schubert cell Ω_λ is isomorphic to $\mathbb{C}^{mn-|\lambda|}$.*

Proof. By definition, every element of Ω_λ admits a unique basis of the form:

$$\{e_{i_1} + v_1, \ldots, e_{i_m} + v_m\},$$

where $i_j = n + j - \lambda_j$ as above and v_j lies in the span of

$$\{e_1, e_2, \ldots, e_{i_j}\} - \{e_{i_1}, \ldots, e_{i_j}\}.$$

We may therefore identify Ω_λ with an affine space of dimension

$$\sum_{j=1}^{m}(n-\lambda_j) = mn - |\lambda|.$$

□

1.4. **Intersection theory on the Grassmanian.** Since the intersections of the Schubert varieties above are highly non-transverse, we must introduce an opposite cellular decomposition in order to understand intersections of the corresponding cycle classes. Let $V^i \subset \mathbb{C}^{n+m}$ be the subspace spanned by $\{e_{n+m-i+1}, \ldots, e_{n+m}\}$ in the standard basis, and let Ω^λ be the Schubert cell defined as above, but where the standard flag is replaced by the flag

$$0 = V^0 \subset \cdots \subset V^{n+m} = \mathbb{C}^{n+m}.$$

Similarly, let $X^\lambda = \overline{\Omega}^\lambda$.

Proposition 1.4.1. *The algebraic cycles X^λ and X_λ are rationally equivalent. If $|\lambda| + |\mu| = nm$, then X^λ and X_μ intersect transversely at a single point if $\lambda = m \times n - \mu$, and do not intersect otherwise. In particular,*

$$[X_\lambda].[X_\mu] = \begin{cases} [pt] & \text{if } \lambda = m \times n - \mu, \\ 0 & \text{otherwise} \end{cases}$$

in $CH^*(\mathbb{G}_{m,n})$.

Proof. The action of $G = SL_{n+m}(\mathbb{C})$ on the Chow group is through a discrete quotient, hence trivial, so $[gX_\lambda] = [X_\lambda]$ for all $g \in G$. If g is the matrix that sends e_i to e_{n+m-i} for each i (possibly up to sign), then

$$\dim(gW \cap V^j) = \dim(W \cap V_j)$$

for all j, so $gX_\lambda = X^\lambda$, and the first claim follows.

Now suppose $W \in X^\lambda \cap X_\mu$ where $|\lambda| + |\mu| = nm$. Then we have:

$$\dim(W \cap V_{n+j-\mu_j}) \geq j$$
$$\dim(W \cap V^{n+m-j+1,-\lambda_{m-j+1}}) \geq m-j+1.$$

Since $\dim W = m$, this implies

$$W \cap V_{n+j-\mu_j} \cap V^{n+m-j+1-\lambda_{m-j+1}} \neq 0$$

for all j. If $V_{n+j-\mu_j} \cap V^{n+m-j+1-\lambda_{m-j+1}} \neq 0$, then necessarily $\mu_j + \lambda_{m+j-1} \leq n$. Since $|\mu| + |\lambda| = nm$, equality holds for all j; hence $\lambda = m \times n - \mu$ and $V_{n+j-\mu_j} \cap V^{n+m-j+1-\lambda_{m-j+1}} = \mathbb{C}e_{n+j-\mu_j}$. It follows that W is the subspace generated by $e_{n+j-\mu_j}$ for all $1 \leq j \leq m$. The transversality of the intersection can be checked using local coordinates near W. □

Corollary 1.4.2. *The classes $[X_\lambda]$ form a basis over $\mathbb{Z}$ of the Chow ring $CH^*(\mathbb{G}_{m,n})$, and of the the singular cohomology $H^*(\mathbb{G}_{m,n}, \mathbb{Z})$.*

Proof. By Proposition 1.3.4 and Proposition 1.3.2, the Schubert cells form an affine paving of $\mathbb{G}_{m,n}$. Hence the $[X_\lambda]$ generate the Chow ring and the singular cohomology, and Proposition 1.4.1 shows that they form a $\mathbb{Z}$-basis. □

Of course, Proposition 1.4.1 does not fully characterize the intersection product. For a thorough discussion of intersections of Schubert varieties and the relation to symmetric functions, see [13].

2. The Weyl group and the Bruhat decomposition

Now let G be any split, simply connected, and semisimple algebraic group over a field k of characteristic zero. Fix a maximal torus $T \subset G$ and a set of positive roots $\Phi^+ \subset \Phi$ for G with respect to T. The Weyl group $W = N_G(T)/T$ is the group of symmetries of Φ; it is generated by the reflections s_α for $\alpha \in \Phi^+$, or more parsimoniously by the simple reflections corresponding to the simple roots $\Delta \subset \Phi^+$. (For more details on the general theory of reductive groups, see [3]; for more on finite reflection groups, see [10].)

2.1. **The Bruhat order.**

Our goal in this section is to establish group-theoretic facts about W which will play a crucial role in the decomposition of the flag variety. Following the notation of [14], we make the following definitions.

Definition 2.1.1. An expression $\mathbf{w}$ for $w \in W$, of length n, is a sequence $(w_{(0)}, \dots, w_{(n)})$ of elements of W such that $w_{(0)} = 1$, $w_{(n)} = w$, and $w_{(j-1)}^{-1} w_{(j)}$ is either the identity or a simple reflection for each $1 \leq j \leq n$. An expression may also be specified by its list of **factors**, i.e. the values $w_{(j-1)}^{-1} w_{(j)}$ (in order). The **length** $\ell(w)$ is the minimal length of an expression $\mathbf{w}$ for w, and an expression for w is called **reduced** if it achieves this minimum. A **subexpression** $\mathbf{v}$ of an expression $\mathbf{w}$ is a sequence $(v_{(0)}, \dots, v_{(n)})$ such that

$$v_{(j-1)}^{-1} v_{(j)} \in \left\{1, w_{(j-1)}^{-1} w_{(j)}\right\}.$$

A **positive expression** $\mathbf{w}$ is one such that $\ell(w_{(i)}) \leq \ell(w_{(i+1)})$ for all $0 \leq i < n$. Finally, we say $v \leq w$ if there exists a reduced expression $\mathbf{w}$ for w and a subexpression $\mathbf{v}$ for v in $\mathbf{w}$.

Remark 2.1.2. The relation $v \leq w$ is not obviously transitive; however, Proposition 2.1.6 below implies transitivity, and so we obtain a well-defined partial order on W known as the Bruhat order.

A reduced expression may also be written as $w = s_{\alpha_1} \cdots s_{\alpha_n}$, where s_{α_i} are the factors of $\mathbf{w}$ and $w_{(i)} = s_{\alpha_1} \cdots s_{\alpha_i}$. The length function on W has a geometric interpretation as the number of "inversions" for the action on the root system. To be precise, we have:

Proposition 2.1.3. *(i) If $\mathbf{w}$ is a reduced expression for $w \in W$ with factors $s_{\alpha_1}, \dots, s_{\alpha_k}$, then*

$$\begin{aligned} w^{-1}(\Phi^+) \cap \Phi^- &= \left\{ s_{\alpha_k} \cdots s_{\alpha_{i+1}}(-\alpha_i),\ 1 \leq i \leq k \right\} \\ &= \left\{ w^{-1} w_{(i)}(-\alpha_i),\ 1 \leq i \leq k \right\}, \end{aligned}$$

and the set has cardinality exactly k.

(ii) If s_α is a simple reflection, then $\ell(ws_\alpha) = \ell(w) \pm 1$.

(iii) If $\beta \in \Phi^+$ is any root, not necessarily simple, then $\ell(ws_\beta) > \ell(w) \implies w(\beta) \in \Phi^+$. Likewise, $\ell(s_\beta w) > \ell(w) \implies w^{-1}(\beta) \in \Phi^+$.

Proof. (i) Let us first show $\Phi^- \cap s_\alpha^{-1}(\Phi^+) = \{-\alpha\}$ if $\alpha \in \Delta$. Indeed, let $\beta \in \Phi^-$. We may write $\beta = -\beta' - m\alpha$, where β' is a sum of simple roots distinct from α with positive coefficients, and $m \geq 0$. Then

$$s_\alpha(\beta) = m\alpha - \beta' - 2\frac{\langle \beta', \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

It is clear that this can be a positive root only if $\beta' = 0$.

For the general case, we proceed by induction on $\ell(w)$. Let $w' = s_{\alpha_2}\cdots s_{\alpha_k}$. For any $\beta \in \Phi$, we have:

$$s_{\alpha_1}w'(\beta) \in \Phi^+ \iff w'(\beta) \in \Phi^+ - \{\alpha_1\} \text{ or } w'(\beta) = -\alpha_1.$$

It therefore suffices to show that $(w')^{-1}(\alpha_1) \notin \Phi^-$. For contradiction, assume otherwise. Then, by induction, $(w')^{-1}\alpha_1 = s_{\alpha_k}\cdots s_{\alpha_{i+1}}(-\alpha_i)$ for some $2 \leq i \leq k$. Using the relation

$$s_{v(\beta)} = vs_\beta v^{-1},$$

is not difficult to see that this implies $\ell(w) < k$, a contradiction.

(ii) The claim follows from (i) by considering the set $(ws_\alpha)^{-1}(\Phi^+) \cap \Phi^-$.

(iii) The second claim follows from the first because $\ell(w) = \ell(w^{-1})$ for all $w \in W$. So suppose that $\ell(ws_\beta) > \ell(w)$. We induct on $\ell(w)$, the base case being trivial. If $w \neq 1$, then there exists a simple reflection s_α such that $\ell(s_\alpha w) < \ell(w)$; for instance, the first element in a reduced word for w. Then

$$\ell(s_\alpha w s_\beta) \geq \ell(ws_\beta) - 1 > \ell(w) - 1 = \ell(s_\alpha w),$$

so $s_\alpha w(\beta) \in \Phi^+$ by induction. It remains to check that $w(\beta) \neq -\alpha$. If so, then $s_\alpha = ws_\beta w^{-1}$, which implies $s_\alpha w = ws_\beta$, a contradiction by the choice of α.

□

Corollary 2.1.4. *If* $\mathbf{w}$ *is any expression for* w*, then it contains a positive subexpression* $\mathbf{v}$ *for* w*. Deleting repeated terms in* $\mathbf{v}$ *yields a reduced expression for* w*.*

Proof. The second claim follows from the first by Proposition 2.1.3(ii). To prove the first, we will show that, if $\mathbf{w}$ is non-reduced and i is the first index such that $\ell(w_{(i)}) > \ell(w_{(i+1)})$, then $\mathbf{w}$ has a subexpression $\mathbf{v}$ for w such that $v_{(i)} = v_{(i+1)}$; an easy induction argument shows that this suffices. Now, if $s_{\alpha_1}, \ldots, s_{\alpha_n}$ are the nontrivial factors of $\mathbf{w}$, then by Proposition 2.1.3 $s_{\alpha_i}\cdots s_{\alpha_{j+1}}(-\alpha_j) = \alpha_{i+1}$ for some $j \leq i$. Then the subexpression $\mathbf{v}$ of $\mathbf{w}$ omitting the factors s_{α_i} and s_{α_j} satisfies

$$v_{(n)} = s_{\alpha_1}\cdots s_{\alpha_{j-1}}s_{\alpha_{j+1}}\cdots s_{\alpha_{i-1}}s_{\alpha_{i+1}}\cdots s_{\alpha_n} = w.$$

□

The following is [1, Lemma 2.2.1].

