MODULARITY OF d-ELLIPTIC LOCI WITH LEVEL STRUCTURE

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Abstract. We consider the generating series of special cycles on $\mathcal{A}_1(N) \times \mathcal{A}_g(N)$, with full level N structure, valued in the cohomology of degree $2g$. The modularity theorem of Kudla-Millson for locally symmetric spaces implies that these series are modular. When $N = 1$, the images of these loci in $\mathcal{A}_g$ are the d-elliptic Noether-Lefschetz loci, which are conjectured to be modular. In the appendix, it is shown that the resulting modular forms are nonzero for $g = 2$ when $N \geq 11$ and $N \neq 12$.

1. Introduction

1.1. ***d*-elliptic loci.** For integers $d, g \geq 1$, let $\widetilde{\mathsf{NL}}_{g,d}$ be the moduli space of morphisms of abelian varieties $h : E \to A$, where E is an elliptic curve, A is a principally polarized abelian variety (PPAV) of dimension g with polarization Θ_A, and $\deg(h^*\Theta_A) = d$. Let $[\widetilde{\mathsf{NL}}_{g,d}] \in \mathrm{CH}^{g-1}(\mathcal{A}_g)$ be the Noether-Lefschetz cycle class associated to the forgetful morphism $\epsilon : \widetilde{\mathsf{NL}}_{g,d} \to \mathcal{A}_g$. (We work throughout with Chow and cohomology groups with rational coefficients.) Set also $[\widetilde{\mathsf{NL}}_{g,0}] = \frac{1}{12}(-1)^g \lambda_{g-1} \in \mathrm{CH}^{g-1}(\mathcal{A}_g)$, see §3.1. These classes have attracted much recent attention; see [17] for a survey. The purpose of this paper is to advance the following conjecture.

Conjecture 1. *The generating series*

$$\sum_{d \geq 0} [\widetilde{\mathsf{NL}}_{g,d}] q^d \in \mathrm{CH}^{g-1}(\mathcal{A}_g) \otimes \mathbb{Q}[[q]]$$

is a cycle-valued modular form of weight $2g$.

Iribar López [9] shows that Conjecture 1 is true upon projection to the tautological ring $R^{g-1}(\mathcal{A}_g)$, in the sense of [2]. Further plausibility checks for Conjecture 1 are afforded by pulling back by the Torelli map $\mathsf{Tor} : \mathcal{M}_g^{\mathrm{ct}} \to \mathcal{A}_g$. It is proven in [7] that the pullbacks $\mathsf{Tor}^![\widetilde{\mathsf{NL}}_{g,d}]$ coincide with the Gromov-Witten virtual classes of the loci of curves $[C] \in \mathcal{M}_g^{\mathrm{ct}}$ admitting a stable map $f : C \to E$ of degree d to *some* genus 1 curve E. In particular, Conjecture 1 would imply that the generating series

$$\sum_{d \geq 0} [\mathcal{M}_{g,1}^{\mathrm{ct};q}(\mathcal{E}, d)]^{\mathrm{vir}} q^d \in \mathrm{CH}^{g-1}(\mathcal{M}_g^{\mathrm{ct}}) \otimes \mathbb{Q}[[q]]$$

is a cycle-valued modular form of weight $2g$, where $\mathcal{M}_{g,1}^{\mathrm{ct};q}(\mathcal{E}, d)$ is the global moduli space of pointed stable maps $f : (C, q) \to (E, p)$ from a compact type curve of genus g to a varying elliptic curve. In some cases, the Torelli pullback $\mathsf{Tor}^![\widetilde{\mathsf{NL}}_{g,d}]$ may be understood more explicitly, see [3].

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The Gromov-Witten classes of stable curves admitting a cover of a *fixed* elliptic curve are shown to be *quasi*-modular in [16]. On the other hand, the closely related (but different) loci of curves $[C] \in \overline{\mathcal{M}}_g$ admitting an *admissible cover* of some elliptic curve $f : C \to E$ of degree d are conjectured to be quasi-modular in [11], and shown to be so when $g \leq 3$. It is furthermore shown in [12] that a certain obstruction to the admissible cover loci being tautological is modular in d, for any g.

We prove the following result in this paper.

Theorem 1.1. *The generating series*

$$\sum_{d \geq 0} [\widetilde{\mathsf{NL}}_{g,d}]^+ q^d \in H^{2g}(\mathcal{A}_1 \times \mathcal{A}_g) \otimes \mathbb{Q}[[q]]$$

is a cycle-valued modular form of weight $2g$.

Here, we consider cycle classes of the maps $\widetilde{\mathsf{NL}}_{g,d} \to \mathcal{A}_1 \times \mathcal{A}_g$, remembering both E and A, rather than only $\widetilde{\mathsf{NL}}_{g,d} \to \mathcal{A}_g$. We use the notation $[\widetilde{\mathsf{NL}}_{g,d}]^+$ to distinguish from the classes $[\widetilde{\mathsf{NL}}_{g,d}]$ on $\mathcal{A}_g$. We also set $[\widetilde{\mathsf{NL}}_{g,0}]^+ = 0$. Theorem 1.1 does not imply Conjecture 1 (even in cohomology), because there is no proper pushforward map $H^{2g}(\mathcal{A}_1 \times \mathcal{A}_g) \to H^{2(g-1)}(\mathcal{A}_g)$. In fact, as stated, Theorem 1.1 is trivial: as we prove in Proposition 3.4, the classes $[\widetilde{\mathsf{NL}}_{g,d}]^+$ are zero in $H^{2g}(\mathcal{A}_1 \times \mathcal{A}_g)$, and are moreover zero in $CH^g(\mathcal{A}_g \times \mathcal{A}_1)$ (Remark 3.5)!

To obtain a non-trivial statement, we add level structure to the moduli problem. Our main result is the following.

Theorem 1.2. *Fix an integer* $N \geq 1$ *and a symplectic group homomorphism* $b : (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^{2g}$. *Let* $\widetilde{\mathsf{NL}}^b_{g,d}(N)$ *be the moduli space of morphisms* $h : E \to A$ *as before, where in addition* E, A *are endowed with full level-*N *structure and the map induced on* N*-torsion by* h *is given by* b.

Then, the generating series

$$\sum_{d \geq 0} [\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+ q^d \in H^{2g}(\mathcal{A}_1(N) \times \mathcal{A}_g(N)) \otimes \mathbb{Q}[[q]]$$

is a cycle-valued modular form of weight $2g$ *and level* N.

In contrast to the situation of Theorem 1.1, the classes $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+$ are proven not all to vanish when $g = 2$ and $N = 11$ and $N \geq 13$ in the appendix.

We will see in Proposition 3.4 that the classes $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+$ are supported in the odd Künneth component $H^1(\mathcal{A}_1(N)) \otimes H^{2g-1}(\mathcal{A}_g(N))$. Thus, Theorem 1.2 witnesses modularity of Noether-Lefschetz cycles in a different part of cohomology as Iribar López's result, which lives in the tautological (hence even) part.

Theorem 1.2 is proven by expressing the cycles $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+$ as pullbacks of special cycles from certain (non-algebraic) symmetric spaces, which we discuss in the next section.

1.2. **Symmetric spaces.** Let (Λ, ω), resp. (Λ', ω') be $\mathbb{Z}^2$, resp. $\mathbb{Z}^{2g}$, equipped with the standard symplectic forms. The tensor product $L = \Lambda \otimes \Lambda'$ has a natural symmetric bilinear pairing γ given by

$$\gamma(v_1 \otimes v_1', v_2 \otimes v_2') := \omega(v_1, v_2)\, \omega'(v_1', v_2').$$

As an integral lattice, we have $L \simeq U^{\oplus 2g}$, where U is the hyperbolic plane lattice with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $V = L \otimes \mathbb{R}$, and let $\Gamma_L \subset SO(V)$ be the subgroup of integral isometries of L that act trivially on $L^\vee/V$. If $K_\infty \subset SO(V)$ is a maximal compact subgroup, then the double quotient

$$\mathrm{km}(L) := \Gamma_L \backslash SO(V)/K_\infty$$

is an example of a non-compact locally symmetric space studied by Kudla-Millson in [10]. In particular, $\mathrm{km}(L)$ contains a countable collection of totally geodesic cycles C_d indexed by positive integers d. Since the lattice L has signature $(2g, 2g)$, these are cycles of real codimension $2g$. The main theorem of [10] implies that their Poincaré duals $[C_d] \in H^{2g}(\mathrm{km}(L), \mathbb{Q})$ are the Fourier coefficients of a classical modular form of weight $2g$, level 1:

Theorem 1.3 (Kudla-Millson [10]). *The generating series*

$$\Phi(q) = \sum_{d \geq 0} [C_d] q^d \in \mathrm{Mod}(2g, \mathrm{SL}_2(\mathbb{Z})) \otimes H^{2g}(\mathrm{km}(L)).$$

By convention, we set $[C_0] := e(\Lambda_g^\vee) \in H^{2g}(\mathrm{km}(L))$, where Λ_g is the tautological real vector bundle of rank $2g$ on $\mathrm{km}(L)$. Recall that we work throughout with rational coefficients.

For $g > 1$, $\mathrm{km}(L)$ is not an algebraic variety, but by the tensor product construction above, it receives a map

$$\phi : \mathcal{A}_1 \times \mathcal{A}_g \to \mathrm{km}(L).$$

defined in §5. We show in Proposition 5.3 that the classes $[C_d] \in H^{2g}(\mathrm{km}(L), \mathbb{Q})$ pull back under ϕ to the Noether-Lefschetz cycles $[\widetilde{\mathsf{NL}}_{g,d}]^+ \in H^{2g}(\mathcal{A}_1 \times \mathcal{A}_g)$, which implies Theorem 1.1.

However, as we have already mentioned, we prove in Proposition 3.4 that in fact the classes appearing in Theorem 1.1 are all zero. To obtain a non-trivial q-series, we need to add level structure to the moduli spaces involved.

Let $N \geq 1$ be an integer, let $\mathcal{A}_1(N)$ and $\mathcal{A}_g(N)$ are the moduli spaces of elliptic curves and PPAVs with full level-N structure. As in Theorem 1.2, let $\widetilde{\mathsf{NL}}^b_{g,d}(N)$ be the moduli space of maps $f : E \to A$ of degree d whose induced map on N-torsion is given by a specified matrix b. Let $\mathrm{km}(L(N))$ be the Kudla-Millson space defined by replacing Γ_L in the definition of $\mathrm{km}(L(N))$ with the subgroup of isometries that reduce to the identity mod N. We have an enhanced map

$$\phi(N) : \mathcal{A}_1(N) \times \mathcal{A}_g(N) \to \mathrm{km}(L(N)).$$

Then, by the theorem of Kudla-Millson [10], we have a generating series

$$\Phi^b(q) := \sum_{d \geq 0} [C^b_d(N)] q^d \in \mathrm{Mod}(2g, \Gamma(N)) \otimes H^{2g}(\mathrm{km}(L(N))$$

lifting $\Phi(q)$. The d-th Fourier coefficient of $\Phi(q)$ pulls back to the Noether-Lefschetz cycle $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+ \in H^{2g}(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$. The difference here is that the modular curve $\mathcal{A}_1(N)$ has non-trivial first cohomology group. We restate Theorem 1.2 as follows.