Lemma 2.1.5. *Suppose that* $\mathbf{w}$ *is a reduced expression of length* n *with a positive subexpression* $\mathbf{v}$*. Then there is a positive subexpression* $\mathbf{v}'$ *of* $\mathbf{w}$ *such that*

$$v_{(n)} \leq v'_{(n)} \text{ and } \ell(v'_{(n)}) = \ell(v_{(n)}) + 1.$$

Proof. Let $v = v_{(n)}$, and, without loss of generality, replace $\mathbf{v}$ with the positive subexpression of $\mathbf{w}$ for v such that

$$\max\left\{i : v_{(i)} = v_{(i+1)}\right\}$$

is minimal. Let $s_{\alpha_1}, \dots, s_{\alpha_n}$ be the factors of $\mathbf{w}$ and $s'_{\alpha_1}, \dots, s'_{\alpha_n}$ the factors of $\mathbf{v}$, so that $s'_{\alpha_i} \in \{s_{\alpha_i}, 1\}$. Let j be the largest index such that $s'_{\alpha_j} = 1$, and let $\mathbf{v}'$ be the subexpression of $\mathbf{w}$ obtained from $\mathbf{v}$ by including the factor s_{α_j}. We claim that $\mathbf{v}'$ has the desired properties; it suffices to show that $\ell(v'_{(n)}) > \ell(v)$. For contradiction, suppose otherwise. Since

$$v'_{(n)} = v \cdot (v^{-1} v_{(j)} s_{\alpha_j} v_{(j)}^{-1} v),$$

by Proposition 2.1.3(iii) we have $v_{(j)}(\alpha_j) \in \Phi^-$. Hence for some $i \le j$ such that $s'_{\alpha_i} = s_{\alpha_i}$, we have

$$v_{(j)}^{-1} v_{(i)}(\alpha_i) = \alpha_j.$$

Substituting $s_{\alpha_i} = v_{(i)}^{-1} v_{(j)} s_{\alpha_j} v_{(j)}^{-1} v_{(i)}$, we obtain

$$v = s'_{\alpha_1} \cdots s'_{\alpha_n} = s'_{\alpha_1} \cdots s'_{\alpha_{i-1}} s'_{\alpha_{i+1}} \cdots s'_{\alpha_{j-1}} s_{\alpha_j} s'_{\alpha_{j+1}} \cdots s'_{\alpha_n},$$

contradicting the choice of $\mathbf{v}$. □

Proposition 2.1.6. *(i) If $v \le w$, then **any** reduced expression for w contains a positive subexpression for v.*

(ii) There is a unique element $w_0 \in W$ of maximal length; w_0 is of order two, and $v \le w$ if and only if $w_0 v \ge w_0 w$. In particular, $w \le w_0$ for all $w \in W$.

Proof. (i) By Corollary 2.1.4 and Lemma 2.1.5, it suffices to consider the case $\ell(w) = \ell(v) + 1$. If there is a reduced expression for w with a subexpression for v, then $w = v s_\beta$ for a root $\beta \in \Phi^+$, not necessarily simple. By Proposition 2.1.3(iii), $w(\beta) \in \Phi^-$; then by Proposition 2.1.3(i), for any reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_k}$, there is an index i such that $\beta = s_{\alpha_k} \cdots s_{\alpha_{i+1}}(\alpha_i)$. This implies

$$v = w s_\beta = s_{\alpha_1} \cdots s_{\alpha_{i-1}} s_{\alpha_{i+1}} \cdots s_{\alpha_k}.$$

This expression is clearly reduced because it has length $\ell(v)$.

(ii) By Proposition 2.1.3(i), no element of W has length more than $|\Phi^+|$, so there is some element w of maximal length. By Proposition 2.1.3 again, if α is a simple root such that $w^{-1}(\alpha) \in \Phi^+$, then $\ell(s_\alpha w) > \ell(w)$. Hence w^{-1} sends all simple roots to negative roots, so $w(\Phi^+) = \Phi^-$. If w_1, w_2 are two elements of maximal length, then $w_1 w_2(\Phi^+) = \Phi^+$, so $\ell(w_1 w_2) = 0$; this implies both uniqueness and $w_0^2 = 1$. For the final claim, again applying Corollary 2.1.4 and Lemma 2.1.5, it suffices to show $v < w \implies w_0 v > w_0 w$ when $\ell(w) = \ell(v) + 1$. Then $w = v s_\beta$ for some $\beta \in \Phi^+$, so $w(\beta) \in \Phi^-$. On the other hand, if $w_0 v < w_0 w$, then $w_0 v(\beta) \in \Phi^-$, a contradiction.

□

2.2. **Example: the Weyl group of SL_n.** If $G = SL_n$ with the standard maximal torus T of diagonal matrices, then the roots are $\{\epsilon_i - \epsilon_j \,:\, i \ne j\}$, where $\epsilon_i : \mathfrak{t} \to \mathbb{C}$ is the map sending the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ to the ith entry λ_i. The standard choice of positive roots is $\Phi^+ = \{\epsilon_i - \epsilon_j \,:\, i < j\}$, and the simple roots are $\epsilon_i - \epsilon_{i+1}$. The Weyl group is isomorphic to S_n, where the ith simple reflection corresponds to the simple transposition $(i, i+1)$. Then Proposition 2.1.3 implies that the length of a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$. We note as well that the longest element w_0 is the order-reversing permutation $\sigma(i) = n + 1 - i$.

2.3. **Parabolic subgroups of the Weyl group.** If $J \subset \Delta$ is a set of simple roots, then the reflections s_α for $\alpha \in J$ generate a **parabolic subgroup** written $W_J \subset W$. (As one might expect, these subgroups are closely related to parabolic subgroups of G; see Proposition 3.1.1.) Define

$$W^J = \{w \in W \,:\, \ell(ws_\alpha) > \ell(w),\ \forall \alpha \in J\}\,;$$

also let $J^\pm$ denote the intersection of the linear span of J with $\Phi^\pm$.

Proposition 2.3.1. *(i) Let $w = w^J w_J$ for some $w^J \in W^J$ and $w_J \in W_J$. Then*

$$\ell(w^J) = \#(\Phi^+ \cap w(\Phi^- \setminus J^-)), \;\; \ell(w_J) = \#(\Phi^+ \cap w(J^-)),$$

and in particular $\ell(w) = \ell(w^J) + \ell(w_J)$.

(ii) The multiplication map is a bijection:

$$W^J \times W_J \to W.$$

In particular, W^J forms a set of coset representatives for W/W_J.

Proof. (i) We first observe that

$$w_J(\Phi^\pm \setminus J^\pm) = \Phi^\pm \setminus J^\pm,$$

because W_J preserves the linear span of J and cannot change the sign of any root not spanned by J. Therefore,

$$\#(\Phi^+ \cap w(\Phi^- \setminus J^-)) = \#(\Phi^+ \cap w^J(\Phi^- \setminus J^-)) = \ell(w^J).$$

The other assertion is similar.

(ii) To show that the multiplication map is surjective, it suffices to see that each coset of W_J contains an element of W^J; and indeed, any element of minimal length in wW_J will lie in W^J by definition. It remains to show injectivity; suppose therefore that $y_1 w_1 = y_2 w_2$ where $y_i \in W^J$ and $w_i \in W_J$. We have

$$y_1 = y_2(w_2 w_1^{-1}),$$

and by (i) the concatenation of a reduced expression for y_2 and a reduced expression for $w_2 w_1^{-1}$ is a reduced expression for y_1. If $w_2 w_1^{-1} \neq 1$, then it has a reduced expression ending with s_α for some $\alpha \in J$; this would contradict $y_1 \in W^J$ because it would imply $\ell(y_1 s_\alpha) < \ell(y_1)$. Hence $w_2 = w_1$, and so $y_1 = y_2$ as well.

□

For any $w \in W$, we write

$$w = w^J w_J$$

for the factorization provided by Proposition 2.3.1. The Bruhat order on W induces a natural partial order on W^J, and in fact this is compatible with the projection $W \to W^J$, in the following sense.

Proposition 2.3.2. *Let $w, v \in W$ be any elements.*

(i) If $w \leq v$, then $w^J \leq v^J$.

(ii) We have $w^J \leq v^J$ if and only if $(w_0 w)^J \geq (w_0 v)^J$, where $w_0 \in W$ is the unique longest element.

In particular, $w^J \leq w_0^J$ for all $w \in W$.

Proof. (i) Since $w^J \leq w$, we may assume without loss of generality that $w^J = w$. We proceed by induction on $\ell(v_J)$, where the base case is trivial. If $v_J \neq 1$, then there exists $\alpha \in J$ such that $\ell(v_J s_\alpha) < \ell(v_J)$, or equivalently $\ell(v s_\alpha) < \ell(v)$. We will show:

$$w \leq v s_\alpha. \tag{2}$$

Indeed, if

$$v s_\alpha = s_{\beta_1} \cdots s_{\beta_k}$$

is any reduced expression, then $s_{\beta_1} \cdots s_{\beta_k} s_\alpha$ contains a reduced expression for w as a subword. Since no reduced expression for w ends in s_α, (2) holds. In particular, by induction, $w \leq (v s_\alpha)^J = v^J$.

(ii) One direction clearly suffices, so suppose $w \leq v$. Since $w^J \leq v^J \leq v$, we have $w_0 v \leq w_0 w^J$. By (i), $(w_0 v)^J \leq (w_0 w^J)^J$. Since $w_0 w^J$ and $w_0 w$ lie in the same W_J-coset, the claim follows.

□

2.4. **Example: $S_m \times S_n \subset S_{m+n}$.** In the notation of the preceding example, consider the subset of simple roots

$$J = \{\epsilon_i - \epsilon_{i+1} \,:\, 1 \leq i \leq n+m-1,\, i \neq m\}\,.$$

The corresponding subgroup $W_J \subset S_{m+n}$ is generated by all simple transpositions except $(m, m+1)$. It is not difficult to see that this subgroup is exactly $S_m \times S_n$, and the set of coset representatives W^J consists of permutations that preserve the internal order of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$. We will see in §3.2 that this is exactly the example relevant to the Grassmanian $\mathbb{G}_{m,n}$.

2.5. **Lifting to G.** The projection $N_G(T) \to W$ is not split in general, but the following well-known result of Tits [19] provides a good approximation.

Proposition 2.5.1. *There exists a system of lifts $\dot{s}_\alpha \in N_G(T)$, for each $\alpha \in \Delta$, such that:*

(i) The natural image of $\dot{s}_\alpha$ in W is s_α.
(ii) For all α, $\dot{s}_\alpha^4 = 1$.
(iii) If $s_{\alpha_1} \cdots s_{\alpha_n}$ and $s_{\beta_1} \cdots s_{\beta_n}$ are two reduced expressions for w, then

$$\dot{s}_{\alpha_1} \cdots \dot{s}_{\alpha_n} = \dot{s}_{\beta_1} \cdots \dot{s}_{\beta_n}.$$

We therefore obtain a well-defined set-theoretic section $W \to N_G(T)$ defined by $w \mapsto \dot{w} = \dot{s}_{\alpha_1} \cdots \dot{s}_{\alpha_n}$, where $w = s_{\alpha_1} \cdots s_{\alpha_n}$ is any reduced expression.

Recall that, for each $\alpha \in \Phi$, there is a unique, one-dimensional unipotent subgroup U_α whose Lie algebra is the root space corresponding to α. By definition, for all $w \in W$ we have:

$$\dot{w} U_\alpha \dot{w}^{-1} = U_{w(\alpha)}. \tag{3}$$

2.6. **The Bruhat decomposition of G.** Let $B^+ \subset G$ be the Borel subgroup associated to the choice of positive roots Φ^+, and U^+ its unipotent radical. Swapping positive and negative roots gives the opposite Borel B^-, with unipotent radical U^-. For each $w \in W$, we define the **Bruhat cell**

$$C(w) = B^+ \dot{w} B^+ \subset G,$$

a locally closed subvariety.