Theorem 1.4. *The pullback*

$$\phi(N)^*\Phi^b(q) = \sum_{d\geq 0}[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+ q^d$$

is a modular form of weight $2g$*, level* N*, valued in* $H^{2g}(\mathcal{A}_1(N)\times\mathcal{A}_g(N))$ *with support in the odd Kunneth component*

$$H^1(\mathcal{A}_1(N))\otimes H^{2g-1}(\mathcal{A}_g(N)).$$

1.3. **Further directions.** In order to gain access to a proper pushforward map relating classes on $\mathcal{A}_1\times\mathcal{A}_g$ (possibly with level structure added) to those on $\mathcal{A}_g$, one needs to add cusps to $\mathcal{A}_1$. We take here $N=1$ for ease of notation, so that $\mathcal{A}_1^* = \overline{\mathcal{M}}_{1,1}$. Then, the natural map $\widetilde{\mathsf{NL}}_{g,d}\to\mathcal{A}_1^*\times\mathcal{A}_g$ remains proper, because a PPAV contains no rational curves. Thus, we may consider the class $[\widetilde{\mathsf{NL}}_{g,d}]^+\in \mathrm{CH}^g(\mathcal{A}_1^*\times\mathcal{A}_g)$, which pushes forward to the class $[\widetilde{\mathsf{NL}}_{g,d}]\in\mathrm{CH}^{g-1}(\mathcal{A}_g)$ appearing in Conjecture 1. We refine the conjecture as follows:

Conjecture 2. *The classes of the compactified d-elliptic cycles* $[\widetilde{\mathsf{NL}}_{g,d}]^+\in CH^g(\mathcal{A}_1^*\times\mathcal{A}_g)$ *are the Fourier coefficients of a modular form of weight* $2g$*, level 1.*

After pushforward, Iribar López's tautological projection calculation [9] shows that the classes $[\widetilde{\mathsf{NL}}_{g,d}]\in\mathrm{CH}^{g-1}(\mathcal{A}_g)$ are non-zero in general, in contrast to the situation on $\mathcal{A}_1\times\mathcal{A}_g$. It follows that the classes $[\widetilde{\mathsf{NL}}_{g,d}]^+\in\mathrm{CH}^g(\mathcal{A}_1^*\times\mathcal{A}_g)$ are non-zero, and because the tautological subspace of $\mathrm{CH}^{g-1}(\mathcal{A}_g)$ maps injectively to $H^{2(g-1)}(\mathcal{A}_g)$, also that the classes $[\widetilde{\mathsf{NL}}_{g,d}]^+\in H^{2g}(\mathcal{A}_1^*\times\mathcal{A}_g)$ are non-zero.

It is natural to expect a passage from modularity to *quasi*-modularity upon extending the cycles $\widetilde{\mathsf{NL}}_{g,d}$ to compactifications of $\mathcal{A}_g$, consistent with results [4, 6] establishing such phenomena at the boundary of orthogonal type Shimura varieties. This is also consistent with the calculations in [11, 16] finding cycled-valued quasi-modular forms on $\overline{\mathcal{M}}_g$.

Note that Conjectures 1 and 2 are formulated with values in the Chow group, following [17, 9]. The methods we employ here are more likely to prove the cohomological version, since the space $\mathrm{km}(L)$ is non-algebraic. One can also formulate both of these conjectures with level structure in the obvious way.

Extending the results of [7] to take level structure into account, the classes $[\mathsf{NL}^b_{g,d}(N)]^+\in H^{2g}(\mathcal{A}_1(N)\times\mathcal{A}_g(N))$ pull back under the pointed Torelli map $\mathsf{Tor}_1(N):\mathcal{M}^{\mathrm{ct}}_{g,1}(N)\to\mathcal{A}_g(N)$ to the virtual (in the sense of Gromov-Witten theory) loci of curves C admitting a cover $f:C\to E$ inducing the map b on N-torsion. After capping with the ψ class on $\mathcal{M}^{\mathrm{ct}}_{g,1}(N)$ and pushing forward to $\mathcal{M}^{\mathrm{ct}}_g(N)$, we obtain a modular series of classes in $H^{2g}(\mathcal{A}_1(N)\times\mathcal{M}^{ct}_g(N),\mathbb{Q})$, again supported on the odd Kunneth component

$$\mathrm{Mod}(2g,\Gamma(N))\otimes H^1(\mathcal{A}_1(N))\otimes H^{2g-1}(\mathcal{M}^{ct}_g(N)).$$

Dually, we obtain a map $\mathrm{Mod}(2g,\Gamma(N))^*\otimes H_1(\mathcal{A}_1(N))\to H^{2g-1}(\mathcal{M}^{\mathrm{ct}}_g(N))$.

Question 1. *What is the image of this map?*

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2. Lattices and theta functions

We follow the exposition in [14] for the definitions of metaplectic groups and vector-valued modular forms. Let $(L, (,))$ be a positive definite, even integral lattice of rank r. The Poisson summation formula implies that

$$\Theta_L(q) := \sum_{x \in L^\vee} q^{\frac{1}{2}(x,x)} e_{[x]} \in \mathrm{Mod}(r/2, \mathrm{Mp}_2(\mathbb{Z}), \mathbb{Q}[L^\vee/L]),$$

where $\mathrm{Mp}_2(\mathbb{Z})$ is the integer metaplectic group, acting on $\mathbb{Q}[L^\vee/L]$ via the Weil representation. This representation factors through a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, where N is the smallest positive integer such that $N \cdot (x,x) \in 2\mathbb{Z}$ for all $x \in L^\vee$. In particular, for any fixed coset $\delta \in L^\vee/L$, we have:

$$\Theta_{L,\delta}(q) := \sum_{x \in \delta} q^{\frac{1}{2}(x,x)} \in \mathrm{Mod}(r/2, \Gamma(N)).$$

When $r \in 2\mathbb{Z}$, this gives a classical modular form of weight $r/2$, level $\Gamma(N)$.

The cohomological theta correspondence of Kudla-Millson allows us to reformulate this story in the setting of the locally symmetric space associated to an indefinite lattice. Let (L, γ) be an indefinite, even integral lattice of signature (r_+, r_-) and rank r, and set $V = L \otimes \mathbb{R}$. Let $\Gamma_L \subset SO(V)$ be the subgroup of integral isometries acting trivially on $L^\vee/L$, and let $K_\infty \subset SO(V)$ be a maximal compact subgroup, isomorphic to $SO(r_+) \times SO(r_-)$.

Definition 2.1. Let $\mathrm{km}(L) = \Gamma_L \backslash SO(V)/K_\infty$. Note that the symmetric space $SO(V)/K_\infty$ is naturally identified with $\mathrm{Gr}^-(r_-, V)$, the space of oriented negative definite real r_--planes in V.

In the symmetric space $SO(V)/K_\infty$, we have an infinite arrangement of totally geodesic submanifolds indexed by dual lattice vectors $v \in V^\vee$ with $\gamma(v,v) > 0$.

Definition 2.2. Let $C_v \subset \mathrm{Gr}^-(r_-, V)$ be the set of negative definite r_--planes that are orthogonal to v. It is a symmetric subspace isomorphic to $\mathrm{Gr}^-(r_-, v^\perp \otimes \mathbb{R})$.

For each integer $d \geq 0$, and $\delta \in V^\vee/V$, the arithmetic subgroup Γ_L acts on the set $\{v \in V^\vee : \gamma(v,v) = 2d, [v] = \delta\}$ with finitely many orbits; see Lemma 4.1. This allows one to define finite type cycles in the quotient $\mathrm{km}(L)$.

Definition 2.3. For each $d > 0$ and $\delta \in V^\vee/V$, let $C_{d,\delta} \subset \mathrm{km}(L)$ be the image of

$$\bigcup_{\substack{\gamma(v,v)=d \\ [v]=\delta}} C_v \subset \mathrm{Gr}^-(r_-, V)$$

under the quotient map to $\Gamma_L \backslash \mathrm{Gr}^-(r_-, V)$. We define smooth uniformizations of $C_{d,\delta}$ using a finite set of Γ_L-orbit representatives $v_1, \ldots, v_{m(d)}$ among the dual lattice vectors of norm d and class δ:

$$\widetilde{C}_d := \bigsqcup_{i=1}^{m(d)} C_{v_i}$$

Theorem 2.4. [10] *For each $\delta \in L^\vee/L$, the power series*

$$\Phi_{L,\delta}(q) := e_0 + \sum_{d>0} [C_{d,\delta}] q^d \in \mathrm{Mod}(r/2, \Gamma(N)) \otimes H^{r_-}(\mathrm{km}(L), \mathbb{Q})$$

is a modular form, where e_0 is the Euler class of the dual tautological bundle of r_--planes.

In this paper, we specialize to the case where $L = U^{\oplus 2g}$, where U is the hyperbolic plane lattice. This lattice is unimodular, so $L^\vee/L = \{0\}$. More generally, we will consider $L(N)$, for some level $N > 0$, which is the lattice L with the quadratic form values multiplied by N.

Proposition 2.5.

$$L(N)^\vee/L(N) \simeq L/NL \simeq (\mathbb{Z}/N\mathbb{Z})^{4g}.$$

Proof. Multiplication by $1/N$ induces horizontal isomorphisms of the abelian groups in the following commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\sim} & L(N)^\vee \\ \uparrow & & \uparrow \\ NL & \xrightarrow{\sim} & L(N). \end{array}$$

The vertical arrows are inclusions of lattices in the same quadratic space. Since L has rank $4g$, the discriminant formula follows. ☐

3. Noether-Lefschetz loci

Fix again an integer $N \geq 1$.

Definition 3.1. Let $\mathcal{A}_g(N)$ be the moduli space of triples (A, Θ_A, ι_A), where (A, Θ_A) is a principally polarized abelian variety (PPAV) of dimension g and $\iota_A : A[N] \to (\mathbb{Z}/N\mathbb{Z})^{2g}$ is a symplectic isomorphism.

When $g = 1$, an elliptic curve is canonically polarized, so we drop Θ from the notation.

Definition 3.2. Let $d, g \geq 1$ be integers. Let $\widetilde{\mathsf{NL}}^b_{g,d}(N)$ be the moduli space of maps of abelian varieties $h : E \to A$, where:

- $(E, \iota_E) \in \mathcal{A}_1(N)$ and $(A, \Theta_A, \iota_A) \in \mathcal{A}_g(N)$,
- $h^*\Theta_A$ has degree d on E,
- $b : (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^{2g}$ is a fixed group homomorphism respecting the standard symplectic forms, and
- the diagram

$$\begin{array}{ccc} E[N] & \xrightarrow{h} & A[N] \\ \downarrow{\scriptstyle \iota_E} & & \downarrow{\scriptstyle \iota_A} \\ (\mathbb{Z}/N\mathbb{Z})^2 & \xrightarrow{b} & (\mathbb{Z}/N\mathbb{Z})^{2g} \end{array}$$

  commutes.

The moduli space $\widetilde{\mathsf{NL}}^b_{g,d}(N)$ may be constructed from $\widetilde{\mathsf{NL}}_{g,d} = \widetilde{\mathsf{NL}}_{g,d}(1)$, as considered in [7], as a union of connected components of the fiber product $\widetilde{\mathsf{NL}}_{g,d} \times_{(\mathcal{A}_g \times \mathcal{A}_1)} (\mathcal{A}_g(N) \times \mathcal{A}_1(N))$. In particular, $\widetilde{\mathsf{NL}}^b_{g,d}(N)$ is smooth of dimension $\binom{g}{2} + 1$.

Let $\epsilon^b(N) : \widetilde{\mathsf{NL}}^b_{g,d}(N) \to \mathcal{A}_g(N)$ and $\mu^b(N) : \widetilde{\mathsf{NL}}^b_{g,d}(N) \to \mathcal{A}_1(N)$ be the forgetful maps remembering the target and source, respectively, of h.

There is a map $\nu_N : \widetilde{\mathsf{NL}}^b_{g,d}(N) \to \widetilde{\mathsf{NL}}_{g,d}$ forgetting level structure, which is surjective onto a union of compoents of $\widetilde{\mathsf{NL}}_{g,d}$. For example, if b is injective, then ν_N surjects onto components of $\widetilde{\mathsf{NL}}_{g,d}$ parametrizing $h : E \to A$ that are injective on N-torsion. However, if $\gcd(d, N) > 1$, then there are components of $\widetilde{\mathsf{NL}}_{g,d}$ parametrizing $h : E \to A$ that factor through an isogeny $E \to E'$ of degree dividing N, which are thus not in the image of ν_N. On the other hand, if $b = 0$ and $\gcd(d, N) = 1$, then $\widetilde{\mathsf{NL}}^b_{g,d}(N)$ is empty.