Remark 2.6.1. The Bruhat cell has a simple description in terms of representation theory, as follows. Let λ be a dominant weight and V the representation of G generated by the highest weight vector ξ_λ; V has a basis of weight vectors. Then $g \in B^+\dot{w}B^+$ implies that the **lowest** weight vector appearing in the basis decomposition of $g\xi_\lambda$ has weight $w\lambda$. Similarly, if $g \in B^-\dot{w}B^+$, then $w\lambda$ is the **highest** weight appearing in $g\xi_\lambda$.

The goal of this subsection is to prove the Bruhat decomposition, our template for decomposing flag varieties:

$$G = \bigsqcup_{w\in W} C(w). \tag{4}$$

Our exposition largely follows [5, Chapter 30].

Proposition 2.6.2. *For each $w \in W$, multiplication induces isomorphisms:*

$$\prod_{\alpha\in\Phi^+\cap w(\Phi^-)} U_\alpha \times B^+ \to C(w),$$
$$(u,b) \mapsto u\dot{w}b;$$
$$\prod_{\beta\in\Phi^-\cap w^{-1}(\Phi^+)} U_\beta \times B^+ \to C(w),$$
$$(u,b) \mapsto \dot{w}ub.$$

Proof. Recall that, by the Baker-Campbell-Hausdorff formula, multiplication is an isomorphism

$$\prod_{\alpha\in\Phi^+} U_\alpha \to U^+,$$

regardless of the order. We therefore have:

$$\begin{aligned} B^+\dot{w}B^+ &= U^+\dot{w}B^+ \\ &= \dot{w}(\dot{w}^{-1}U^+\dot{w})B^+ \\ &= \dot{w}\prod_{\alpha\in\Phi^+} U_{w^{-1}(\alpha)}B^+ \\ &= \dot{w}\prod_{\alpha\in\Phi^+\cap w(\Phi^-)} U_{w^{-1}(\alpha)}B^+, \end{aligned}$$

where the last expression has no redundancy because $B^- \cap B^+ = T$. The desired expressions follow immediately. □

Lemma 2.6.3. *For each $w \in W$ and each simple reflection s_α, we have:*

$$C(s_\alpha)C(w) = \begin{cases} C(s_\alpha w) & \text{if } \ell(s_\alpha w) > \ell(w), \\ C(s_\alpha w) \cup C(w) & \text{if } \ell(s_\alpha w) < \ell(w). \end{cases}$$

Proof. We calculate directly using Proposition 2.6.2:

$$\begin{aligned} C(s_\alpha)C(w) &= U_\alpha\dot{s}_\alpha B^+\dot{w}B^+ \\ &= \dot{s}_\alpha U_{-\alpha}\dot{w}\prod_{\beta\in\Phi^-\cap w^{-1}(\Phi^+)} U_{w^{-1}(\beta)}B^+ \\ &= \dot{s}_\alpha\dot{w}U_{-w^{-1}(\alpha)}\prod_{\beta\in\Phi^-\cap w^{-1}(\Phi^+)} U_\beta B^+. \end{aligned}$$

Now, if $\ell(s_\alpha w) > \ell(w)$, we may conclude using Proposition 2.1.3(i). So suppose that $\ell(s_\alpha w) < \ell(w)$. If we set $w' = s_\alpha w$, then $\ell(s_\alpha w') > \ell(w')$, so

$$C(s_\alpha)C(s_\alpha w') = C(s_\alpha)C(s_\alpha)C(w').$$

It therefore suffices to show

$$C(s_\alpha)C(s_\alpha) = C(1) \cup C(s_\alpha)$$

for each simple reflection s_α. Taking $w = s_\alpha$ in our calculation above, we have:

$$C(s_\alpha)C(s_\alpha) = U_\alpha U_{-\alpha} B^+.$$

Now, using only a computation in SL_2, one can show

$$U_\alpha U_{-\alpha} U_\alpha T = U_\alpha T \cup U_\alpha \dot{s}_\alpha U_\alpha T,$$

which suffices to conclude. □

Corollary 2.6.4. *We have*

$$G = \cup_{w \in W} C(w).$$

Proof. The lemma shows that the right-hand side is a subgroup of G. Since it contains B^+ and each $\dot{s}_\alpha$, it must be the entire group. □

Now, to establish (4), it suffices to show:

Proposition 2.6.5. *If $w \neq w'$, then $C(w) \neq C(w')$.*

Proof. We induct on $\min\{\ell(w), \ell(w')\}$. The base case is clear; for the inductive step, suppose $\ell(w) \leq \ell(w')$ and let s_α be a simple reflection such that $\ell(s_\alpha w) < \ell(w)$. Then

$$s_\alpha w \in C(s_\alpha)C(w') \subset C(s_\alpha w') \cup C(w')$$

by Lemma 2.6.3. The inductive hypothesis implies either $w = w'$, as desired, or $s_\alpha w = w'$. The latter case is impossible because $\ell(s_\alpha w) < \ell(w) \leq \ell(w')$. □

The geometric significance of the Bruhat order is encapsulated by the following fundamental fact.

Proposition 2.6.6. *For each $w \in W$, the Zariski closure of $C(w)$ is:*

$$\overline{C(w)} = \bigsqcup_{v \leq w} C(v).$$

Proof. The following elegant proof may be found in [18], and uses the Bott-Samelson varieties further studied by Demazure [8]. Let $w = s_{\alpha_1} \dots s_{\alpha_k}$ be a reduced decomposition. For each $\alpha \in \Delta$, define

$$P_\alpha = C(s_\alpha) \cup C(1),$$

which is a subgroup of G by Lemma 2.6.3; it is not difficult to check using Proposition 2.6.2 and a calculation in SL_2 that $P_\alpha / B \simeq \mathbb{P}^1$. For varieties X, Y with left and right actions by B, define an equivalence relation on $X \times Y$ by $(x, y) \sim (xb^{-1}, by)$. When it exists, the quotient $X \times_B Y$ by this equivalence relation again has left and right actions by B.

We claim that the multiplication map

$$\pi : P_{\alpha_1} \times_B P_{\alpha_2} \times_B \cdots \times_B P_{\alpha_k} \to G$$

is proper. Indeed, it is the base change along $G \to G/B$ of the multiplication map $P_{\alpha_1} \times_B P_{\alpha_2} \times_B \cdots \times_B P_{\alpha_k}/B \to G/B$, whose source is an iterated $\mathbb{P}^1$-bundle. Therefore the image of π is an irreducible, closed subvariety of G.

Restricted to a dense open subvariety, π agrees with the isomorpism onto $C(w)$ given by Proposition 2.6.2. Hence the image of π is exactly the closure of $C(w)$. On the other hand, by Proposition 2.1.6(i) the image of π is $\sqcup_{v\leq w} C(v)$. □

Remark 2.6.7. This reasoning could also be used to give a geometric proof of Proposition 2.1.6(i).

Corollary 2.6.8. *The subset $B^- B^+$ is open and dense in G.*

Proof. The preceding proposition, along with Proposition 2.1.6, shows that the "big cell" $B^+ \dot{w}_0 B^+$ is open and dense, where $w_0 \in W$ is the unique longest element. Hence $\dot{w}_0 B^+ \dot{w}_0 B^+ = B^- B^+$ is open and dense as well. □

For each $w \in W$, let $\Omega_w \subset G/B^+$ be the natural image of the Bruhat cell $C(w)$. We refer to Ω_w as a Schubert cell in G/B^+; the closure $X_w = \overline{\Omega}_w$ is called a Schubert variety. Combining the preceding proposition with Proposition 2.6.2, we conclude:

Corollary 2.6.9. *For each $w \in W$, Ω_w is isomorphic to $\mathbb{A}^{\ell(w)}$ and*

$$X_w = \bigsqcup_{v \leq w} \Omega_v.$$

In particular, the Schubert cells define a cellular decomposition of the complete flag variety G/B^+.

In the next section, we will generalize this result to all flag varieties G/P.

3. Generalized flag varieties

The homogeneous space G/P, where P is a parabolic subgroup, has the structure of a smooth, projective algebraic variety. In this section, we give a decomposition of G/P generalizing the one we have seen for the complex Grassmanian.

3.1. Classifying parabolic subgroups.

Recall that a parabolic subgroup is a closed subgroup $P \subset G$ containing a Borel subgroup B, which we may assume without loss of generality is B^+.

Proposition 3.1.1. *(i) If $J \subset \Delta$ is any set of simple roots, then*

$$P_J = \bigsqcup_{w \in W_J} C(w)$$

is a parabolic subgroup of G. Conversely, any parabolic subgroup $B^+ \subset P \subset G$ is of this form.

(ii) If $J_1, J_2 \subset \Delta$, then $P_{J_1} \cap P_{J_2} = P_{J_1 \cap J_2}$.

Proof. (i) By Lemma 2.6.3 and Proposition 2.6.6, to show that P_J is a closed subgroup it suffices to check that W_J is downwards closed for the Bruhat order on W. Indeed, if $w \in W_J$ and $v \leq w$, then $v^J \leq w^J = 1$ by Proposition 2.3.2.

Conversely, suppose P is any closed parabolic containing B^+; then

$$P = \bigsqcup_{w \in W_0} C(w)$$

for some subset $W_0 \subset W$. Since P is closed, if s_α is a simple reflection appearing in the reduced decomposition of any $w \in W_0$, we have $s_\alpha \in W_0$. It follows that W_0 is a subgroup generated by simple reflections, hence of the form W_J for some J.

(ii) It suffices to show that $W_{J_1} \cap W_{J_2} = W_{J_1 \cap J_2}$. The inclusion $\supset$ is obvious, so suppose that $w \in W_{J_1} \cap W_{J_2}$. Fix any reduced expression for w, and suppose it contains s_α for a simple root α. Then by definition $s_\alpha \leq w$. As we saw in the proof of (i), this implies $s_\alpha \in W_{J_1} \cap W_{J_2}$, so by Proposition 2.3.1(i) $\alpha \in J_1 \cap J_2$. Hence $w \in W_{J_1 \cap J_2}$.

□

We define the **(generalized) Schubert cell** Ω_w^J to be the image of $C(w)$ in G/P_J, and the **(generalized) Schubert variety** X_w^J to be its Zariski closure. In general, X_w^J may be singular, but it is always Cohen-Macaulay (even in positive characteristic [15]). When $J = \emptyset$, we omit the superscript.

Proposition 3.1.2. *(i) For each $J \subset \Delta$, there is a decomposition*

$$G/P_J = \bigsqcup_{w \in W^J} \Omega_w^J.$$

(ii) For all $w \in W$, Ω_w^J is isomorphic to $\mathbb{A}^{\ell(w^J)}$.

(iii) For each $w \in W^J$,

$$X_w^J = \bigsqcup_{\substack{v \in W^J \\ v \leq w}} \Omega_v^J.$$

In particular, $\dim(G/P_J) = \ell(w_0^J) = \#\Phi^+ \setminus J^+$.

Proof. (i) Using the Bruhat decomposition along with Lemma 2.6.3 and Proposition 2.3.1, we have:

$$\begin{aligned} G &= \bigsqcup_{w \in W} C(w) \\ &= \bigsqcup_{w \in W^J} \bigsqcup_{v \in W_J} C(w)C(v) \\ &= \bigsqcup_{w \in W^J} C(w)P_J. \end{aligned}$$

(ii) It suffices to show that the map

$$\prod_{\beta \in \Phi^- \cap w^{-1}(\Phi^+)} U_\beta \to \Omega_w,$$

defined by $u \mapsto \dot{w}uP_J$, is an isomorphism when $w \in W^J$. By Proposition 2.6.2, this morphism is well-defined and surjective. It is also injective: the intersection

$$\left(\prod_{\beta \in \Phi^- \cap w^{-1}(\Phi^+)} U_\beta \right) \cap P_J$$

is trivial because none of the βs lie in the linear span of J (cf. 2.3.1(i)).

(iii) Since the quotient map $\pi : G \to G/P_J$ is open, we may compute the closures inside G:

$$\begin{aligned}\overline{\Omega}^J_w &= \pi\left(\overline{\pi^{-1}(\Omega^J_w)}\right)\\ &= \pi\left(\overline{B^+\dot{w}P_J}\right)\\ &= \pi\left(\overline{\bigsqcup_{v\in W_J} B^+\dot{w}\dot{v}B^+}\right)\\ &= \pi\left(\cup_{v\in W_J}\cup_{w'\leq wv} B^+\dot{w}'B^+\right)\\ &= \cup_{v\in W_J}\cup_{w'\leq wv}\Omega^J_{w'}.\end{aligned}$$

By Proposition 2.3.2, since $w \in W^J$, $w' \leq wv \implies (w')^J \leq w$, which completes the proof.

□

Each Bruhat cell Ω^J_w contains a distinguished point $e(w) = \dot{w}P_J$, which is fixed by the left action of the torus $T \subset G$. The T-action stabilizes Ω^J_w, and we have the following:

Lemma 3.1.3. *Let $w \in W^J$; then $e(w)$ is the unique T-stable point in Ω^J_w.*

Proof. Let $\dot{w}uP_J$ be a fixed point for T, where

$$u \in \prod_{\beta\in\Phi^-\cap w^{-1}(\Phi^+)} U_\beta.$$

Then, for all $t \in T$,

$$u^{-1}\dot{w}^{-1}t\dot{w}u \in P_J.$$

Since $\dot{w}$ normalizes T, this is equivalent to

$$u^{-1}Tu \subset P_J.$$

The left-hand side is contained in

$$T \prod_{\beta\in\Phi^-\cap w^{-1}(\Phi^+)} U_\beta,$$

where there is no redundancy in the expressions, and this subgroup intersects P_J only in T because $w \in W^J$. Hence it suffices to show that, if $u \neq 1$, there exists $t \in T$ such that $u^{-1}tu$ has nonzero U_β component for some β. But this is clearly true because no nontrivial unipotent elements normalize T. □

3.2. **Example: the Grassmanian.** We now translate our first example, the Grassmanian, into the more general language of parabolic subgroups. Let $G = SL_{n+m}/\mathbb{C}$; the parabolic subgroup stabilizing the coordinate subspace $\mathbb{C}^m \subset \mathbb{C}^{m+n}$ corresponds to the set of simple roots $J = \Delta - \{\epsilon_m - \epsilon_{m+1}\}$ considered in §2.4. Indeed, the lift $S_{n+m} \to G$ sends a permutation to the corresponding $(n+m)\times(n+m)$ permutation matrix, up to sign; the permutation matrices preserving $\mathbb{C}^m$ are precisely the ones corresponding to the parabolic subgroup $S_m \times S_n \subset S_{m+n}$.

Let us now identify the Schubert cells on G/P_J. The T-fixed point $e(w)$ corresponds to the subspace $W_w = \dot{w}(\mathbb{C}^m)$. If $V_i \subset \mathbb{C}^{m+n}$ is the i-dimensional coordinate subspace $\langle e_1, \ldots, e_i \rangle$, then

$$\dim(V_i \cap W_w) = \# w(\{1, \ldots, m\}) \cap \{1, \ldots, i\}\,.$$

This gives a recipe for the jump sequence associated to W_w; in particular, as expected, the partition $\lambda(W_w)$ depends only on the coset of w in W/W_J. We leave to the reader the verification that the inclusion relation $\lambda \subset \mu$ corresponds exactly to the partial order on W^J. We also remark that $\dim(G/P_J) = \#(\Phi^+/J^+) = mn$: indeed, Φ^+/J^+ consists of roots $\epsilon_i - \epsilon_j$ with $1 \le i \le m < m+1 \le j \le m+n$.

3.3. **Intersections on G/P_J: dual Schubert varieties.** As in the case of Grassmanians, the intersections of Schubert varieties are best understood by considering the opposite (or dual) Schubert varieties, i.e. $w_0 X_w^J$, where $w_0 \in W$ is the unique longest element. The intersection $w_0 X_w^J \cap X_v^J$ is sometimes called a **Richardson variety**. Note that, since G is generated by one-parameter subgroups, X_w^J and $w_0 X_w^J$ are rationally equivalent. The following proposition generalizes Proposition 1.4.1.

Proposition 3.3.1. *(i) The Richardson variety $w_0 X_{w_0 v}^J \cap X_w^J$ is nonempty if and only if $v^J \le w^J$.*

(ii) The co-dimension of $X_{w_0 v}^J$ is equal to $\ell(v^J)$, and for all $w, v \in W$ with $\ell(w^J) = \ell(v^J)$, we have:

$$[X_w^J].[X_{w_0 v}^J] = \begin{cases} 0 & \text{if } w \neq v, \\ [pt] & \text{if } w = v. \end{cases}$$

In light of this proposition, $w_0 X_{w_0 w}^J$ is sometimes called the dual Schubert variety to X_w^J.

Proof. (i) The intersection $X_w^J \cap w_0 X_{w_0 v}^J$ is a T-stable closed subvariety, so if it is nonempty it contains a T-fixed point. By Lemma 3.1.3, such a point is of the form $e(y)$ for some $y \in W^J$. But by Proposition 3.1.2, any such y satisfies both $y \le w^J$ and $(w_0 y)^J \le (w_0 v)^J$. By Proposition 2.3.2, the latter equality implies $v^J \le y$, so by transitivity of the Bruhat order $v^J \le w^J$. The converse is immediate.

(ii) The co-dimension claim follows from Proposition 2.3.1. To compute the intersection product, we consider again $w_0 X_{w_0 v}^J \cap X_w^J$. By (i), if the intersection is nonempty then $w = v$. In this case, the intersection at $e(w)$ is transverse, as we can check using the coordinates of Proposition 2.6.2; since any other irreducible component of this intersection would have a T-fixed point, we conclude $X_w^J \cap X_{w_0 w}^J$ is indeed a single point with multiplicity one.

□

Corollary 3.3.2. *The algebraic cycle classes $[X_w^J]$ form a $\mathbb{Z}$-basis of the Chow ring $CH^*(G/P_J)$. If $k = \mathbb{C}$, then their images in the ring of singular cohomology $H^*(G/P_J)$ likewise form a $\mathbb{Z}$-basis.*

Proof. That these cycle classes generate follows from Proposition 3.1.2; Proposition 3.3.1 implies that they are also linearly independent over $\mathbb{Z}$. □

Corollary 3.3.3. *The divisor class group of G/P_J is freely generated by the cycle classes $[X^J_{w_0 s_\alpha}]$, where α ranges over $\Delta \setminus J$.*

Proof. In light of the previous corollary, we must show that:

$$\left\{w^J \in W^J \,:\, \ell(w^J) = \ell(w_0^J) - 1\right\} = \left\{(w_0 s_\alpha)^J \,:\, \alpha \notin \Delta \setminus J\right\},$$

and that the $(w_0 s_\alpha)^J$ are distinct. Our first claim is that

$$(w_0 s_\alpha)_J = (w_0)_J \tag{5}$$

for all $\alpha \in \Delta \setminus J$. Indeed, applying Proposition 2.3.1(i), $(w_0)_J$ is the unique longest element of W_J, and $\ell((w_0 s_\alpha)_J) = \ell(w_0)_J = \#J^+$ because $s_\alpha(J^-) \subset \Phi^-$.

Now, (5) implies that the $(w_0 s_\alpha)^J$ are distinct elements of length $\ell(w_0^J) - 1$. It remains to show that every w^J of length $\ell(w_0^J) - 1$ is of this form. For any such w^J, we have:

$$\ell(w^J (w_0)_J) = \ell(w^J) + \ell((w_0)_J) = \ell(w_0) - 1.$$

However, any element of W of length $\ell(w_0) - 1$ is of the form $w_0 s_\alpha$ for some $\alpha \in \Delta$, by Proposition 2.1.3(i). It follows that

$$w^J = (w^J (w_0)_J)^J = (w_0 s_\alpha)^J.$$

If $\alpha \in J$, then of course $(w_0 s_\alpha)^J = w_0^J$, which would imply $\ell(w^J) = w_0^J$. So $\alpha \in \Delta \setminus J$. □

4. Line bundles on G/P_J and representation theory

4.1. **The Borel presentation of cohomology.** Let $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ be the character group of T. For any $\lambda \in X(T)$, one may define an associated line bundle $L(\lambda)$ on the complete flag variety G/B^+, with total space

$$G \times_{B^+,\lambda} \mathbb{A}^1.$$

Here, B^+ acts on the affine line by the usual extension of λ to $B^+ = TU^+$, and the notation is as in the proof of Proposition 2.6.6. Sending each character to the Chern class of the associated line bundle, we obtain a ring map:

$$\mathrm{Sym}(X(T)) \to CH^*(G/B^+). \tag{6}$$

Theorem 4.1.1 (Borel, 1953)**.** *The map above induces an isomorphism:*

$$CH^*(G/B^+) \otimes \mathbb{Q} \simeq \mathbb{Q} \otimes \frac{\mathrm{Sym}(X(T))}{\mathrm{Sym}(X(T))^W_+},$$

where the quotient is by the ideal generated by W-invariant elements of positive degree. Furthermore, the natural map

$$CH^*(G/P_J) \otimes \mathbb{Q} \to CH^*(G/B^+) \otimes \mathbb{Q}$$

identifies

$$CH^*(G/P_J) \otimes \mathbb{Q} \xrightarrow{\sim} \left(CH^*(G/B^+)^{W_J}\right).$$

Borel's proof [2] (which we do not reproduce here) is purely topological; we have translated the result to the Chow ring via Corollary 3.3.2.

In the case of degree-one elements of $\mathrm{Sym}(X(T))$, the second assertion of the theorem may be viewed as follows. Let

$$\lambda = \sum_{\alpha \in \Delta} c_\alpha \omega_\alpha \in X(T)$$

be a dominant weight, where ω_α is the fundamental weight associated to the simple root α. (Recall that the fundamental weights form a $\mathbb{Z}$-basis of the character group, and λ is dominant if and only if $c_\alpha \geq 0$ for all α.) Then λ is associated to a highest weight representation of G, with highest weight vector ξ_λ. If $J_\lambda \subset \Delta$ is the set of α with $c_\alpha = 0$, then ξ_λ is an eigenvector for any parabolic subgroup P_J such that $J \subset J_\lambda$ and in particular λ extends to a character of P_J. We may therefore form the associated bundle $L_J(\lambda)$ on G/P_J, whose pullback to G/B^+ via the natural projection is exactly $L(\lambda)$.

Conversely, if λ extends to P_J, then it is fixed by W_J, which implies $J \subset J_\lambda$ since $s_\beta\omega_\alpha = \omega_\alpha - \delta_{\alpha\beta}\alpha$ for all simple roots α, β. The associated line bundle construction therefore defines a map

$$\mathrm{Sym}((X(T)^{W_J}) \to CH^*(G/P_J).$$

However, when $J \neq \emptyset$, it is not true that $\mathrm{Sym}((X(T)^{W_J}) = \mathrm{Sym}(X(T))^{W_J}$, so characteristic classes of line bundles on G/P_J do not generate the cohomology in general, even up to torsion. Although we expect all cycle classes on G/P_J to be exhausted by the characteristic classes of vector bundles associated to higher-dimensional representations of G/P_J, we have not verified this fact.

4.2. **Computing characteristic classes on G/P_J.** In this section, we identify the characteristic classes of the line bundles $L_J(\lambda)$ in terms of Schubert varieties; this relates the Borel presentation of the cohomology to the one given in Corollary 3.3.2. The first step is the following observation:

Proposition 4.2.1. *Let V be a representation of G with highest weight λ; for each $J \subset J_\lambda$, V defines a map $f^J_\lambda : G/P_J \to \mathbb{P}(V)$ by $g \mapsto [g \cdot \xi_\lambda]$, where ξ_λ is a nonzero highest weight vector. Then*

$$L_J(-\lambda) = (f^J_\lambda)^*(\mathcal{O}(1)),$$

where $\mathcal{O}(1)$ is the hyperplane line bundle on $\mathbb{P}(V)$.