By the same proof as in [7, Lemma 3.4] (the level structure does not affect the arguments), the morphism $\epsilon^b(N)$ is proper, as is the morphism $(\epsilon^b(N), \nu^b(N)) : \widetilde{\mathsf{NL}}^b_{g,d}(N) \to \mathcal{A}_1(N) \times \mathcal{A}_g(N)$. Thus, one may consider the cycle classes associated to these morphisms.

Definition 3.3. We define the cycle classes $[\widetilde{\mathsf{NL}}^b_{g,d}(N)] \in \mathrm{CH}^{g-1}(\mathcal{A}_g(N))$ and $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+ \in \mathrm{CH}^g(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$, as well as their images in cohomology, to be the classes associated to the morphisms $\epsilon^b(N), (\epsilon^b(N), \nu^b(N))$, respectively.

In [7], the pullbacks by the pointed Torelli map $\mathsf{Tor}_1 : \mathcal{M}^{\mathrm{ct}}_{g,1} \to \mathcal{A}_g$ of $[\widetilde{\mathsf{NL}}_{g,d}]$ and $[\widetilde{\mathsf{NL}}_{g,d}]^+$ to $\mathrm{CH}^{g-1}(\mathcal{M}^{\mathrm{ct}}_{g,1})$ and $\mathrm{CH}^g(\mathcal{M}^{\mathrm{ct}}_{g,1} \times \mathcal{M}_{1,1})$, respectively, are shown to agree with the Gromov-Witten virtual classes on the moduli spaces of stable maps $\mathcal{M}^{\mathrm{ct},q}_{g,1}(\mathcal{E}, d)$ to the universal elliptic curve, where the superscript q denotes that the stable maps $f : C \to E$ are required to send the marked point of C to the origin of E. Identical arguments show that the classes $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]$ and $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+$ pull back to virtual classes on the spaces of stable maps with full level-N structure and whose induced maps on N-torsion are prescribed by b.

3.1. **The case $d = 0$.** Let $\lambda_i \in \mathrm{CH}^i(\mathcal{A}_g(N))$ denote the i-th Chern class of the Hodge bundle on $\mathcal{A}_g(N)$. By convention, we set

$$[\widetilde{\mathsf{NL}}^b_{g,0}] = (-1)^g \frac{1}{12}\lambda_{g-1} \in \mathrm{CH}^{g-1}(\mathcal{A}_g(N)),$$
$$[\widetilde{\mathsf{NL}}^b_{g,0}]^+ = (-1)^g \lambda_g = 0 \in \mathrm{CH}^g(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$$

if $b \equiv 0 \pmod N$, and both cycle classes $[\widetilde{\mathsf{NL}}^b_{g,0}], [\widetilde{\mathsf{NL}}^b_{g,0}]^+$ to be zero otherwise. (See [21, Proposition 1.2] for the vanishing of λ_g.)

The definitions are explained as follows. The moduli space of maps $f : E \to A$ of degree zero inducing b on H_1 is empty if $b \not\equiv 0 \pmod N$, and isomorphic to $\mathcal{A}_1(N) \times \mathcal{A}_g(N)$ if $b \equiv 0 \pmod N$. We also add cusps in the first factor, passing to $\mathcal{A}_1(N)^* \times \mathcal{A}_g(N)$. A constant map $f : E \to A$ has obstruction space

$$H^1(E, f^*T_A) \cong H^1(E, \mathcal{O}_E) \otimes T_0 A \cong H^0(E, \omega_E)^\vee \otimes H^0(A, \Omega_A)^\vee.$$

Thus, the product $\mathcal{A}_1(N)^* \times \mathcal{A}_g(N)$ is equipped naturally with the global obstruction bundle $\mathbb{E}_1^\vee \otimes \mathbb{E}_g^\vee$ given by the tensor product of Hodge bundles on each factor. The virtual class $[\widetilde{\mathsf{NL}}^b_{g,0}]^+ \in \mathrm{CH}^g(\mathcal{A}_1(N)^* \times \mathcal{A}_g(N))$ is therefore naturally given by the top Chern class

$$\begin{aligned} c_g(\mathbb{E}_1^\vee \otimes \mathbb{E}_g^\vee) &= c_g(\mathbb{E}_g^\vee) + c_1(\mathbb{E}_1^\vee) c_{g-1}(\mathbb{E}_g^\vee) \\ &= (-1)^g \lambda_g + \left(-\frac{1}{12} \cdot [E_0]\right) \cdot ((-1)^{g-1}\lambda_{g-1}), \end{aligned}$$

where $[E_0] \in \mathrm{CH}^1(\mathcal{A}_1^*)$ is the class of a geometric point. Pushing forward to $\mathrm{CH}^{g-1}(\mathcal{A}_g(N))$ gives the formula for $[\widetilde{\mathsf{NL}}^b_{g,0}] \in \mathrm{CH}^{g-1}(\mathcal{A}_g(N))$ above, and restricting to the interior gives the formula for $[\widetilde{\mathsf{NL}}^b_{g,0}]^+ \in \mathrm{CH}^g(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$.

3.2. **Vanishing.**

Proposition 3.4. *The classes $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+ \in H^{2g}(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$ in cohomology are supported in the odd Künneth component $H^1(\mathcal{A}_1(N)) \otimes H^{2g-1}(\mathcal{A}_g(N))$. In particular, they vanish when $N = 1$.*

Proof. We may assume that $d \geq 1$. Passing from $\mathcal{A}_1(N)$ to $\mathcal{A}_1(N)^*$, we have that the map $\widetilde{\mathsf{NL}}^b_{g,d}(N) \to \mathcal{A}_1(N)^* \times \mathcal{A}_g(N)$ is proper, by the same argument as in [7, Lemma 3.4]. Thus, we may consider the cycle class $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+$ as an element of $H^{2g}(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$ or of $H^{2g}(\mathcal{A}_1(N)^* \times \mathcal{A}_g(N))$; the pullback of the former is equal to the latter.

Consider now the projection of $[\widetilde{\mathsf{NL}}^b_{g,d}(N)] \in H^{2g}(\mathcal{A}_1(N)^* \times \mathcal{A}_g(N))$ to the Künneth component $H^0(\mathcal{A}_1(N)^*) \otimes H^{2g}(\mathcal{A}_g(N))$. Up to a constant, the projection map is given by $\alpha \mapsto 1 \otimes p_*(\alpha \cap ([E] \times \mathcal{A}_g(N)))$, where $[E] \in H^2(\mathcal{A}_1(N)^*)$ is the class of any point and $p_* : H^{2g+2}(\mathcal{A}_1(N)^* \times \mathcal{A}_g(N)) \to H^{2g}(\mathcal{A}_g(N))$ is proper pushforward. Indeed, this map is easily seen to be the identity on $H^0(\mathcal{A}_1(N)^*) \otimes H^{2g}(\mathcal{A}_g(N))$ and zero on the other two Künneth components.

In particular, we may take $[E]$ to be any cusp of $\mathcal{A}_1(N)^*$, and in this case we have that $\widetilde{\mathsf{NL}}^b_{g,d}(N) \cap ([E] \times \mathcal{A}_g(N))$ is empty in $\mathcal{A}_1(N)^* \times \mathcal{A}_g(N)$, because smooth PPAVs contain no rational curves. Thus, $[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+$ projects to zero in $H^0(\mathcal{A}_1(N)^*) \otimes H^{2g}(\mathcal{A}_g(N))$, so the same is true in $H^0(\mathcal{A}_1(N)) \otimes H^{2g}(\mathcal{A}_g(N))$. Moreover, the Künneth component $H^2(\mathcal{A}_1(N)) \otimes H^{2(g-1)}(\mathcal{A}_g(N))$ is identically zero, so the claim follows.

Finally, when $N = 1$, we have $H^1(\mathcal{A}_1) = 0$, as $\mathcal{A}_1$ is contractible. Thus, the class $[\widetilde{\mathsf{NL}}_{g,d}]^+ \in H^{2g}(\mathcal{A}_1 \times \mathcal{A}_g)$ is identically zero. ☐

Remark 3.5. When $N = 1$, the same argument shows that the classes $[\widetilde{\mathsf{NL}}_{g,d}]^+$ vanish in $\mathrm{CH}^g(\mathcal{A}_1 \times \mathcal{A}_g)$. Indeed, because the coarse space of $\mathcal{A}_1^*$ is isomorphic to $\mathbb{P}^1$, the Chow group $\mathrm{CH}^g(\mathcal{A}_1^* \times \mathcal{A}_g)$ admits a Künneth decomposition

$$\mathrm{CH}^g(\mathcal{A}_1^* \times \mathcal{A}_g) \cong \mathrm{CH}^1(\mathcal{A}_1^*) \otimes \mathrm{CH}^g(\mathcal{A}_g) \oplus \mathrm{CH}^0(\mathcal{A}_1^*) \otimes \mathrm{CH}^{g-1}(\mathcal{A}_g),$$

and one can proceed as in the proof of Proposition 3.4.

4. Uniformization

Let $\Gamma(N) \subset \mathrm{SL}(2, \mathbb{Z})$ be the subgroup of matrices congruent to the identity modulo N, and more generally let $\Gamma(N)_g \subset \mathrm{Sp}(2g, \mathbb{Z})$ be the subgroup of such matrices for any g. Then, we have $\mathcal{A}_1(N) = \Gamma(N)\backslash\mathbb{H}$ and $\mathcal{A}_g(N) = \Gamma(N)_g\backslash\mathbb{H}_g$.

We also denote by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, J_{2g} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

the matrices of the standard symplectic forms, where Id is the $g \times g$ identity matrix.

Lemma 4.1. *Let $M_{2g\times 2,d}$ be the set of integer $2g\times 2$ matrices whose columns span a rank 2 sublattice of discriminant d^2 in the standard symplectic lattice $\mathbb{Z}^{2g}$.*

Then, for any $N\geq 1$, the set $\Gamma(N)\backslash M_{2g\times 2,d}/\Gamma(N)_g$ is finite.

Proof. Let $V_{\mathbb{Q}}=\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^{2g}$ be the standard symplectic $\mathbb{Q}$-vector space. The symplectic group $G_{\mathbb{Q}}=\mathrm{Sp}(2g,\mathbb{Q})$ acts on V, so it also acts diagonally on $W=V\oplus V$. Let O be the $G_{\mathbb{Q}}$-orbit of an element of $M_{2g\times 2,d}$; this orbit clearly contains all of $M_{2g\times 2,d}$. By [1, Theorem 9.11], $O\cap W_{\mathbb{Z}}$ is composed of finitely many orbits for $G_{\mathbb{Z}}$. □

Lemma 4.2. *Let $E=\mathbb{C}/\Lambda_2$ be an elliptic curve and let $A=\mathbb{C}^g/\Lambda'_{2g}$ be a PPAV. Choose symplectic bases $H_1(E)\simeq\Lambda_2$ and $H_1(A)\simeq\Lambda'_{2g}$ with respect to the polarization forms.*

Let $h:E\to A$ be a map of degree d, and let B_h be the matrix of the induced map on homology $H_1(E)\to H_1(A)$ with respect to the chosen bases of Λ_2,Λ'_{2g}. Then, we have $B_h\in M_{2g\times 2,d}$.

Proof. If $h:E\to A$ has degree d, then the composition

$$E\xrightarrow{h}A\to A^\vee\xrightarrow{h^\vee}E^\vee\to E$$

is equal to the multiplication by d map $[d]$. The induced maps on first homology groups are given by

$$H_1(E)\to H_1(A)\xrightarrow{J_{2g}}H_1(A^\vee)\to H_1(E^\vee)\xrightarrow{J_2^{-1}}H_1(E),$$

and the composition is $d\,\mathrm{Id}$,

With respect to their chosen bases, the map $H_1(E)\to H_1(A)$ is given by $B_h\in M_{2g\times 2}$, and the map $H_1(A^\vee)\to H_1(E^\vee)$ by its transpose B_h^T, so we have

$$J_2^{-1}B_h^TJ_{2g}B_h=d\,\mathrm{Id}\,.$$

This is equivalent to

$$B_h^TJ_{2g}B_h=dJ_2,$$

so $B_h\in M_{2g\times 2,d}$. □

Lemma 4.3. *Given a pair $(\tau,\tau')\in\mathbb{H}\times\mathbb{H}_g$, there exists a degree d map $E_\tau\to A_{\tau'}$ between the associated abelian varieties if and only if*

$$(B\tilde{\tau})^T\cdot\tilde{\tau}'=0\in\mathbb{C}^g$$

for some matrix $B\in M_{2g\times 2,d}$. Here, $\tilde{\tau}=\begin{pmatrix}\tau\\ \mathrm{Id}\end{pmatrix}$ denotes the Siegel augmentation.