Proof. Dualizing, we consider instead the pullback of the tautological bundle $V \to \mathbb{P}(V)$. It then suffices to note that the following diagram is Cartesian:

$$\begin{array}{ccc} G \times_{P_J,\lambda} \mathbb{A}^1 & \longrightarrow & V \\ \downarrow & & \downarrow \\ G/P_J & \longrightarrow & \mathbb{P}(V). \end{array}$$

□

On the other hand, consider the divisor

$$D = \sum_{\alpha \in \Delta \setminus J} c_\alpha X^J_{w_0 s_\alpha}$$

on G/P_J, where c_α is a nonnegative integer for each α. (By Corollary 3.3.3, all effective divisors on G/P_J are linearly equivalent to one of this form.) For each $g \in G$, the translate gD is linearly equivalent to D. Thus the linear system $|D|$ has no base points, and it defines a morphism f_D of G/P_J into a projective space $\mathbb{P}(V)$ by the usual recipe; if $t_i \in H^0(G/P_J, L(D))$ are a basis of global sections, $0 \leq i \leq n$, then $\dim V = n+1$ and $f_D(y) = [t_0(y) : \ldots : t_n(y)] \in \mathbb{P}(V)$.

Proposition 4.2.2. *There exists an irreducible representation $\rho : G \to V$ whose projectivization $\overline{\rho}$ satisfies*

$$f_D(gy) = \overline{\rho}(g) f_D(y)$$

for all $g \in G$ and $y \in G/P_J$. The highest weight of V is

$$\lambda_D = \sum_{\alpha \in \Delta \setminus J} c_\alpha \omega_\alpha,$$

where ω_α is the fundamental weight corresponding to the root α; in particular,

$$L(D) = f_D^*(\mathcal{O}(1)) = L_J(-\lambda_D).$$

Remark 4.2.3. With more care, this construction (in fact, just the special case $P_J = B^+$) can be adapted to a proof of the highest weight theorem classifying all representations of G. This was the approach taken in the Séminaire Chevalley [6]. The proof here is adapted from Chevalley's notes as well as from Serre's proof of the Borel-Weil theorem [17].

Proof. We begin by constructing the action of G on $\mathbb{P}(V)$. Choose any $g \in G$; since gD is linearly equivalent to D, there exists a function $u_g : G/P_J \to \mathbb{P}^1$, defined up to a scalar, such that $\operatorname{div} u_g = gD - D$. Each section t_i may be considered as a function on G/P_J such that $\operatorname{div} t_i \geq -D$. Then t_i^g, the function such that $t_i^g(y) = t_i(gy)$, satisfies

$$\operatorname{div} t_i^g + u_{g^{-1}} \geq -g^{-1}D + g^{-1}D - D = -D.$$

Hence

$$t_i^g u_g = \sum a_{ij}(g) t_j$$

for some coefficients $a_{ij}(g)$ defined up to a scalar. The matrix $a_{ij}(g)$ defines the sought-after action of g on $\mathbb{P}(V)$, because $f_D(gy) = [t_0^g(y) : \ldots : t_n^g(y)]$.

Now that we have the projective representation $\overline{\rho}$, it may be lifted to an honest representation $\rho : G \to V$ because G is simply connected. The irreducibility of V is clear: the image of f_D is generated by $f_D(1 \cdot P_J)$ under the action of g, and by definition the image of f_D is not contained in any hyperplane of $\mathbb{P}(V)$. Therefore V is a highest weight representation, with some weight λ_D which is fixed by W_J, and by Proposition 4.2.1 we have $L(D) = L_J(-\lambda_D)$. By linearity, to establish the formula for λ_D, it suffices to consider $D = X^J_{w_0 s_\alpha}$.

By definition, D is the cycle class of $f_D^{-1}(H)$, where H is a hyperplane section of $\mathbb{P}(V)$. Let $\{\xi_\mu\}$ be a basis of weight vectors for V, and consider the hyperplane of V generated by all weight vectors except the lowest one $\xi_{w_0\lambda_D}$. If H is its image in $\mathbb{P}(V)$, then $f_D^{-1}(H)$ is, at least set-theoretically, the set of cosets gP_J such that $\xi_{w_0\lambda_D}$ does not appear in the basis expansion of $g\xi_{\lambda_D}$. As observed in Remark 2.6.1, this is exactly the collection of Schubert cells $B^+\dot{w}P_J$ such that $w\lambda_D \neq w_0\lambda_D$. We therefore have:

$$\left\{ w \in W^J \,:\, w\lambda_D \neq w_0\lambda_D \right\} = \left\{ w \in W^J \,:\, w \leq w_0 s_\alpha \right\}.$$

One can check directly that this implies

$$\lambda_D = n_\alpha \omega_\alpha$$

for some positive integer n_α. We claim that in fact $n_\alpha = 1$; indeed, we have

$$L_J(\omega_\alpha)^{\otimes n_\alpha} = L(X^J_{w_0 s_\alpha}),$$

and by Corollary 3.3.3 the line bundles $L(X^J_{w_0 s_\alpha})$ freely generate the Picard group of G/P_J as α ranges over $\Delta \setminus J$. □

4.3. **Example: tautological bundles.** Suppose that $G = SL_n/\mathbb{C}$, and consider the standard upper-triangular Borel subgroup B. The complete flag variety G/B parameterizes flags:

$$0 = V_0 \subset \cdots \subset V_n = \mathbb{C}^n, \;\; \dim V_i = i.$$

It therefore carries tautological bundles $L_i = V_i/V_{i-1}$. It is not difficult to check that these are the bundles associated to the weights

$$\lambda_i : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i.$$

Since these weights additively generate $X(T)$, Borel's result implies that the characteristic classes of the tautological bundles generate the cohomology as a ring. For a direct proof of this fact, see [13]. We also can make sense of the kernel $\mathrm{Sym}(X(T))^W_+$ of the natural map $\mathrm{Sym}(X(T)) \to CH^*(G/B)$. For instance, the trivial bundle $C = \mathbb{C}^n \times SL_n/B$ is an iterated extension of all of the L_i. By the splitting principle and the multiplicative property of total Chern classes, we obtain:

$$c(C) = \prod_{i=1}^{n} (1 + c_1(L_i)) = 1.$$

The nonconstant terms of this polynomial are the elementary symmetric functions in $c_1(L_i)$, so their vanishing corresponds to the Weyl-invariance of the nontrivial symmetric functions in λ_i, considered as elements of $\mathrm{Sym}(X(T))$.

Now consider the Grassmanian $\mathbb{G}_{m,n}$, corresponding to the parabolic subgroup $P_{m,n} \subset SL_{m+n}$. It carries a tautological vector bundle $\mathcal{W}$ of rank m, whose fiber over a point of the Grassmanian is the corresponding subspace of $\mathbb{C}^{m+n}$. This vector bundle is also the associated bundle to the standard representation of GL_m, considered as a representation of $P_{m,n}$. Let $\pi : SL_{m+n}/B \to SL_{m+n}/P_{m,n}$ be the natural projection; then $\pi^*(\mathcal{W})$ is the iterated extension of the bundles $L_1, \ldots, L_m$, and so

$$\pi^*(c(\mathcal{W})) = \prod_{i=1}^{m} (1 + c_1(L_i)).$$

From the Borel perspective, the total Chern class of $\mathcal{W}$ is the image of the $S_m \times S_n$-invariant element

$$\prod_{i=1}^{m} (1 + \lambda_i) \in \mathrm{Sym}(X(T)).$$

In fact, the nonconstant terms in this polynomial generate all of $(\mathrm{Sym}(X(T))^{S_m \times S_n}$, so the higher Chern classes of $\mathcal{W}$ generate the cohomology of the Grassmanian. Again, a more concrete proof of this fact may be found in [13].

On the other hand, there is a paucity of line bundles on the Grassmanian compared to the complete flag variety. Since $\Delta \setminus J$ is the single root $\epsilon_m - \epsilon_{m+1}$, the Picard group of $\mathbb{G}_{m,n}$ is generated by just one line bundle: the one associated to the corresponding fundamental weight of SL_{m+n}. This weight is

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{m+n} \end{pmatrix} \mapsto t_1 \cdots t_m,$$

so the corresponding line bundle is the top exterior power of $\mathcal{W}$.

4.4. **Intersections on G/P_J.** The Borel presentation implies that the intersections with divisors fully determine the multiplicative structure of the Chow ring on G/B^+, although not on G/P_J. In this section, we will prove Chevalley's formula [7] (Proposition 4.4.2), which evaluates the intersection product of any Schubert cycle with a cycle of codimension one. The first step is to classify the T-stable curves, partially following the exposition of [18].

For any root β, not necessarily simple, let G_β be the subgroup of G generated by U_β and $\dot{s}_\beta$. Equivalently, G_β is the image of the homomorphism $SL_2 \to G$ associated to β.

Lemma 4.4.1. *If $\beta \in \Phi^+$ is a root not in the linear span of J, and $w \in W^J$ is any element, then the natural image $C_{w,\beta}$ of*

$$\dot{w} G_\beta P_J$$

in G/P_J is a T-stable curve isomorphic to $\mathbb{P}^1$. Furthermore:

- *(i) Any closed, irreducible T-stable curve is of this form.*
- *(ii) The T-fixed points on $C_{w,\beta}$ are $e(w)$ and $e(ws_\beta)$.*
- *(iii) For each $v \in W^J$, $C_{w,\beta} \subset X_v^J$ if and only if $w, (ws_\beta)^J \leq v$.*
- *(iv) The cycle class of $C_{w,\beta}$ is:*

$$[C_\beta] = \sum_{\alpha \in \Delta - J} \langle \omega_\alpha, \beta \rangle \cdot [X_{s_\alpha}^J].$$

Proof. It is clear that $C_{w,\beta}$ is T-stable. Since $G_\beta \cap P_J$ is a Borel subgroup B_β of $G_\beta \simeq SL_2$, $C_{w,\beta}$ is isomorphic to $G_\beta / B_\beta \simeq \mathbb{P}^1$.

- (i) Suppose that C is a closed, irreducible T-stable curve. It possesses a T-fixed point, which by Lemma 3.1.3 is of the form $\dot{w} P_J$ for some $w \in W^J$. By Corollary 2.6.8, $\dot{w} B^- B^+ P_J$ is an open neighborhood of $\dot{w} P_J$. On the other hand,

$$B^- B^+ P_J = \left(\prod_{\alpha \in \Phi^- \setminus J^-} U_\alpha \right) P_J.$$

  A T-stable curve in this open affine subset must correspond to some U_α, as can be seen clearly on the Lie algebra level. Let $\beta = -\alpha$; then

$$\dot{w}^{-1} C \cap B^- B^+ P_J = U_{-\beta} P_J,$$

  so, since C is closed, it contains $C_{w,\beta}$, hence is equal to it.
- (ii) If $e(v) \in \dot{w} G_\beta P_J$, then $\dot{v}^{-1} \dot{w} \in G_\beta P_J$. If $w' = (v^{-1} w)^J$, it follows that $\dot{w}' \in G_\beta P_J = U_{-\beta} P_J \sqcup \dot{s}_\beta P_J$. The same argument as the one in Lemma 3.1.3 shows that $w' \in \{s_\beta, 1\}$.
- (iii) Suppose that $ws_\beta \leq w$. Then

$$\dot{w} U_{-\beta} P_J \subset B^+ \dot{w} P_J,$$

so we have $C_{w,\beta} \subset X_w^J$. It follows by Proposition 3.1.2 that $C_{w,\beta} \subset X_v^J$ for all $v \geq w$. On the other hand, if $C_{w,\beta} \subset X_v^J$, then $e(w) \in X_v^J$, which by Lemma 3.1.3 implies $w \leq v$. Similarly, if $w \leq ws_\beta$, then $C_{w,\beta} \subset X_{ws_\beta}^J$ because $\dot{w}\dot{s}_\beta U_{-\beta} P_J \subset B^+ \dot{w}\dot{s}_\beta P_J$, and the rest of the proof is the same.

(iv) By Proposition 3.3.1, it suffices to check that

$$[C_{w,\beta}].[X_{w_0 s_\alpha}^J] = \langle \omega_\alpha, \beta \rangle [pt].$$

Since cycle classes are translation-invariant, assume $w = 1$ and consider the inclusion $\iota : C_{1,\beta} \hookrightarrow G/P_J$. By the usual projection formula,

$$[X_{w_0 s_\alpha}^J].[C_{1,\beta}] = \iota_*(\iota^*[X_{w_0 s_\alpha}^J]).$$

Now, $[X_{w_0 s_\alpha}^J]$ is the characteristic class of the line bundle $L_J(-\omega_\alpha)$ by Proposition 4.2.2, and the restriction of $L_J(-\omega_\alpha)$ to $C_{1,\beta} \simeq \mathbb{P}^1$ is $\mathcal{O}(\langle \omega_\alpha, \beta \rangle)$. (Since $C_{1,\beta}$ can be naturally identified with the SL_2 flag variety G_β/B_β, this is just another application of Proposition 4.2.2.) Since $\iota_*([pt]) = [pt]$, this completes the proof.

□

We are now ready to prove Chevalley's intersection formula:

Proposition 4.4.2. *For any $w \in W^J$ and $\alpha \in \Delta \setminus J$, we have:*

$$[X_w^J].[X_{w_0 s_\alpha}^J] = \sum_{\substack{v = (ws_\beta)^J,\ \beta \in \Phi^+ \\ \ell(v) = \ell(w)-1}} \langle \omega_\alpha, \beta \rangle [X_v^J].$$

Proof. By Proposition 3.3.1, it suffices to consider the triple intersections

$$[X_w^J].[X_{w_0 s_\alpha}^J].[X_{w_0 v}^J],$$

where $v \in W^J$ and $\ell(v) = \ell(w) - 1$. We first compute $[X_w^J].[X_{w_0 v}^J]$, by considering the intersection $X_w^J \cap w_0 X_{w_0 v}^J$. If the intersection is nonempty, then it is T-stable, so it has a T-fixed point. By Lemma 3.1.3, such a point is of the form $e(y)$ with $y \in W^J$, $y \leq w$, and $(w_0 y)^J \leq (w_0 v)^J$; the latter inequality is equivalent to $v \leq y$ by Proposition 2.3.2(ii). Hence $v \leq w$, which implies $v = ws_\beta$ for some $\beta \in \Phi^+$. After an easy check of transversality and an application of Lemma 4.4.1, we therefore have, for all $v \in W^J$ such that $\ell(v) = \ell(w) - 1$:

$$[X_w^J].[X_{w_0 v}^J] = \begin{cases} [C_\beta] & \text{if } v = (ws_\beta)^J, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4.1(iv) now completes the proof. □

5. The Deodhar decomposition

In this section, we will discuss the work of Marsh and Rietsch [14], which builds on a result of Deodhar [9]. The set up is as before: G is a split, simply connected semisimple algebraic group over a field k; we invite the reader to recall the definitions in §2.1.1, which will be used extensively. Also, from now on, we restrict our consideration to the complete flag variety G/B^+.

5.1. **Relative position.** If $y_1 = g_1B^+$, $y_2 = g_2B^+ \in G/B^+$, then the double coset $B^+g_1^{-1}g_2B^+$ is well-defined and, according to the Bruhat decomposition, equal to $B^+\dot{w}B^+$ for a unique $w \in W$. Following [14], we call w the relative position of y_1, y_2 and write

$$y_1 \xrightarrow{w} y_2.$$

Proposition 5.1.1. *(i) If $y_1 \xrightarrow{w} y_2 \xrightarrow{s_\alpha} y_3$ for a simple reflection s_α, then either:*

$$y_1 \xrightarrow{ws_\alpha} y_3 \text{ or } y_1 \xrightarrow{w} y_3.$$

If the latter holds, then necessarily $\ell(ws_\alpha) < \ell(w)$.

(ii) If $y_1 \xrightarrow{w} y_2$, then $y_2 \xrightarrow{w^{-1}} y_1$.

Proof. (i) is a direct translation of Lemma 2.6.3, and (ii) is clear from definition. □

Proposition 5.1.2. *Suppose that* **w** *is a reduced expression for $w \in W$ with factors $s_{\alpha_1}, \dots, s_{\alpha_n}$. Then for all $y \in \Omega_w$, there is a unique sequence $y_0, \dots, y_n$ such that:*

$$e(1) = y_0 \xrightarrow{s_{\alpha_1}} y_1 \xrightarrow{s_{\alpha_2}} \cdots \xrightarrow{s_{\alpha_n}} y_n = y.$$

The map $\pi^w_{w_{(i)}} : \Omega_w \to \Omega_{w_{(i)}}$ given by $y \mapsto y_{(i)}$ is algebraic and defined on cosets by

$$b\dot{w}B^+ \mapsto b\dot{w}_{(i)}B^+,\ b \in B$$

Proof. For existence, we just need to check that $b\dot{w}B^+ \to b\dot{w}_{(i)}B^+$ is well-defined, or equivalently that $B^+ \cap \dot{w}B^+\dot{w}^{-1} \subset B^+ \cap \dot{w}_{(i)}B^+\dot{w}_{(i)}^{-1}$. This follows from Proposition 2.1.3(i).

For uniqueness, it suffices to show that if there are two such sequences y_i, y_i' then $y_{n-1} = y'_{n-1}$. By Proposition 5.1.1, we have $y_{n-1} \xrightarrow{s_{\alpha_n}} y_n \xrightarrow{s_{\alpha_n}} y'_{n-1}$, hence either $y_{n-1} = y'_{n-1}$ or $y_{n-1} \xrightarrow{s_{\alpha_n}} y'_{n-1}$. In the latter case, by Proposition 5.1.1 again we have $y_0 \xrightarrow{w} y'_{n-1}$, which contradicts the uniqueness of relative position. □

5.2. **Pinning G.** To work in coordinates on G, we establish some more notation. For each $\alpha \in \Delta$, fix an associated group homomorphism

$$\phi_\alpha : SL_2 \to G.$$

The choice of ϕ_α yields explicit one-parameter subgroups defined by

$$x_\alpha(t) = \phi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\quad y_\alpha(t) = \phi_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

considered as homomorphisms $\mathbb{G}_a \to G$ of algebraic groups over k. The images are the root subgroups U_α, $U_{-\alpha}$, respectively. Taken together, the data of T, Φ^+, and the maps ϕ_α, x_α, y_α are sometimes called a pinning of G. In our coordinates, the lifts of s_α guaranteed by Proposition 2.5.1 may be given by

$$\dot{s}_\alpha = \phi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

5.3. **Deodhar components.** Fix a reduced expression $\mathbf{w}$ for w. For each $y \in \Omega_w$, define a sequence $v_{(i)} \in W$ by the diagram of relative positions:

$$\begin{array}{ccccccccc}
e(1) = y_0 & \xrightarrow{s_{\alpha_1}} & y_1 & \xrightarrow{s_{\alpha_2}} & \cdots & \xrightarrow{s_{\alpha_{n-2}}} & y_{n-1} & \xrightarrow{s_{\alpha_n}} & y_n = y \\
{\scriptstyle w_0 v_{(0)}(y)}\Big\uparrow & & {\scriptstyle w_0 v_{(1)}(y)}\Big\uparrow & & & & {\scriptstyle w_0 v_{(n-1)}(y)}\Big\uparrow & & {\scriptstyle w_0 v_{(n)}(y)}\Big\uparrow \\
e(w_0) & & e(w_0) & & & & e(w_0) & & e(w_0)
\end{array}$$

Equivalently, if we define for any $v, v' \in W$ the cell

$$\mathcal{R}_{v,v'} = w_0 \Omega_{w_0 v} \cap \Omega_{v'} = B^+ e(v') \cap B^- e(v) \subset G/B^+, \tag{7}$$

then $v_{(i)}(y)$ is the unique element of W such that

$$y_i \in \mathcal{R}_{v_{(i)}(y), w_{(i)}}.$$

For any sequence $\mathbf{v} \in W^n$, define the Deodhar component:

$$\begin{aligned}
\mathcal{R}_{\mathbf{v},\mathbf{w}} &= \left\{ y \in \mathcal{R}_{v,w} \ : \ v_{(i)}(y) = v_{(i)} \right\} \\
&= \left\{ y \in \mathcal{R}_{v,w} \ : \ \pi^w_{w_{(i)}}(y) \in \mathcal{R}_{v_{(i)}, w_{(i)}} \right\}.
\end{aligned} \tag{8}$$

5.4. **Distinguished subexpressions.** By definition, we have

$$\Omega_w = \bigsqcup_{\mathbf{v}} \mathcal{R}_{\mathbf{v},\mathbf{w}}.$$

But we do not yet even know for which $\mathbf{v}$ the Deodhar component is nonempty! The key combinatorial condition is the following. If $\mathbf{v}$ is a subexpression of $\mathbf{w}$, then $\mathbf{v}$ is called **distinguished** if, for all $1 \le i \le n$, we have:

$$\ell(v_{(i-1)} s_{\alpha_i}) < \ell(v_{(i-1)}) \implies v_{(i)} = v_{(i-1)} s_{\alpha_i}.$$

Proposition 5.4.1. *If $\mathcal{R}_{\mathbf{v},\mathbf{w}} \neq \emptyset$, then $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.*

Proof. Suppose $y \in \mathcal{R}_{\mathbf{v},\mathbf{w}}$ and consider for each i the diagram of relative positions:

$$\begin{array}{ccc}
y_{i-1} & \xrightarrow{s_{\alpha_i}} & y_i \\
{\scriptstyle w_0 v_{(i-1)}} \nwarrow & & \nearrow {\scriptstyle w_0 v_{(i)}} \\
& e(w_0) &
\end{array}$$

Then by Proposition 5.1.1, we have either $w_0 v_{(i)} = w_0 v_{(i-1)} s_{\alpha_i}$ or $w_0 v_{(i)} = w_0 v_{(i-1)}$, and the latter occurs only if $\ell(w_0 v_{(i-1)} s_{\alpha_i}) < \ell(w_0 v_{(i-1)})$. Equivalently, either $v_{(i)} = v_{(i-1)} s_{\alpha_i}$ or $v_{(i)} = v_{(i-1)}$, and the latter occurs only if $\ell(v_{(i-1)} s_{\alpha_i}) > \ell(v_{(i-1)})$. Since $v_{(0)} = 1$, this is exactly the condition for $\mathbf{v}$ to be a distinguished subexpression of $\mathbf{w}$. □

5.5. **Recursively parameterizing Deodhar components.** Suppose that $\mathbf{w}$ is a reduced expression of length n for $w \in W$, and $\mathbf{v}$ is a distinguished subexpression. Let α be a simple root such that $\ell(ws_\alpha) = n+1$, and let $\mathbf{w}^\alpha$ be the reduced expression obtained by appending $w_{(n+1)} = ws_\alpha$ to $\mathbf{w}$. There are two natural extensions of $\mathbf{v}$ to a subexpression of $\mathbf{w}^\alpha$, obtained by appending either $v_{(n+1)} = v_{(n)} s_\alpha$ or $v_{(n+1)} = v_{(n)}$ to $\mathbf{v}$. Let us label them $\mathbf{v}^\alpha$ and $\mathbf{v}^\circ$, respectively. Then we have:

$$(\pi^{ws_\alpha}_w)^{-1}(\mathcal{R}_{\mathbf{v},\mathbf{w}}) = \mathcal{R}_{\mathbf{v}^\alpha,\mathbf{w}^\alpha} \sqcup \mathcal{R}_{\mathbf{v}^\circ,\mathbf{w}^\alpha}. \tag{9}$$

Note that, if $v_{(n)} \dot{s}_\alpha < v_{(n)}$, then $\mathcal{R}_{\mathbf{v}^\circ,\mathbf{w}^\alpha} = \emptyset$ by Proposition 5.4.1.