Proof. Let $h:E\to A$ be a degree d map. Let $B_h\in M_{2g\times 2,d}$ be the matrix for

$$h_*:H_1(E)\to H_1(A)$$

with respect to symplectic bases $\langle\alpha,\beta\rangle$ for $H_1(E)$ and $\langle\alpha_j,\beta_j\rangle_{j=1,\dots,g}$ for $H_1(A)$. The graph $\Gamma_h\subset E\times A$ has homology class

$$\alpha\times h_*\beta-\beta\times h_*\alpha+E\times[\mathrm{pt}]+[\mathrm{pt}]\times h_*[E]\in H_2(E\times A,\mathbb{Z}).$$

If $\omega\in H^{1,0}(E)$ and $\omega_j\in H^{1,0}(A)$ are the normalized holomorphic 1-forms, then their external wedge product $\omega\wedge\omega_j\in H^{2,0}(E\times A)$ integrates to 0 on any algebraic curve, so on Γ_h in particular. In terms of the decomposition above, this implies that

$$\int_\alpha\omega\int_\beta h^*\omega_j-\int_\beta\omega\int_\alpha h^*\omega_j=0$$

for $j = 1, 2, \ldots, g$.

The Siegel augmented period matrices are given by

$$\tilde{\tau} = \begin{pmatrix} \int_\alpha \omega \\ \int_\beta \omega \end{pmatrix},$$

$$\tilde{\tau}' = \begin{pmatrix} \int_{\alpha_i} \omega_j \\ \int_{\beta_i} \omega_j \end{pmatrix}.$$

Multiplying $\tilde{\tau}'$ by B_h^T on the left has the effect of replacing α_i and β_j with the pushforwards of α and β under h:

$$B_h^T \tilde{\tau}' = \begin{pmatrix} \int_{h_*\alpha} \omega_j \\ \int_{h_*\beta} \omega_j \end{pmatrix} = \begin{pmatrix} \int_\alpha h^*\omega_j \\ \int_\beta h^*\omega_j \end{pmatrix}.$$

Now, we also have

$$J_2^{-1}\tilde{\tau} = \begin{pmatrix} -\int_\beta \omega \\ \int_\alpha \omega \end{pmatrix}$$

and the vanishing above is equivalent to

$$\tilde{\tau}^T J_2 B_h^T \tilde{\tau}' = (B_h J_2^{-1} \tilde{\tau})^T \cdot \tilde{\tau}' = 0.$$

Taking $B = B_h J_2^{-1}$ yields the first direction of the Lemma.

Conversely, given $B \in M_{2g\times 2,d}$, define $B_h = BJ_2$, and a linear subtorus

$$\Gamma \subset E \times A = \mathbb{C}/\Lambda_\tau \times \mathbb{C}^g/\Lambda'_{\tau'}$$

by $\Gamma = \{(z, B_h z) \mid z \in \mathbb{C}\}$, where we extend the symplectic bases of the lattices $\Lambda_\tau, \Lambda'_{\tau'}$ to $\mathbb{C}, \mathbb{C}^g$, respectively. Reversing the previous calculation, the vanishing $(B\tilde{\tau})^T \cdot \tilde{\tau}' = 0$ implies that integrals of holomorphic 2-forms on Γ vanish, which in turn implies that $\Gamma \cong E$ is a complex subtorus. Post-composing with the projection to A gives the desired map $h : E \to A$. □

Corollary 4.4. *We have an isomorphism*

$$\widetilde{\mathsf{NL}}^b_{g,d}(N) \cong \Gamma(N)\backslash \left\{ (\tau, \tau', B) \in \mathbb{H} \times \mathbb{H}_g \times M_{2g\times 2,d} \,\middle|\, \begin{matrix} (B\tilde{\tau})^T \cdot \tilde{\tau}' = 0 \\ b \equiv BJ_2 \pmod N \end{matrix} \right\} /\Gamma(N)_g$$

Proof. Lemmas 4.2 and 4.3 show that, given $E_\tau \in \mathcal{M}_{1,1}(N)$ and $A_{\tau'} \in \mathcal{A}_g(N)$, the data of $f : E_\tau \to A_{\tau'}$ of degree d is equivalent to the data of a matrix $B \in M_{2g\times 2,d}$, up to the actions of $\Gamma(N), \Gamma(N)_g$, satisfying $(B\tilde{\tau})^T \cdot \tilde{\tau}' = 0$. Moreover, the induced map $b : (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^{2g}$ on N-torsion is given, by the calculation of Lemma 4.3, by the matrix $B_h = BJ_2$. □

5. Kudla-Millson Modularity

5.1. **Symmetric spaces.** Let (W_2, ω) and (W'_{2g}, ω') be real symplectic vector spaces. The tensor product $V = W_2 \otimes W'_{2g}$ has a natural symmetric pairing γ given by the product $\omega \cdot \omega'$. Note that all pure tensors in V are isotropic with respect to γ. Choose Darboux bases $\langle e, f\rangle$ and $\langle e_i, f_i\rangle$ for W_2 and W'_{2g}, respectively, so that

$$e \otimes e'_i + f \otimes f'_i \;\; (1 \le i \le g)$$

$$e \otimes f'_i - f \otimes e'_i \;\; (1 \le i \le g)$$

form a basis for a maximal positive definite subspace $P_0 \subset V$. Similarly,

$$e \otimes e'_i - f \otimes f'_i \ \ (1 \leq i \leq g)$$

$$e \otimes f'_i + f \otimes e'_i \ \ (1 \leq i \leq g)$$

form a basis for a maximal negative definite subspace $N_0 \subset V$, with $N_0 = P_0^\perp$. Hence, the symmetric pairing γ is non-degenerate of signature $(2g, 2g)$.

Next, consider the map

$$\varphi : \mathrm{SL}_2(\mathbb{R}) \times \mathrm{Sp}_{2g}(\mathbb{R}) \to SO(V)_0 \simeq SO(2g, 2g)_0$$

defined by $\varphi(M, M') = M \otimes M'$. Its kernel is $\{\pm(\mathrm{Id}, \mathrm{Id})\}$, which is contained in the maximal compact $K = SO(2) \times U(g)$. The restriction of φ to K lands in K':

$$\varphi|_K : K \to K' = SO(2g) \times SO(2g) \subset SO(2g, 2g)_0.$$

Hence φ induces an embedding ϕ on the associated symmetric spaces:

$$\phi : \mathbb{H} \times \mathbb{H}_g \to \mathrm{Gr}^-(2g, V).$$

The symmetric space for $SO(2g, 2g)_0$ may be identified with the positive definite Grassmannian or the negative definite Grassmannian; we choose the latter. Explicitly, given $(\tau, \tau') \in \mathbb{H} \times \mathbb{H}_g$, there exist matrices $(M, M') \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{Sp}_{2g}(\mathbb{R})$ sending $(i, iI_g) \mapsto (\tau, \tau')$.

$$\phi(\tau, \tau') = (M \otimes M')(N_0) \in \mathrm{Gr}^-(2g, V).$$

One easily checks that $\mathrm{Stab}(i, iI_g)$ preserves N_0, so the map ϕ is well-defined.

Proposition 5.1. *Let*

$$B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_{2g} & b_{2g} \end{pmatrix} \in M_{2g \times 2}$$

be an integer $2g \times 2$ matrix. Let

$$B_\phi = \sum_{k=1}^{g} b_{g+k}(e \otimes e'_k) - b_k(e \otimes f'_k) - a_{g+k}(f \otimes e'_k) + a_k(f \otimes f'_k) \in W_2 \otimes W'_{2g}.$$

Then, for any $(\tau, \tau') \in \mathbb{H} \times \mathbb{H}_g$, the following are equivalent:

- *$\phi(\tau, \tau')$ is orthogonal to B_ϕ in V.*
- *$(B\tilde{\tau})^T \cdot \tilde{\tau}' = 0 \in \mathbb{C}^g$. Here, $\tilde{\tau} = \begin{pmatrix} \tau \\ \mathrm{Id} \end{pmatrix}$ denotes the Siegel augmentation.*

Proof. Let

$$M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, M' = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix},$$

where W, X, Y, Z are $g \times g$ real matrices. then, we have

$$\tau = \frac{wi + x}{yi + z}, \tau' = (iW + X)(iY + Z)^{-1}.$$

For $j = 1, 2, \ldots, g$, the j-th entry of the $1 \times g$ matrix $(yi+z)(B\tilde{\tau})^T \cdot \tilde{\tau}'(iY+Z)$ is

$$\beta_j = \sum_{k=1}^{g} (a_k(wi+x) + b_k(yi+z))(iW+X)_{kj} + \sum_{k=1}^{g} (a_{g+k}(wi+x) + b_{g+k}(yi+z))(iY+Z)_{kj}$$

where $(iW+X)_{kj}, (iY+Z)_{kj}$ denote the entries in the k-th row and j-th column of the respective matrices. Thus, we have

$$\Re(\beta_j) = \sum_{k=1}^{g} [a_k(xx_{kj} - ww_{kj}) + b_k(zx_{kj} - yw_{kj}) + a_{g+k}(xz_{kj} - wy_{kj}) + b_{g+k}(zz_{kj} - yy_{kj})],$$

$$\Im(\beta_j) = \sum_{k=1}^{g} [a_k(wx_{kj} + xw_{kj}) + b_k(yx_{kj} + zw_{kj}) + a_{g+k}(wz_{kj} + xy_{kj}) + b_{g+k}(yz_{kj} + zy_{kj})].$$

On the other hand, we compute that $\phi(\tau, \tau') = (M \otimes M')(N_0)$ is spanned by

$$\begin{aligned} r_j &= (we+yf) \otimes (We'_j + Yf'_j) - (xe+zf) \otimes (Xe'_j + Zf'_j), \\ &= -\sum_{k=1}^{g} [(e \otimes e'_k)(xx_{kj} - ww_{kj}) + (e \otimes f'_k)(xz_{kj} - wy_{kj}) \\ &\qquad + (f \otimes e'_k)(zx_{kj} - yw_{kj}) + (f \otimes f'_k)(zz_{kj} - yy_{kj})], \\ s_j &= (we+yf) \otimes (Xe'_j + Zf'_j) + (xe+zf) \otimes (We'_j + Yf'_j), \\ &= \sum_{k=1}^{g} [(e \otimes e'_k)(wx_{kj} + xw_{kj}) + (e \otimes f'_k)(wz_{kj} + xy_{kj}) \\ &\qquad + (f \otimes e'_k)(yx_{kj} + zw_{kj}) + (f \otimes f'_k)(yz_{kj} + zy_{kj})]. \end{aligned}$$

for $j = 1, 2, \ldots, g$. Here, the matrices W, X are taken to act on the basis $\{e'_1, \ldots, e'_g\}$ and the matrices Y, Z are taken to act on the basis $\{f'_1, \ldots, f'_g\}$.

Finally, we see that $\gamma(B_\phi, -r_j) = \Re(\beta_j)$ and $\gamma(B_\phi, s_j) = \Im(\beta_j)$, which yields the needed equivalence. $\square$

5.2. **Arithmetic Quotients.** Fix principal lattices $\Lambda \subset W_2$ and $\Lambda' \subset W'_{2g}$, which give rise to an even unimodular lattice $L := \Lambda \otimes \Lambda' \subset V$. The map of ϕ between symmetric spaces descends to a map on arithmetic quotients:

$$\phi : \mathrm{Aut}(\Lambda)_N \backslash \mathbb{H} \times \mathrm{Aut}(\Lambda')_N \backslash \mathbb{H}_g \to \mathrm{Aut}(L)_N \backslash \mathrm{Gr}^-(2g, V) =: \mathrm{km}(L(N)).$$

Here, $\mathrm{Aut}(\Lambda)_N, \mathrm{Aut}(\Lambda')_N, \mathrm{Aut}(L)_N$ denote the automorphism groups of the respective lattices that reduce to the identity mod N, so that ϕ is a map $\mathcal{A}_1(N) \times \mathcal{A}_g(N) \to \mathrm{km}(L(N))$.

If $v \in L$ is a vector of positive norm, then set

$$\widetilde{C}_v := \{P \in \mathrm{Gr}^-(2g, V) : P \subset v^\perp\}.$$

This is a non-empty sub-symmetric space of codimension $2g$. By Lemma 4.1, $\mathrm{Aut}(L)_N$ acts on the lattice vectors of norm $2d > 0$ with finitely many orbits.

Fix now an abelian group homomorphism $b : (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^{2g}$. Choose symplectic bases $\{e, f\}$ and $\{e'_1, \ldots, e'_g, f'_1, \ldots, f'_g\}$ of Λ, Λ', respectively, and given

$$v = \sum_{k=1}^{g} b_{g+k}(e \otimes e'_k) - b_k(e \otimes f'_k) - a_{g+k}(f \otimes e'_k) + a_k(f \otimes f'_k) \in L$$

define

$$v_h = \begin{pmatrix} -b_1 & a_1 \\ -b_2 & a_2 \\ \vdots & \vdots \\ -b_{2g} & a_{2g} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_{2g} & b_{2g} \end{pmatrix} \cdot J_2.$$

Here, the vector $v \in L$ plays the role of B_ϕ in Propostion 5.1, and v_h plays the role of the matrix B_h in Lemmas 4.2 and 4.2. The intermediate matrix B appearing in Lemma 4.2 and Propostion 5.1 satisfies $BJ_2 = B_h$. Given a d-elliptic map $h : E \to A$, the matrix of the induced map on first homology is given by B_h, which will be required to reduce modulo N to the prescribed map b.

Definition 5.2. We define the special cycles

$$C_d^b(N) := \mathrm{Aut}(L)_N \backslash \left(\bigcup_{\substack{v \in L/\mathrm{Aut}(L)_N. \\ v^2 = 2d \\ v_h \equiv b \pmod N}} \widetilde{C}_v \right) \subset \mathrm{km}(L(N)).$$

Proposition 5.3. *We have a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathsf{NL}}^b_{g,d}(N) & \longrightarrow & C_d^b(N) \\ \Big\downarrow{\scriptstyle (\epsilon(N),\mu(N))} & & \Big\downarrow \\ \mathcal{A}_1(N) \times \mathcal{A}_g(N) & \xrightarrow{\phi} & \mathrm{km}(L(N)) \end{array}$$

inducing a birational map $\widetilde{\mathsf{NL}}^b_{g,d}(N) \to \phi^{-1}(C_d^b(N))$. *In particular, we have*

$$[\widetilde{\mathsf{NL}}^b_{g,d}(N)]^+ = \phi^*[C_d^b(N)]$$

in $H^{2g}(\mathcal{A}_1(N) \times \mathcal{A}_g(N))$.

Proof. Propostion 5.1 shows that the composition $\phi \circ (\epsilon(N), \mu(N))$ has image equal to $C_d^b(N)$, and furthermore that $\phi^{-1}(C_d^b(N))$ is equal to the locus of (E, A) (with full level-N structure) for which there *exists* a map $h : E \to A$ inducing the map b on N-torsion. To identify this pullback generically with $\widetilde{\mathsf{NL}}^b_{g,d}(N)$, it therefore suffices to show that the map $(\epsilon(N), \mu(N))$ is birational onto its image.

Note that $(\epsilon(N), \mu(N))$ is unramified by [7, Proposition 3.6] (the level structure does not affect the local arguments), so it suffices to show to show that $(\epsilon(N), \mu(N))$ is generically of degree 1 on geometric points. This amounts to the statement that, at a general point $h : E \to A$ of $\widetilde{\mathsf{NL}}^b_{g,d}(N)$, the map h is the only one (up to isomorphism) from E to A. By a dimension count, we may assume that A splits up to isogeny into the product of E and a simple abelian variety B of dimension $g - 1$, and when $g = 2$, we can assume further that B is not isogeneous to E. We may furthermore assume that E is general.

Now, if there are two non-isomorphic maps $h_1, h_2 : E \to A$, then the induced map $E \times E \to A$ must have a 1-dimensional kernel E', by the assumption on the splitting of A. The two maps $i_1, i_2 : E' \to E$ must have the same degree δ, as the two compositions

$E' \to E \times \{0\} \to A$ and $E' \to \{0\} \times E \to A$ agree up to sign, and $\deg(h_1) = \deg(h_2) = d$. Thus, the two maps $i_1 \circ i_1^\vee$ and $i_2 \circ i_1^\vee$ must both have map δ^2. The first map is multiplication by δ. If E is general, then $\mathrm{End}(E) \cong \mathbb{Z}$, so $i_2 \circ i_1^\vee$ must either be multiplication by δ or $-\delta$. In particular, we have $i_1 \circ i_1^\vee = \pm i_2 \circ i_2^\vee$, and hence $i_1 = \pm i_2$, implying that h_1, h_2 are the same geometric point of $\widetilde{\mathsf{NL}}^b_{g,d}(N)$. □

The main input into Theorem 1.2 is the modularity of Kudla-Millson.

Theorem 5.4 (Kudla-Millson [10]). *Let e_0 be the Euler class of the dual of the rank $2g$ tautological bundle on $\mathrm{km}(L(N))$ of negative definite $2g$-planes. For any $b \in L(N)^\vee/L(N)$, the power series*

$$\Phi^b(q) := e_0 + \sum_{d \geq 1} [C^b_d(N)] q^d \in \mathrm{Mod}(2g, \Gamma(N)) \otimes H^{2g}(\mathrm{km}(L(N)))$$

is a modular form of weight $2g$ and level N.

Proof of Theorem 1.2. The pullback under ϕ of the tautological bundle is a real vector bundle on $\mathcal{A}_1 \times \mathcal{A}_g$ whose complexification has fiber at (E, X) equal to $H^{1,0}(E) \otimes H^{1,0}(X) \oplus H^{0,1}(E) \otimes H^{0,1}(X)$. The two direct summands are duals and complex conjugates of each other. The inclusion of the real $2g$-plane followed by projection onto the first summand gives an isomorphism of real oriented vector bundles, so

$$\phi^*(e_0) = (-1)^g c_g(\mathbb{E}_1 \boxtimes \mathbb{E}_g) \in H^{2g}(\mathcal{A}_1 \times \mathcal{A}_g, \mathbb{Q}).$$

□

Appendix A. Nonvanishing of certain Noether-Lefschetz classes

by N. Sweeting

A.1. **Overview.** In this appendix, we prove the nonvanishing of the Noether-Loefschetz classes $[\widetilde{\mathsf{NL}}^b_{2,1}(N)]^+$ for suficiently large N when b is an embedding (Theorem A.11 below). The strategy of the proof is to produce, using theta lifts from $\mathrm{GSO}_{2,2}$ to GSp_4, explicit classes in $H^4_c(\mathcal{A}_1(N) \times \mathcal{A}_2(N), \mathbb{C})$ with nonzero pairing against $[\widetilde{\mathsf{NL}}^b_{2,1}(N)]^+$ under Poincaré duality. Most of the theoretical work is contained in [20, Theorems A and C], but without any precise control of the level N; thus we must supplement the methods of *loc. cit.* with a number of additional computations to make the level structures explicit.

A.2. **Conventions.**

A.2.1. If G is an algebraic group over $\mathbb{Q}$, then $[G] := G(\mathbb{Q})\backslash G(\mathbb{A})$ denotes the usual adelic quotient. We denote by $\mathcal{A}(G)$ the space of automorphic forms on $[G]$, and by $\mathcal{A}_0(G)$ the subspace of cusp forms. If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup, then we write $\mathcal{A}_0(G; K)$ for the space of K-invariant cusp forms.

A.2.2. For an integer $N \geq 1$, we consider the compact open subgroup

$$K_1(N) = \prod K_1(N)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \, : \, c \in N\widehat{\mathbb{Z}}, \; d \in 1 + N\widehat{\mathbb{Z}} \right\}$$

of $\mathrm{GL}_2(\mathbb{A}_f)$. It is clear that $K_1(N)_p$ depends only on the p-adic valuation of N.

A.2.3. We denote by $B \subset \mathrm{GL}_2$ the upper triangular Borel subgroup, and by $U \subset B$ the unipotent radical. We define a map of algebraic groups $\mathbb{G}_m \to \mathrm{GL}_2$ by $c \mapsto h_c = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$.

A.2.4. Let $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}$ be the unique everywhere unramified character such that $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$, and let ψ_k be the local component of ψ for every completion k of $\mathbb{Q}$.

A.2.5. Let $\mathrm{SO}(2) \subset \mathrm{GL}_2(\mathbb{R})$ be the standard maximal compact subgroup; we denote by χ_m the weight-m character of $\mathrm{SO}(2)$ defined by

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto (\cos\theta + i\sin\theta)^m.$$

A.3. **Shimura varieties.**

A.3.1. *Disconnected moduli spaces of abelian varieties.* Fix $N \geq 1$, and define $K_N = K_{N,g} = \prod_p K_{N,g,p} \subset \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ to be the compact open subgroup of matrices that are congruent to the identity modulo N. Let $\mathcal{A}'_g(N)_{\mathbb{Q}}$ be the complex Shimura variety for GSp_{2g} of level K_N:

$$\mathcal{A}'_g(N) = \mathrm{GSp}_{2g}(\mathbb{Q})\backslash \mathrm{GSp}_{2g}(\mathbb{A}_f) \times \mathbb{H}_g/K_N. \tag{1}$$

There is a natural projection $\mathcal{A}'_g(N) \to \mathbb{Q}^\times\backslash\mathbb{A}_f^\times/(1+N\widehat{\mathbb{Z}})^\times \simeq \mu_N$, whose fibers are the geometric connected components, each isomorphic to $\mathcal{A}_g(N)$. We also have the natural embedding

$$\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_{g-1}(N) \hookrightarrow \mathcal{A}'_1(N) \times \mathcal{A}'_g(N) \tag{2}$$

corresponding to the embedding of groups $\mathrm{GSp}_2 \times_{\mathbb{G}_m} \mathrm{GSp}_{2g-2} \hookrightarrow \mathrm{GSp}_2 \times \mathrm{GSp}_{2g}$.

Proposition A.1. *For all $b : (\mathbb{Z}/N\mathbb{Z})^2 \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^{2g}$, we have*

$$0 \neq [\widetilde{\mathsf{NL}}^b_{g,1}]^+ \in H^4(\mathcal{A}_1(N) \times \mathcal{A}_g(N), \mathbb{Q})$$

if and only if

$$0 \neq [\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_{g-1}(N)] \in H^4(\mathcal{A}'_1(N) \times \mathcal{A}'_g(N), \mathbb{Q}).$$

Proof. Because all choices of the embedding b differ by an element of $\mathrm{Sp}_4(\mathbb{Z}/N\mathbb{Z})$, the resulting classes $[\widetilde{\mathsf{NL}}^b_{g,1}]^+$ are transitively permuted by the natural action of $\mathrm{Sp}_4(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{A}_g(N)$. Hence we may assume without loss of generality that b is the embedding sending $(\mathbb{Z}/N\mathbb{Z})^2$ isomorphically onto the last two coordinates of $(\mathbb{Z}/N\mathbb{Z})^{2g}$.