Lemma 5.5.1. *Suppose given a subset $G_0 \subset U^- \dot{v}_n$ such that the natural projection $G \to G/B^+$ induces an isomorphism*

$$G_0 \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\mathbf{w}}.$$

Then:

(i) If $\ell(v_{(n)} s_\alpha) < \ell(v_{(n)})$, then there is a commutative diagram:

$$\begin{array}{ccc} G_0 \times U_\alpha \dot{s}_\alpha & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v}^\alpha,\mathbf{w}^\alpha} \\ \big\downarrow{\scriptstyle pr_1} & & \big\downarrow{\scriptstyle \pi_w^{ws_\alpha}} \\ G_0 & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v},\mathbf{w}} \end{array}$$

where the top arrow is induced by multiplication and projection $G \to G/B^+$.

(ii) If $\ell(v_{(n)} s_\alpha) > \ell(v_{(n)})$, then there are commutative diagrams:

$$\begin{array}{ccc} G_0 \times U_{-\alpha}^* & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v}^\circ,\mathbf{w}^\alpha} \\ \big\downarrow{\scriptstyle pr_1} & & \big\downarrow{\scriptstyle \pi_w^{ws_\alpha}} \\ G_0 & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v},\mathbf{w}} \end{array} \qquad \begin{array}{ccc} G_0 \times \{\dot{s}_\alpha\} & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v}^\alpha,\mathbf{w}^\alpha} \\ \big\downarrow{\scriptstyle pr_1} & & \big\downarrow{\scriptstyle \pi_w^{ws_\alpha}} \\ G_0 & \xrightarrow{\sim} & \mathcal{R}_{\mathbf{v},\mathbf{w}} \end{array}$$

where the top arrows are again induced by multiplication and projection, and $U_{\pm\alpha}^* = U_{\pm\alpha} \setminus \{1\}$.

Proof. Our first claim is that the map $G_0 \times U_\alpha \dot{s}_\alpha \to G/B^+$ is an isomorphism onto $(\pi_w^{ws_\alpha})^{-1}(\mathcal{R}_{\mathbf{v},\mathbf{w}})$. Indeed, it has the correct image because $gB^+ \xrightarrow{s_\alpha} gus_\alpha B^+$ for all $g \in G$ and $u \in U^+$, and injectivity can be checked on the restriction to a single value of $g \in G_0$, where it is clear. By Proposition 5.4.1, (i) follows.

For (ii), suppose that $y = gu\dot{s}_\alpha \in G/B^+$, where $u \in U_\alpha$ and $g \in G_0$. If $u = 1$, then

$$y = g\dot{s}_\alpha B^+ \in U^- \dot{v}_{(n)} \dot{s}_\alpha B^+,$$

so by definition $y \in \mathcal{R}_{\mathbf{v}^\alpha,\mathbf{w}^\alpha}$.

Suppose now that $u \neq 1$. There is an isomorphism (of varieties) $\iota : U_\alpha^* \to U_{-\alpha}^*$ such that

$$u\dot{s}_\alpha B^+ = \iota(u) B^+.$$

This can be checked at the level of SL_2, where it is given by

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}.$$

Therefore the natural map $G_0 \times U_{-\alpha}^* \to G/B^+$ is an isomorphism onto the image of $G_0 \times U_\alpha^* \dot{s}_\alpha$. Since

$$G_0 U_{-\alpha}^* \subset U^- \dot{v}_n U_{-\alpha} \subset U^- \dot{v}_{(n)},$$

the image of $G_0 \times U_{-\alpha}^*$ lands in $\mathcal{R}_{\mathbf{v}^\circ,\mathbf{w}^\alpha}$, and (ii) follows. □

We can use this lemma to give an explicit parameterization of each Deodhar component.

Definition 5.5.2. For each expression $\mathbf{v}$ of length n, define index sets

$$J_{\mathbf{v}}^+ = \left\{ i \in \{1, \dots, n\} \,:\, v_{(i)} > v_{(i-1)} \right\},$$
$$J_{\mathbf{v}}^\circ = \left\{ i \in \{1, \dots, n\} \,:\, v_{(i)} = v_{(i-1)} \right\},$$
$$J_{\mathbf{v}}^- = \left\{ i \in \{1, \dots, n\} \,:\, v_{(i)} < v_{(i-1)} \right\}.$$

For every reduced expression $\mathbf{w}$ with factors $s_{\alpha_1}, \dots, s_{\alpha_n}$, and every distinguished subexpression $\mathbf{v}$, define a subvariety $G_{\mathbf{v},\mathbf{w}}$ of G to be the space of products

$$g = g_1 \cdots g_n, \quad g_i = \begin{cases} x_{\alpha_i}(m_i)\dot{s}_{\alpha_i}^{-1}, & \text{if } i \in J^-_{\mathbf{v}} \\ y_{\alpha_i}(t_i), \quad t_i \neq 0 & \text{if } i \in J^\circ_{\mathbf{v}} \\ \dot{s}_{\alpha_i} & \text{if } i \in J^+_{\mathbf{v}}, \end{cases}$$

where m_i and t_i are considered as parameters in $\mathbb{A}^1$ and $\mathbb{A}^1 - \{0\}$, respectively. Let

$$\rho_{\mathbf{v},\mathbf{w}} : \left(\mathbb{A}^1\right)^{J^-_{\mathbf{v}}} \times \left(\mathbb{A}^1 - \{0\}\right)^{J^\circ_{\mathbf{v}}} \to G_{\mathbf{v},\mathbf{w}}$$

be the map defined by the parameters m_i and t_i.

Proposition 5.5.3 ([14] Proposition 5.2)**.** *Let $\mathbf{w}$ be a reduced expression and $\mathbf{v}$ a distinguished subexpression. Then $\rho_{\mathbf{v},\mathbf{w}}$ is an isomorphism, and the natural map induces an isomorphism:*

$$G_{\mathbf{v},\mathbf{w}} \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\mathbf{w}}$$
$$g \mapsto gB^+.$$

Moreover, if $g = g_1 \dots g_n$ is the factorization witnessing $g \in G_{\mathbf{v},\mathbf{w}}$, the partial products satisfy $g_1 \dots g_i \in U^- \dot{v}_{(i)}$ and $\pi^w_{w_{(i)}}(g_1 \dots g_n B^+) = g_1 \dots g_i B^+$.

Proof. The strategy is to induct on n; the base case is trivial, and for the inductive step we must apply Lemma 5.5.1. So assume that the proposition holds for $\mathbf{v}, \mathbf{w}$ and consider a reduced expression of the form $\mathbf{w}^\alpha$ for a simple root α. By induction, we may apply the lemma to

$$G_0 = G_{\mathbf{v},\mathbf{w}} \subset U^- \dot{v}^n.$$

The only point requiring explanation is the substitution of $U_\alpha \dot{s}_\alpha^{-1}$ for $U_\alpha \dot{s}_\alpha$ in case (i). This is needed to ensure $G_{\mathbf{v}^\alpha,\mathbf{w}^\alpha} = G_{\mathbf{v},\mathbf{w}} U_\alpha \dot{s}_{\alpha_i}^{-1} \subset U^- \dot{v}_{(n+1)}$ because $\dot{v}_{(n+1)} = \dot{v}_{(n)} \dot{s}_\alpha^{-1}$; but it does no other damage because $\dot{s}_\alpha^2 \in T$. □

The following corollaries are originally due to Deodhar; less explicit proofs of them are contained in [9].

Corollary 5.5.4. *(i) Given a reduced expression $\mathbf{w}$ for w of length n, $\mathcal{R}_{\mathbf{v},\mathbf{w}} \neq \emptyset$ if and only if $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.*
(ii) If $\mathbf{v}$ is a distinguished subexpression for v, then

$$\dim(\mathcal{R}_{\mathbf{v},\mathbf{w}}) = \ell(w) - \ell(v) - |J^-_{\mathbf{v}}|,$$

and on the level of points there is an isomorphism

$$k^{J^-_{\mathbf{v}}} \times (k^*)^{J^\circ_{\mathbf{v}}} \cong \mathcal{R}_{\mathbf{v},\mathbf{w}}(k).$$

This corollary suggests that we should consider for which distinguished $\mathbf{v}$ the set $|J^-_{\mathbf{v}}|$ is a small is possible; recall that a subexpression $\mathbf{v}$ of $\mathbf{w}$ is **positive** if $v_{(i)} \geq v_{(i-1)}$ for all i.

Proposition 5.5.5 ([14], Lemma 3.5)**.** *Given a reduced expression $\mathbf{w}$ for w, for every $v \leq w$ there is a unique distinguished, positive subexpression $\mathbf{v}_+$ for v in $\mathbf{w}$.*

Proof. We proceed by induction on $\ell(w) = n$. If $\mathbf{v}$ is both positive and distinguished, then we must have $v_{(n-1)} s_{\alpha_n} > v_{(n-1)}$. So set $v_{(n-1)} = v_{(n)} s_{\alpha_n}$ if $v s_{\alpha_n} < v$ and $v_{(n-1)} = v$ otherwise. We then have $v_{(n-1)} \leq w_{(n-1)}$, so by induction there is a unique positive, distinguished subexpression for $v_{(n-1)}$ in the reduced expression $(w_{(0)}, \ldots, w_{(n-1)})$. Then the resulting sequence, with $v_{(n)} = v$ appended, is the unique positive, distinguished subexpression for $\mathbf{v}$ in $\mathbf{w}$. □

Corollary 5.5.6. *The cell $\mathcal{R}_{v,w}$ is irreducible of dimension $\ell(w)-\ell(v)$, and $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}(k)$ is Zariski dense in $\mathcal{R}_{v,w}(\overline{k})$. If $k = \mathbb{R}$ or $\mathbb{C}$, then $\mathcal{R}_{\mathbf{v}_+,\mathbf{w}}(k)$ is also dense in $\mathcal{R}_{v,w}(k)$ for the analytic topology.*

Proof. In light of Corollary 5.5.4 and Proposition 5.5.5, we need only observe that k is Zariski dense in $\overline{k}$ for any characteristic zero field k. □

5.6. **Positivity.** Suppose for this subsection that $k = \mathbb{R}$. We give a brief review of the notion of positivity following [14, §11] as well as [12]. In 1930, Schoenberg [16] defined a matrix to be totally ≥ 0 if all of its minors are ≥ 0; this definition was found to have powerful combinatorial applications.

For any reductive group G with a pinning as above, we may define

$$U^+_{\alpha,\geq 0} = x_\alpha(\mathbb{R}_{\geq 0}) = \phi_\alpha \begin{pmatrix} 1 & \mathbb{R}_{\geq 0} \\ 0 & 1 \end{pmatrix}, \quad U^-_{\alpha,\geq 0} = y_\alpha(\mathbb{R}_{\geq 0}) = \phi_\alpha \begin{pmatrix} 1 & 0 \\ \mathbb{R}_{\geq 0} & 1 \end{pmatrix}$$

for each simple root α. Likewise, let

$$T_{\alpha,>0} = \left\{ \phi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \,:\, t \in \mathbb{R}_{\geq 0} \right\}.$$

If we denote by $U^{\pm}_{\geq 0}$ the submonoid of G generated by the $U^{\pm}_{\alpha,\geq 0}$ for all simple roots α, and by $T_{>0}$ the submonoid of G generated by the $T_{\alpha,>0}$, then the totally nonnegative part of G is defined as:

$$G_{\geq 0} = U^-_{\geq 0} T_{>0} U^+_{\geq 0}.$$

This is again a monoid, and a classical theorem due to Whitney [20] shows that, when $G = GL_n$, then $G_{\geq 0}$ consists exactly of the positive matrices in the sense of Schoenberg.