The embedding $\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_{g-1}(N) \hookrightarrow \mathcal{A}'_1(N) \times \mathcal{A}'_g(N)$ factors through the open and closed subvariety $\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_g(N) \subset \mathcal{A}'_1(N) \times \mathcal{A}'_g(N)$. For each connected component $\mathcal{A}_1(N) \times \mathcal{A}_g(N)$ of $\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_g(N)$, our choice of b implies that

$$(\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_{g-1}(N)) \cap (\mathcal{A}_1(N) \times \mathcal{A}_g(N)) = \widetilde{\mathsf{NL}}^b_{g,1},$$

and the proposition follows. □

Lemma A.2. *Let $N, M \geq 1$ be integers, and suppose*
$$0 \neq \left[\mathcal{A}_1'(N) \times_{\mu_N} \mathcal{A}_{g-1}'(N)\right] \in H^4(\mathcal{A}_1'(N) \times \mathcal{A}_g'(N), \mathbb{Q}).$$
Then
$$0 \neq \left[\mathcal{A}_1'(NM) \times_{\mu_{NM}} \mathcal{A}_{g-1}'(NM)\right] \in H^4(\mathcal{A}_1'(NM) \times \mathcal{A}_g'(NM), \mathbb{Q}).$$

Proof. The pullback map $H^4(\mathcal{A}_1'(N) \times \mathcal{A}_g'(N), \mathbb{Q}) \to H^4(\mathcal{A}_1'(NM) \times \mathcal{A}_g'(NM), \mathbb{Q})$ is injective because the projection $\pi_{N,M} : \mathcal{A}_1'(NM) \times \mathcal{A}_g'(NM) \to \mathcal{A}_1'(N) \times \mathcal{A}_g'(N)$ is finite. Hence by assumption, $\pi_{N,M}^* \left[\mathcal{A}_1'(N) \times_{\mu_N} \mathcal{A}_{g-1}'(N)\right] \neq 0$. On the other hand, the preimage $\pi_{N,M}^{-1}(\mathcal{A}_1'(N) \times_{\mu_N} \mathcal{A}_{g-1}'(N))$ is a union of $\mathrm{GSp}_2(\mathbb{Z}/NM\mathbb{Z}) \times \mathrm{GSp}_{2g}(\mathbb{Z}/NM\mathbb{Z})$-translates of $\mathcal{A}_1'(NM) \times_{\mu_{NM}} \mathcal{A}_{g-1}'(NM)$, and the lemma follows. □

A.3.2. *Automorphic forms in cohomology.* Fix $\tau = i\,\mathrm{Id} \in \mathbb{H}_g$. The stabilizer of τ in $\mathrm{GSp}_{2g}(\mathbb{R})$ is the subgroup $\mathbb{R}^\times \cdot U(g)$, and the tangent space to the real manifold $\mathbb{H}_g$ at τ is $\mathfrak{p} := \mathfrak{sp}_{2g,\mathbb{R}}/\mathfrak{u}(g)$. As a $U(g)$-module, $\mathfrak{p}_\mathbb{C}$ is isomorphic to $\mathrm{Sym}^2 \oplus (\mathrm{Sym}^2)^\vee$, where Sym^2 is the symmetric square of the g-dimensional defining representation of $U(g)$.

Thus we have a canonical map

$$(\mathcal{A}(\mathrm{GSp}_{2g}) \otimes \wedge^i \mathfrak{p}_\mathbb{C}^*)^{\mathbb{R}^\times \cdot U(g)} \to H^i(\mathcal{A}_g'(N), \mathbb{C}), \tag{3}$$

which on cusp forms restricts to a map

$$(\mathcal{A}_0(\mathrm{GSp}_{2g}) \otimes \wedge^i \mathfrak{p}_\mathbb{C}^*)^{\mathbb{R}^\times \cdot U(g)} \to H_c^i(\mathcal{A}_g'(N), \mathbb{C}). \tag{4}$$

A.4. **Newforms and Whittaker models for $\mathbf{GL_2}$.**

A.4.1. Let π be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for prime p. The conductor of π is the least n such that $\pi^{K_1(p^n)_p} \neq 0$; it is well-known that such an n always exists, and, if n is minimal, then $\pi^{K_1(p^n)_p}$ is one-dimensional. A generator of this space is called a local newform for π.

A.4.2. Recall the nontrivial additive character $\psi_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, which is trivial on $\mathbb{Z}_p$ but not on $p\mathbb{Z}_p$. We also view $\psi_{\mathbb{Q}_p}$ as a character of
$$U(\mathbb{Q}_p) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},\ a \in \mathbb{Q}_p \right\} \subset \mathrm{GL}_2(\mathbb{Q}_p).$$
Then π has a Whittaker model
$$W_{\psi_{\mathbb{Q}_p}}(\pi) \subset \mathrm{Ind}_{U(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \psi_{\mathbb{Q}_p}.$$
Let $W^0_{\pi,\psi_{\mathbb{Q}_p}} \in W_{\psi_{\mathbb{Q}_p}}(\pi)$ be a local newform.

Proposition A.3. *Suppose π has conductor $n \geq 1$, so that $L(s,\pi) = (1 - \alpha p^{-s})$ for some $\alpha \in \mathbb{C}$. Then up to rescaling $W^0_{\pi,\psi_{\mathbb{Q}_p}}$, we have*
$$W^0_{\pi,\psi_k}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 0, & \mathrm{ord}_p(t) < 0, \\ 1, & \mathrm{ord}_p(t) = 0, \\ |t|^{1/2} \alpha^{\mathrm{ord}_p(t)}, & \mathrm{ord}_p(t) > 0. \end{cases}$$

Proof. This is a special case of [15, Theorem 4.1]. □

A.5. **Induced representations and Eisenstein series on $\mathbf{GL_2}$.**

A.5.1. For each place v of $\mathbb{Q}$, let

$$I_v(s) = \operatorname{Ind}_{B(\mathbb{Q}_v)}^{\mathrm{GL}_2(\mathbb{Q}_v)} \delta_B^s \tag{5}$$

be the normalized induction, and let $I(s) = \otimes'_v I_v(s)$.

For $\varphi(s) \in I(s)$, we have the Eisenstein series

$$E(g, s; \varphi) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} \varphi(s)(\gamma g), g \in \mathrm{GL}_2(\mathbb{A}),$$

which converges for $\Re(s) \gg 0$.

A.5.2. For $N \geq 1$, we define a section $\varphi_N^0 = \otimes_v \varphi_{N,v}^0 \in I(1/2)$ as follows:

- For $v = p$, $\varphi_{N,p}^0$ is the unique $K_1(N)_p$-invariant section supported on $B(\mathbb{Q}_p) \cdot K_1(N)_p$ and satisfying $\varphi_{N,p}^0(1) = 1$.
- For $v = \infty$, $\varphi_{N,\infty}^0$ is the unique SO(2)-spherical section satisfying $\varphi_{N,\infty}^0(1) = 1$.

We can extend φ_N^0 uniquely to a section $\varphi_N^0(s) \in I(s)$ so that the restriction of φ_N^0 to $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \cdot \mathrm{SO}(2)$ is independent of s.

Proposition A.4. *The Eisenstein series $E(g, s; \varphi_N^0)$ has a pole at $s = 1/2$, with residue a nonzero constant function of g.*

Proof. By the well-known theory of Eisenstein series for GL_2, it suffices to show that $\varphi_{N,v}^0$ has nontrivial image under the intertwining operator

$$M_v : I_v(1/2) \to I_v(-1/2)$$

for all primes v. At $v = \infty$ and $v = p \nmid N$, this is clear because $\varphi_{N,v}^0$ is the unique spherical vector for a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_v)$, so we consider the case of $v = p|N$. The intertwining operator is given explicitly by

$$M_p(\varphi)(g) = \int_{\mathbb{Q}_p} \varphi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}y, \quad \varphi \in I_v(1/2), g \in \mathrm{GL}_2(\mathbb{Q}_p).$$

Normalizing the measure so that $\mathbb{Z}_p$ has unit volume, we therefore calculate:

$$\begin{aligned} M_p(\varphi_{N,p}^0)(1) &= \varphi_{N,p}^0\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} \varphi_{N,p}^0\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix}\right) \mathrm{d}y \\ &= 0 + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |y|^{-2} \varphi_{N,p}^0\left(\begin{pmatrix} 1 & 0 \\ y^{-1} & 1 \end{pmatrix}\right) \mathrm{d}y \\ &= \sum_{n \geq \mathrm{ord}_p(N)} \int_{p^{-n}\mathbb{Z}_p^\times} |y|^{-2} \mathrm{d}y = \sum_{n \geq \mathrm{ord}_p(N)} p^{-n-1}(p-1) \neq 0, \end{aligned}$$

where in the second line we have used that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \notin B(\mathbb{Q}_p)K_1(N)_p$ for all $p|N$. □

A.6. **Weil representation and theta lifting.**

A.6.1. Let $\epsilon = \pm 1$, and let V, W be vector spaces over a field k equipped with nondegenerate ϵ-symmetric and $(-\epsilon)$-symmetric pairings, respectively. We assume $\dim W = 2n$ and $\dim V = 2m$ are even, that V has trivial discriminant character, and that W is equipped with a complete polarization

$$W = W_1 \oplus W_2, \quad W_2 = W_1^*. \tag{6}$$

A.6.2. Let $G_1 = G_1(V)$, $G = G(V)$ be the connected isometry and similitude groups, respectively, of V, and likewise $H_1 = H_1(W)$ and $H = H(W)$; we have the natural similitude characters $\nu_G : G \to \mathbb{G}_m$ and $\nu_H : H \to \mathbb{G}_m$.

A.6.3. *The local Weil representation.* Suppose that k is a local field, and let ψ_k be a nontrivial additive character of k. Then following Roberts' construction [18], the similitude Weil representation $\omega = \omega_{V,W,\psi_k}$ of $(H \times_{\mathbb{G}_m} G)(k)$ is realized on the Schwartz space $\mathcal{S}(W_2 \otimes V)$ of compactly supported, complex-valued functions on $W_2 \otimes V$. Concise descriptions of this representation can be found in [5, §2] or [20, §4], but all we require are the following two facts:

- The action of
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & \nu_G(g) \end{pmatrix}, g \right) \in (H \times_{\mathbb{G}_m} G)(k)$$
on $\mathcal{S}(W_2 \otimes V)$ is given by $\phi \mapsto |\nu_G(g)|^{-mn/2} \phi \circ g^{-1}$.
- Suppose $V = V_1 \oplus V_2$ is a polarization of V. Then the Fourier transform
$$\begin{aligned} &\mathcal{S}(W_2 \otimes V) \to \mathcal{S}(W \otimes V_2) \\ &\phi \mapsto \widehat{\phi}, \quad \widehat{\phi}(x_1, x_2) = \int_{W_2 \otimes V_1} \phi(z, x_2) \psi(z \cdot x_1) \mathrm{d}z, \end{aligned} \tag{7}$$
with $\mathrm{d}z$ the self-dual Haar measure, defines an $(H \times_{\mathbb{G}_m} G)(k)$-linear isomorphism from ω_{V,W,ψ_k} to ω_{W,V,ψ_k}.

A.6.4. *The global Weil representation.* Now turn to the global situation, and take $k = \mathbb{Q}$ in (A.6.1). The adelic Schwartz space $\mathcal{S}_{\mathbb{A}}(W_2 \otimes V)$ is the restricted tensor product of the local Schwartz spaces $\mathcal{S}_k(W_2 \otimes V) = \mathcal{S}(W_2 \otimes V \otimes k)$ as k ranges over completions of $\mathbb{Q}$. The global Weil representation $\omega = \omega_{V,W,\psi}$ of $(H \times_{\mathbb{G}_m} G)(\mathbb{A})$ is realized on $\mathcal{S}_{\mathbb{A}}(W_2 \otimes V)$ as the restricted tensor product of the local Weil representations.