To return to flag varieties, the totally nonnegative part of G/B^+ is by definition

$$(G/B^+)_{\geq 0} = \overline{\left\{ uB^+ \,:\, u \in U^-_{\geq 0} \right\}}, \tag{10}$$

where the closure is in the analytic topology. By [12, Proposition 8.12], $G_{\geq 0}$ preserves $(G/B^+)_{\geq 0}$ under the group action. However, it is **not** true that $(G/B^+)_{\geq 0}$ is the image of $G_{\geq 0}$ under the projection to G/B^+. In other words, taking the closure in the flag variety is a truly nontrivial operation.

For each $v \leq w$, the nonnegative part of the cell $\mathcal{R}_{v,w}$ is defined as:

$$\mathcal{R}^{>0}_{v,w} = \mathcal{R}_{v,w} \cap (G/B^+)_{\geq 0}. \tag{11}$$

Lusztig conjectured [12] that each $\mathcal{R}^{>0}_{v,w}$ is a semi-algebraic cell, i.e. isomorphic to $\mathbb{R}^{\ell(w)-\ell(v)}_{>0}$. The following theorem resolves this conjecture in a completely explicit way.

Theorem 5.6.1 ([14], Theorem 11.3)**.** *Let $\mathbf{w}$ be a reduced expression for $w \in W$ with factors $s_{\alpha_1}, \ldots, s_{\alpha_n}$, and suppose $v \leq w$. Let $\mathbf{v}_+$ be the unique positive subexpression for v in $\mathbf{w}$. Then $\mathcal{R}^{>0}_{v,w} \subset \mathcal{R}_{\mathbf{v}_+,\mathbf{w}}$, and, if $G^{>0}_{\mathbf{v}_+,\mathbf{w}} \subset G_{\mathbf{v}_+,\mathbf{w}}(\mathbb{R})$ is the pre-image of $\mathcal{R}^{>0}_{v,w}$, then $\rho_{\mathbf{v}_+,\mathbf{w}}$ induces an isomorphism*

$$(\mathbb{R}_{>0})^{\ell(w)-\ell(v)} \xrightarrow{\sim} G^{>0}_{\mathbf{v}_+,\mathbf{w}}.$$

Remark 5.6.2. Since $G_{\mathbf{v}_+,\mathbf{w}}(\mathbb{R})$ is the set of products

$$g = g_1 \cdots g_n, \quad g_i = \begin{cases} y_{\alpha_i}(t_i),\ t_i \in \mathbb{R}^*, & \text{if } i \in J^\circ_{\mathbf{v}_+} \\ \dot{s}_{\alpha_i}, & \text{if } i \in J^+_{\mathbf{v}_+}, \end{cases}$$

the theorem says that the positive part of $\mathcal{R}_{v,w}$ is exactly the image of the subset of $G_{\mathbf{v}_+,\mathbf{w}}$ where all the parameters t_i are positive.

Note that $G^{>0}_{\mathbf{v}_+,\mathbf{w}} \not\subset G_{\geq 0}$ in general; but if $v = 1$, then the inclusion clearly holds.

In the rest of this section, we describe some of the tools used in the proof.

5.7. **Minors.** Marsh and Rietsch use a method they call the "generalized chamber Ansatz" to construct an inverse to the isomorphisms

$$(12) \qquad \left(\mathbb{A}^1\right)^{J^-_{\mathbf{v}}} \times \left(\mathbb{A}^1 - \{0\}\right)^{J^\circ_{\mathbf{v}}} \xrightarrow[\rho_{\mathbf{v},\mathbf{w}}]{\sim} G_{\mathbf{v},\mathbf{w}} \xrightarrow{\sim} \mathcal{R}_{\mathbf{v},\mathbf{w}}.$$

This is a key step in the proof of Theorem 5.6.1, because it allows one to extract the parameters t_i, m_i using only information about a point in G/B^+. Perhaps unsurprisingly, doing so requires a generalization of matrix minors from SL_n to our group G.

Definition 5.7.1 ([14], Definition 6.2)**.** Let λ be a dominant weight and V_λ the corresponding highest weight representation with highest weight vector ξ_λ. For $w, w' \in W$, define

$$\Delta^{w\lambda}_{w'\lambda} : G \to \mathbb{A}^1$$

to be the map sending g to the $\dot{w}\xi_\lambda$-component of $g\dot{w}'\xi_\lambda$.

Actually, one should check that this is well-defined, i.e. depends only on $w\lambda$, $w'\lambda$, and g. First of all, λ can be recovered from $w\lambda$ because any weight is in the Weyl group orbit of a unique dominant weight; and $\dot{w}\xi_\lambda$ can be recovered from $w\lambda$ and λ since, if s_α stabilizes λ, then $\dot{s}_\alpha \in U_\alpha U_{-\alpha} U_\alpha$ stabilizes ξ_λ [14, Lemma 6.1].

As an example, one may check directly that if $G = SL_n$, and $\lambda = \omega_i$ is a fundamental weight, the $\Delta^{w\lambda}_{w'\lambda}$ compute exactly the minors of g.

Remark 5.7.2. Suppose $z\dot{w}B^+ \in \mathcal{R}_{\mathbf{v},\mathbf{w}}$, where $\mathbf{w}$ is a reduced expression for w and $z \in U^+$; we claim

$$\Delta^{w_{(i)}\lambda}_{v_{(i)}\lambda}(z) \neq 0$$

for all indices $0 \leq i \leq \ell(w)$ and any dominant weight λ. Indeed, by definition $z\dot{w}_{(i)}B^+ \in B^- v_{(i)} B^+$, so, for any highest weight λ, the vector $z\dot{w}_{(i)}\xi_\lambda$ has nonzero component in the $v_{(i)}\lambda$ weight space.

The explicit inverse to (12) is:

Theorem 5.7.3 ([14], Theorem 7.1)**.** *With notation as above, suppose that $z \in U^+$ and $z\dot{w}B^+ \in \mathcal{R}_{\mathbf{v},\mathbf{w}}(k)$. Then, by Proposition 5.5.3, $z\dot{w}B^+ = gB^+$ for some $g = g_1 \cdots g_n$, where*

$$g_i = \begin{cases} x_{\alpha_i}(m_i)\dot{s}_{\alpha_i}^{-1}, & \text{if } i \in J^-_{\mathbf{v}} \\ y_{\alpha_i}(t_i), \quad t_i \neq 0 & \text{if } i \in J^\circ_{\mathbf{v}} \\ \dot{s}_{\alpha_i} & \text{if } i \in J^+_{\mathbf{v}}, \end{cases}$$

and t_i, m_i are considered as parameters in k. We have:

(i) For each $i \in J^{\circ}_{\mathbf{v}}$,

$$t_i = \frac{\prod_{\beta \neq \alpha_i} \Delta^{v_{(i)}\omega_\beta}_{w_{(i)}\omega_\beta}(z)^{-\langle \alpha_i, \beta^\vee \rangle}}{\Delta^{v_{(i)}\omega_{\alpha_i}}_{w_{(i)}\omega_{\alpha_i}}(z)\Delta^{v_{(i-1)}\omega_{\alpha_i}}_{w_{(i-1)}\omega_{\alpha_i}}(z)},$$

where in the numerator β ranges over simple roots, ω_β is the corresponding fundamental weight, and $\beta^\vee$ is the coroot.

(ii) For each $i \in J^{-}_{\mathbf{v}}$ and with the same conventions,

$$m_i = \frac{\Delta^{v_{(i-1)}\omega_{\alpha_i}}_{w_{(i)}\omega_{\alpha_i}}(z)\Delta^{v_{(i-1)}\omega_{\alpha_i}}_{w_{(i-1)}\omega_{\alpha_i}}(z)}{\prod_{\beta \neq \alpha_i} \Delta^{v_{(i)}\omega_\beta}_{w_{(i)}\omega_\beta}(z)^{-\langle \alpha_i, \beta^\vee \rangle}} - \Delta^{v_{(i-1)}\omega_{\alpha i}}_{s_{\alpha_i}\omega_{\alpha_i}}(g_1 \cdots g_{i-1}).$$

The proof is somewhat involved, but we will give some indications, starting with a simple lemma.

Lemma 5.7.4 ([14], Lemma 7.3(1)). *With notation as above, suppose λ is a dominant weight. Then:*

$$\Delta^{v_{(i)}\lambda}_{w_{(i)}\lambda}(z) = \frac{1}{\Delta^{w_{(i)}}_{\lambda}(g_1 \cdots g_i)}.$$

Proof. By Proposition 5.5.3,

$$g_1 \cdots g_i B^+ = \pi^w_{w_{(i)}}(gB^+) = \pi^w_{w_{(i)}}(z\dot{w}B^+) = z\dot{w}_{(i)}B^+.$$

So in the highest weight representation V_λ, the lines spanned by $z\dot{w}_{(i)}\xi_\lambda$ and $g_1 \cdots g_i \xi_\lambda$ agree. It follows that:

$$\frac{\Delta^{v_{(i)}\lambda}_{w_{(i)}\lambda}(z)}{\Delta^{v_{(i)}\lambda}_{\lambda}(g_1 \cdots g_i)} = \frac{\Delta^{w_{(i)}\lambda}_{w_{(i)}\lambda}(z)}{\Delta^{w_{(i)}\lambda}_{\lambda}(g_1 \cdots g_i)}.$$

Since $z \in U^+$ and $g_1 \cdots g_i \in U^- \dot{v}_{(i)}$ by Proposition 5.5.3, we have

$$\Delta^{v_{(i)}\lambda}_{\lambda}(g_1 \cdots g_i) = \Delta^{w_{(i)}\lambda}_{w_{(i)}\lambda}(z) = 1,$$

and the lemma follows. □

Similar formulas can be proven for the other minors appearing in Theorem 5.7.3. The upshot is that all of $\Delta^{v_{(i)}\omega_\beta}_{w_{(i)}\omega_\beta}(z)$, $\Delta^{v_{(i-1)}\omega_{\alpha_i}}_{w_{(i-1)}\omega_{\alpha_i}}(z)$, $\Delta^{v_{(i-1)}\omega_{\alpha_i}}_{w_{(i-1)}\omega_{\alpha_i}}(z)$, etc. can ultimately be written in terms of the partial products $g_1 \cdots g_i$. These minors can then be calculated directly in terms of the parameters t_i, m_i, and the theorem is reduced to a difficult but tractable exercise in algebra. Notice that Proposition 5.5.3 is an essential input, not just motivation: one needs to know that the parameters t_i, m_i exist in order to compute them. This is the origin of the name "ansatz."

The link between Theorem 5.7.3 and positivity is the following fact, which recalls the classical definition of positivity:

Lemma 5.7.5 ([14], Lemma 11.4). *If $k = \mathbb{R}$ and $z\dot{w}B^+ \in (G/B^+)_{\geq 0}$, where $z \in U^+$, then*

$$\Delta^{v\lambda}_{w\lambda}(z) \geq 0$$

for all $v \in W$ and all dominant weights λ.

The key point in the proof of Theorem 5.6.1 is the following. One shows, using the formulas of Theorem 5.7.3 and some related identities, that if $z\dot{w}B^+ = gB^+$ for any $g \in G_{\mathbf{v},\mathbf{w}}$ with $\mathbf{v}_+ \neq \mathbf{v}$ (or any $g \in G_{\mathbf{v}_+,\mathbf{w}}$ with a negative parameter t_i), then z must have some negative minor.

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