Recall the automorphic realization of ω, given by the theta kernel:

$$\theta(h, g; \phi) = \sum_{x \in W_2(\mathbb{Q}) \otimes V(\mathbb{Q})} \omega(h, g)\phi(x), \quad (h, g) \in (H \times_{\mathbb{G}_m} G)(\mathbb{A}), \;\; \phi \in \mathcal{S}_{\mathbb{A}}(W_2 \otimes V). \tag{8}$$

A.6.5. *Theta lifts of automorphic forms.* Let $f \in \mathcal{A}_0(G(\mathbb{A}))$ be an automorphic cusp form and choose any $\phi \in \mathcal{S}_{\mathbb{A}}(W_2 \otimes V)$. Then, fixing a Haar measure $\mathrm{d}g_1$ on $G_1(\mathbb{A})$, the similitude theta lift $\theta_\phi(f)$ to H is the automorphic function

$$h \mapsto \int_{[G_1]} \theta(g_1 g_0, h; \phi) f(g_1 g_0) \mathrm{d}h_1, \quad h \in H(\mathbb{A}), \tag{9}$$

where $g_0 \in G(\mathbb{A})$ is any element such that $\nu_G(g_0) = \nu_H(h)$.

For any compact open subgroup $K \subset H(\mathbb{A}_f)$, we say $\phi_f \in \mathcal{S}_{\mathbb{A}_f}(W_2 \otimes V)$ is K-invariant, which we write as

$$\phi_f \in \mathcal{S}_{\mathbb{A}_f}(W_2 \otimes V)^K,$$

if for all $k \in K$, there exists $g_0 \in G(\mathbb{A}_f)$ with $\nu_G(g_0) = \nu_H(k)$ such that

$$\omega(g_0, k)\phi_f = \phi_f.$$

Note that, if we fix $\phi_\infty \in \mathcal{S}_{\mathbb{R}}(W_2 \otimes V)$, then

$$\theta_{\phi_f \otimes \phi_\infty}(f) \text{ is } K\text{-invariant for all } \phi_f \in \mathcal{S}_{\mathbb{A}_f}(W_2 \otimes V)^K. \tag{10}$$

A.7. **Some explicit Schwartz functions.**

A.7.1. *The split four-dimensional quadratic space.* We briefly recall the conventions of [20, §5.1]. Let $V = M_2$, with its canonical involution $x \mapsto x^*$ and quadratic form given by $(x, y) = \mathrm{tr}(x\overline{y}^*)$. We have the map of algebraic groups over $\mathbb{Q}$:

$$\boldsymbol{p}_Z : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GO}(V) \tag{11}$$

defined by $\boldsymbol{p}_Z(g_1, g_2) \cdot x = g_1 x g_2^*$. The kernel of $\boldsymbol{p}_Z$ is the antidiagonally embedded $\mathbb{G}_m$, and $\boldsymbol{p}_Z$ is a surjection onto the connected similitude group $\mathrm{GSO}(V) \subset \mathrm{GO}(V)$.

A.7.2. For any pair of automorphic forms f_1, f_2 on $\mathrm{GL}_2(\mathbb{A})$ with the same central character, we obtain an automorphic form $f_1 \boxtimes f_2$ on $\mathrm{GSO}(V)$ defined by

$$(f_1 \boxtimes f_2)(\boldsymbol{p}_Z(g_1, g_2)) = f_1(g_1) f_2(g_2), \quad g_1, g_2 \in \mathrm{GL}_2(\mathbb{A}). \tag{12}$$

A.7.3. *Symplectic spaces.* For all g, we consider the standard symplectic space of dimension $2g$ over $\mathbb{Q}$, with basis $e_1, e_2, \ldots, e_{2g}$ such that

$$e_{2n-1} \cdot e_{2n} = -e_{2n} \cdot e_{2n-1} = 1, \; \forall 1 \leq n \leq g,$$

and all other pairings of basis vectors are trivial. We will always take the complete polarization

$$\langle e_1, e_2, \ldots, e_{2n} \rangle = \langle e_1, e_3, \ldots, e_{2g-1} \rangle \oplus \langle e_2, e_4, \ldots, e_{2g} \rangle.$$

A.7.4. *Nonarchimedean Schwartz functions.* For each prime p, define the Schwartz function

$$\phi_{N,p} \in \mathcal{S}_{\mathbb{Q}_p}(V)$$

to be the indicator function of the subset

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \subset V \otimes \mathbb{Q}_p = M_2(\mathbb{Q}_p).$$

Clearly $\phi_{N,p}$ depends only on the p-adic valuation of N. Also identify $\mathcal{S}_{\mathbb{Q}_p}(V)$ with the Schwartz space $\mathcal{S}_{\mathbb{Q}_p}(\langle e_2 \rangle \otimes V)$, which realizes the Weil representation of $(\mathrm{GSp}_2 \times_{\mathbb{G}_m} \mathrm{GSO(V)})(\mathbb{Q}_p)$.

Fix the polarization $V = V_1 \oplus V_2$, where

$$V_1 = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}. \tag{13}$$

Proposition A.5. *Under the Fourier transform of (7),*

$$\widehat{\phi}_{N,p} \in \mathcal{S}_{\mathbb{Q}_p}(\langle e_1, e_2\rangle \otimes V_2)$$

is the indicator function of the set

$$e_1 \otimes \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \subset \langle e_1, e_2\rangle \otimes V_2.$$

Proof. By definition,

$$\widehat{\phi}_{N,p}\left(e_1 \otimes \begin{pmatrix} 0 & 0 \\ z_1 & w_1 \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 \\ z_2 & w_2 \end{pmatrix}\right) = \int_{\mathbb{Q}_p^2} \phi_{N,p}\left(\begin{pmatrix} x & y \\ z_2 & w_2 \end{pmatrix}\right) \cdot \psi\left(xw_1 - yz_1\right),$$

and the proposition follows. □

Proposition A.6. *Fix integers N and M and consider the Schwartz function*

$$\phi_{N,M,p} := \phi_{N,p} \otimes \phi_{M,p} \in \mathcal{S}_{\mathbb{Q}_p}(\langle e_2\rangle \otimes V) \otimes \mathcal{S}_{\mathbb{Q}_p}(\langle e_4\rangle \otimes V) \subset \mathcal{S}_{\mathbb{Q}_p}(\langle e_2, e_4\rangle \otimes V).$$

Then we have

$$\phi_{N,M,p} \in \mathcal{S}_{\mathbb{Q}_p}(\langle e_2, e_4\rangle \otimes V)^{K_{N,2,p}\cap K_{M,2,p}}.$$

Proof. By the same calculation as Proposition A.5, the Fourier transform

$$\widehat{\phi}_{N,M,p} \in \mathcal{S}_{\mathbb{Q}_p}(\langle e_1, e_2, e_3, e_4\rangle \otimes V_2)$$

is the indicator function of the set

$$e_1 \otimes \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_p & N\mathbb{Z}_p \end{pmatrix} + e_3 \otimes \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} + e_4 \otimes \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_p & M\mathbb{Z}_p \end{pmatrix}.$$

Because the Fourier transform is equivariant for $(\mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GSO}(V))(\mathbb{Q}_p)$, the proposition follows from the stability of this set under the action of

$$K_{N,2,p} \cap K_{M,2,p} = \left\{g \in \mathrm{GSp}_4(\mathbb{Z}_p) : g \equiv \mathrm{Id} \pmod{p^{\max\{\mathrm{ord}_p(N), \mathrm{ord}_p(M)\}}}\right\}.$$

□

A.7.5. *The local Siegel-Weil map.* For each place v of $\mathbb{Q}$, we have a map

$$M_{1,v}[\cdot] : \mathcal{S}_{\mathbb{Q}_v}(V) \to I_v(1/2)$$

given by

$$M_{1,v}[\phi](g) = \omega(h_{\det(g)}, \boldsymbol{p}_Z(g,1))\widehat{\phi}(0) = |\det(g)|^{-1} \int_{\mathbb{Q}_v} \phi\left(g^{-1}\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) \mathrm{d}x\mathrm{d}y,$$

cf. [20, §6.4.6].

Proposition A.7. *Suppose $p|N$. Then*

$$M_{1,p}[\phi_{N,p} - p^{-1}\phi_{N/p,p}] = (1-p^{-1})\varphi^0_{N,p}.$$

Proof. First, we calculate, for $i \geq 0$:

$$\begin{aligned} M_{1,p}[\phi_{N,p}]\left(\begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix}\right) &= \int_{\mathbb{Q}_p} \phi_{N,p}\left(\begin{pmatrix} x & y \\ p^i x & p^i y \end{pmatrix}\right) \mathrm{d}x\mathrm{d}y \\ &= \mathrm{Vol}(\mathbb{Z}_p \cap p^{\mathrm{ord}_p(N)-i}\mathbb{Z}_p) = \begin{cases} p^{i-\mathrm{ord}_p(N)}, & i \leq \mathrm{ord}_p(N) \\ 1, & i > \mathrm{ord}_p(N). \end{cases} \end{aligned}$$

In particular,

$$M_{1,p}[\phi_{N,p} - p^{-1}\phi_{N/p,p}]\left(\begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix}\right) = \begin{cases} 1-p^{-1}, & i \geq \mathrm{ord}_p(N) \\ 0, & 0 \leq i < \mathrm{ord}_p(N). \end{cases} \tag{14}$$

On the other hand, it is clear that $M_{1,p}[\phi_{N,p} - p^{-1}\phi_{N/p,p}]$ is invariant under $K_1(N)_p$. By the Iwasawa decomposition $\mathrm{GL}_2(\mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot \mathrm{GL}_2(\mathbb{Z}_p)$ and the coset decomposition

$$\mathrm{GL}_2(\mathbb{Z}_p) = \bigsqcup_{0 \leq i \leq \mathrm{ord}_p(N)} \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} K_1(N)_p,$$

we conclude from (14) that $M_{1,p}[\phi_{N,p} - p^{-1}\phi_{N/p,p}]_p = (1-p^{-1})\varphi^0_{N,p}$. □

A.7.6. *Archimedean Schwartz function.* Let τ be the representation of $U(2)$ of highest weight $(3,-1)$. We fix the nontrivial vector-valued archimedean Schwartz function

$$\phi_\infty \in \left(\mathcal{S}_{\mathbb{R}}(W_2 \otimes V) \otimes \tau \otimes (\chi_2^\vee \boxtimes \chi_4^\vee)\right)^{U(2) \times \boldsymbol{p}_Z(\mathrm{SO}(2)\times\mathrm{SO}(2))},$$

denoted $\varphi^-_{2,4}$ in [20, §7.1.6].

A.8. **Proof of Theorem A.11.**

A.8.1. *Construction of cohomology classes.* Fix new cuspidal Hecke eigenforms f_1 and f_2 for $\Gamma_1(N)$ of weights 4 and 2, respectively, and of equal nebetype character ε. Then f_1 and f_2 correspond to automorphic forms

$$f_{1,\mathbb{A}} \in \left(\mathcal{A}_0(\mathrm{GSp}_2; K_{N,1}) \otimes \chi_4\right)^{\mathbb{R}^\times \cdot U(1)}, \quad f_{2,\mathbb{A}} \in \left(\mathcal{A}_0(\mathrm{GSp}_2; K_{N,1}) \otimes \chi_2\right)^{\mathbb{R}^\times \cdot U(1)}.$$

For any Schwartz function

$$\phi_f \in \mathcal{S}_{\mathbb{A}_f}(\langle e_2, e_4\rangle \otimes V)^{K_{N,2}},$$

we consider the vector-valued lift

$$\Theta_{\phi_f \otimes \phi_\infty}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}}) \in \left(\mathcal{A}(\mathrm{GSp}_4; K_{N,2}) \otimes \tau\right)^{\mathbb{R}^\times \cdot U(2)}.$$

Remark A.8. Assuming it is nonzero, the vector-valued automorphic form $\Theta_{\phi_f\otimes\phi_\infty}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}})$ generates the unique generic member of the endoscopic Yoshida lift L-packet on GSp_4 associated to f_1 and f_2, cf. [19].

By [19, Theorem 8.3], $\Theta_{\phi_f\otimes\phi_\infty}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}})$ is a cusp form. Moreover, an easy calculation shows that $\mathrm{Hom}_{U(2)}(\tau, \wedge^3\mathfrak{p}^*_{\mathbb{C}})$ is one-dimensional in the notation of (A.3.2) with $g = 2$. Hence from (4), we obtain a class

$$\left[\Theta_{\phi_f\otimes\phi_\infty}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}})\right] \in H^3_c(\mathcal{A}'_2(N), \mathbb{C}) \tag{15}$$

which is well-defined up to a scalar multiple.

When $g = 1$, the space $\mathfrak{p}^*_{\mathbb{C}}$ is a direct sum $\chi_2 \oplus \chi_2^\vee$ as a $U(1)$-module. In particular,

$$\overline{f_{2,\mathbb{A}}} \in (\mathcal{A}_0(\mathrm{GSp}_2; K_{N,1}) \otimes \chi_2^\vee)^{\mathbb{R}^\times \cdot U(1)}$$

defines a class

$$[\overline{f_{2,\mathbb{A}}}] \in H^1_c(\mathcal{A}'_1(N), \mathbb{C}).$$

This is the usual cohomology class attached to the holomorphic modular form $f_2 \otimes \varepsilon^{-1}$. By the Künneth formula, we also have the cohomology class

$$[\overline{f_{2,\mathbb{A}}}] \boxtimes \left[\Theta_{\phi_f\otimes\phi_\infty}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}})\right] \in H^4_c(\mathcal{A}'_1(N) \times \mathcal{A}'_2(N), \mathbb{C}). \tag{16}$$

Proposition A.9. *Up to a nonzero scalar depending on the normalizations, the Poincaré duality pairing*

$$\left\langle [\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_1(N)]\,, [\overline{f_{2,\mathbb{A}}}] \boxtimes \left[\Theta_{\phi_f \otimes \phi_\infty}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}})\right]\right\rangle \in H^8_c(\mathcal{A}'_1(N) \times \mathcal{A}'_2(N), \mathbb{C}) \simeq \mathbb{C}$$

is given by

$$\int_{[H]} \Theta_{\phi_f \otimes \overline{\phi_\infty}}(f_{1,\mathbb{A}} \boxtimes f_{2,\mathbb{A}})(\iota(h_1, h_2))\overline{f_{2,\mathbb{A}}}(h_1)\mathrm{d}(h_1, h_2),$$

where $H = \mathrm{GSp}_2 \times_{\mathbb{G}_m} \mathrm{GSp}_2$ *is given the coordinates* (h_1, h_2) *and* $\iota : H \hookrightarrow \mathrm{GSp}_4$ *is the standard embedding.*

Proof. See [20, Proposition 7.2.4]. □

Lemma A.10. *Suppose* $N > 1$ *is an integer such that there exist cuspidal newforms* f_1 *and* f_2 *for* $\Gamma_1(N)$ *of weights 4 and 2, respectively, of equal nebentype character* ε*. Then*

$$0 \neq [\mathcal{A}'_1(N) \times_{\mu_N} \mathcal{A}'_1(N)] \in H^4(\mathcal{A}'_1(N) \times \mathcal{A}'_2(N), \mathbb{Q}).$$

Proof. Without loss of generality, we may assume f_1 and f_2 are Hecke eigenforms. Then Proposition A.9 reduces us to showing the nonvanishing of the period that appears therein, for some choice of $\phi_f \in \mathcal{S}_{\mathbb{A}_f}(\langle e_2, e_4\rangle \otimes V)^{K_{N,2}}$.

We now fix the Schwartz function $\phi_f \in \mathcal{S}_{\mathbb{A}_f}(\langle e_2, e_4\rangle \otimes V)$ to be of the form $\phi_f^{(1)} \otimes \phi_f^{(2)}$ for $\phi_f^{(i)} = \otimes_p \phi_p^{(i)} \in \mathcal{S}_{\mathbb{A}_f}(\langle e_{2i}\rangle \otimes V)$, $i = 1, 2$. By [20, Theorem 6.5.2, Proposition 7.1.9], it suffices to show that, for all $p|N$, we may choose $\phi_p^{(i)}$ such that the following local zeta integrals are all nonzero:

$$\begin{aligned}\int_{(U\backslash \mathrm{PGL}_2 \times U\backslash \mathrm{PGL}_2)(\mathbb{Q}_p)} \int_{\mathrm{SL}_2(\mathbb{Q}_p)} & W^0_{\pi_{1,p},\psi_{\mathbb{Q}_p}}(g_1) W^0_{\pi_{2,p},\psi_{\mathbb{Q}_p}}(g_2) W^0_{\pi^\vee_{2,p},\psi^{-1}_{\mathbb{Q}_p}}(h_1 h_c) \\ & \omega(h_1 h_c, g)\widehat{\phi}^{(1)}_p(1,0,0,-1)\varphi^0_{N,p}(g_2) M_1[\phi^{(2)}_p](g_1)\mathrm{d}h_1 \mathrm{d}g_1 \mathrm{d}g_2, \\ & c = \det(g_1 g_2), \\ & g = \boldsymbol{p}_Z(g_1, g_2).\end{aligned} \tag{17}$$

Here, $(1, 0, 0, -1) \in \langle e_1, e_2\rangle \otimes V_2$ is the vector $e_1 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$; $\pi_{1,p}$, $\pi_{2,p}$, and $\pi^\vee_{2,p}$ are the local components of the automorphic representations generated by $f_{1,\mathbb{A}}$, $f_{2,\mathbb{A}}$, and $\overline{f_{2,\mathbb{A}}}$, respectively; and $W^0_{\pi_{1,p},\psi_{\mathbb{Q}_p}}$, etc. are the corresponding local newforms in the Whittaker models. In fact, in [20, Theorem 6.5.2], φ^0_N is replaced with $\varphi^0 := \varphi^0_1$; however, the proof of *loc. cit.* still applies, as long as Proposition A.4 is used to replace the explicit calculation of the residue in [20, Proposition 6.4.10]. We choose our Schwartz functions as follows for $p|N$:

- $\phi^{(1)}_p = \phi_{N,p}$ for all p.
- $\phi^{(2)}_p = \phi_{N,p} - p^{-1}\phi_{N/p,p}$ for all p.

With these choices, we now show that (17) is nonzero.

We first consider the inner integral:

$$I(g) = \int_{\mathrm{SL}_2(\mathbb{Q}_p)} W^0_{\pi^\vee_{2,p},\psi_{\mathbb{Q}_p}}(h_1 h_c)\omega(h_1 h_c, g)\widehat{\phi}_{N,p}(1, 0, 0, -1)\mathrm{d}h_1. \tag{18}$$

Now, [20, Lemma 6.3.3] and its proof identifies (18) with a function in the $\psi_{\mathbb{Q}_p}^{-1} \boxtimes \psi_{\mathbb{Q}_p}^{-1}$-Whittaker model of the representation $\pi_{2,p}^\vee \boxtimes \pi_{2,p}^\vee$ of $\mathrm{GL}_2(\mathbb{Q}_p) \boxtimes \mathrm{GL}_2(\mathbb{Q}_p)$. Because $\phi_{N,p}$ is clearly invariant by $\boldsymbol{p}_Z(K_1(N)_p \times K_1(N)_p)$, and because $\mathrm{ord}_p(N)$ is the conductor of $\pi_{2,p}$, (18) is a scalar multiple of the local newform $W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}} \boxtimes W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}$. To show this scalar multiple is nonzero, we evaluate

$$I(1) = \int_{\mathrm{SL}_2(\mathbb{Q}_p)} W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}}(h_1) \omega(h_1, 1) \widehat{\phi}_{N,p}(1, 0, 0, -1) \mathrm{d}h_1. \tag{19}$$

Now by Proposition A.5, we can calculate directly that $\omega(h_1, 1)\widehat{\phi}_{N,p}(1, 0, 0, -1)$ is the indicator function of $K_1(N)_p$. Hence

$$I(1) = \mathrm{Vol}(K_1(N)_p) \cdot W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}}(1) \neq 0$$

by Proposition A.3. Hence, up to a nonzero scalar, (17) becomes, after using Proposition A.3:

$$\int_{(U\backslash \mathrm{PGL}_2 \times U\backslash \mathrm{PGL}_2)(\mathbb{Q}_p)} W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}(g_1) W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}(g_2) W^0_{\pi_1, \psi_{\mathbb{Q}_p}}(g_1) W^0_{\pi_{2,p}, \psi_{\mathbb{Q}_p}}(g_2) \varphi^0_{N,p}(g_1) \varphi^0_{N,p}(g_2) \mathrm{d}g_1 \mathrm{d}g_2. \tag{20}$$

This factors into the product of the two integrals

$$\int_{(U\backslash \mathrm{PGL}_2)(\mathbb{Q}_p)} W^0_{\pi_1, \psi_{\mathbb{Q}_p}}(g_1) W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}(g_1) \varphi^0_{N,p}(g_1) \mathrm{d}g_1, \tag{21}$$

$$\int_{(U\backslash \mathrm{PGL}_2)(\mathbb{Q}_p)} W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}(g_2) W^0_{\pi_{2,p}, \psi_{\mathbb{Q}_p}}(g_2) \varphi^0_{N,p}(g_2) \mathrm{d}g_2. \tag{22}$$

Now, because $\varphi^0_{N,p}$ is supported on $B(\mathbb{Q}_p)K_1(N)_p$ by definition, and the local newforms are all $K_1(N)_p$-invariant, (21) and (22) become, up to nonzero scalars:

$$\int_{\mathbb{Q}_p^\times} W^0_{\pi_1, \psi_{\mathbb{Q}_p}}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) |t| \mathrm{d}^\times t,$$
$$\int_{\mathbb{Q}_p^\times} W^0_{\pi_2, \psi_{\mathbb{Q}_p}}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) W^0_{\pi_{2,p}^\vee, \psi_{\mathbb{Q}_p}^{-1}}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) |t| \mathrm{d}^\times t$$

Now an easy computation using Proposition A.3 shows that both of these integrals are nonzero, which proves the lemma. ☐

Theorem A.11. *For $N = 11$ and all $N \geq 13$, and for all $b : (\mathbb{Z}/N\mathbb{Z})^2 \hookrightarrow (\mathbb{Z}/N\mathbb{Z})^{2g}$, we have*

$$0 \neq [\widetilde{\mathsf{NL}}^b_{2,1}(N)]^+ \in H^4(\mathcal{A}_1(N) \times \mathcal{A}_2(N), \mathbb{Q}).$$

Proof. Combining Proposition A.1 with Lemmas A.2 and A.10, it suffices to show that, for all such N, there exists $N_0 | N$ satisfying the following condition:

(∗) There exist cuspidal newforms f_1 and f_2 of weights 4 and 2, respectively, for $\Gamma_1(N_0)$, with equal central characters ε.

First, we note that $p = 11$ and all primes $p > 13$ satisfy (∗). Indeed, for such p it is known that there exists a cuspidal newform f of weight 2 for $\Gamma_0(p)$, cf. [8, Proposition B.3]; then f^2 is a cuspidal eigenform of weight 4 for $\Gamma_0(p)$, which is necessarily new because there are no cusp forms of weight 4 for $\mathrm{SL}_2(\mathbb{Z})$.

To exhibit more integers satisfying $(*)$, we give the following table (sorted by prime factorization of N_0), in which all the data and labels are taken from [13].

N_0	f_1	f_2	ε
$16 = 2^4$	16.4.e.a	16.2.e.a	16.e
$27 = 3^3$	27.4.a.a	27.2.a.a	triv
$25 = 5^2$	25.4.d.a	25.2.d.a	25.d
$49 = 7^2$	49.4.a.a	49.2.a.a	triv
13	13.4.e.a	13.2.e.a	13.e
$24 = 2^3 \cdot 3$	24.4.a.a	24.2.a.a	triv
$18 = 2 \cdot 3^2$	18.4.c.a	18.2.c.a	18.c
$20 = 2^2 \cdot 5$	20.4.a.a	20.2.a.a	triv
$14 = 2 \cdot 7$	14.4.a.a	14.2.a.a	triv
$15 = 3 \cdot 5$	15.4.a.a	15.2.a.a	triv
$21 = 3 \cdot 7$	21.4.a.a	21.2.a.a	triv
$35 = 5 \cdot 7$	35.4.a.a	35.2.a.a	triv

Now a direct calculation shows that all N as in the theorem are divisible by either 11, a prime $p > 13$, or one of the N_0 appearing in the table, which completes the proof. $\square$

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