ON THE BLOCH-KATO CONJECTURE FOR SOME FOUR-DIMENSIONAL SYMPLECTIC GALOIS REPRESENTATIONS

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ABSTRACT. The Bloch-Kato conjecture predicts a far-reaching connection between orders of vanishing of L-functions and the ranks of Selmer groups of p-adic Galois representations. In this article, we consider the four-dimensional, symplectic Galois representations arising from automorphic representations π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with trivial central character and with the lowest cohomological archimedean weight. Under mild technical conditions, we prove that the Selmer group vanishes when the central value $L(\pi, \mathrm{spin}, 1/2)$ is nonzero. In the spirit of bipartite Euler systems, we bound the Selmer group by using level-raising congruences to construct ramified Galois cohomology classes. The relation to L-values comes via the $\mathrm{GSpin}_3 \hookrightarrow \mathrm{GSpin}_5$ periods on a compact inner form of GSp_4. We also prove a result towards the rank-one case: if the π-isotypic part of the Abel-Jacobi image of any of Kudla's one-cycles on the Siegel threefold is nonzero, it generates the full Selmer group. These cycles are linear combinations of embedded quaternionic Shimura curves, and under the conjectural arithmetic Rallis inner product formula, their heights are related to $L'(\pi, \mathrm{spin}, 1/2)$.

CONTENTS

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0. INTRODUCTION

0.1. **Main results.** Let $\pi = \otimes' \pi_v$ be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of trivial central character and conductor $N(\pi)$, such that π_∞ belongs to the discrete series L-packet of parallel weight $(3,3)$. In particular, π appears in the étale cohomology of the GSp_4 Shimura variety with trivial coefficients. Let E be a sufficiently large coefficient field; then we have a compatible family of p-adic Galois representations

$$\rho_{\pi,\mathfrak{p}} : G_\mathbb{Q} \to \mathrm{GSp}_4(E_\mathfrak{p})$$

indexed by primes $\mathfrak{p}|p$ of E, and normalized so that the similitude character of $\rho_{\pi,\mathfrak{p}}$ is cyclotomic. Let $V_{\pi,\mathfrak{p}}$ be the underlying four-dimensional $E_\mathfrak{p}[G_\mathbb{Q}]$-module of $\rho_{\pi,\mathfrak{p}}$, and consider the Bloch-Kato Selmer group

$$H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = \ker\left(H^1(\mathbb{Q}, V_{\pi,\mathfrak{p}}) \to H^1(\mathbb{Q}_p, V_{\pi,\mathfrak{p}} \otimes B_{\mathrm{cris}}) \times \prod_{\ell \neq p} H^1(I_{\mathbb{Q}_\ell}, V_{\pi,\mathfrak{p}})\right),$$

where $I_{\mathbb{Q}_\ell}$ is the local inertia subgroup. The Bloch-Kato conjecture applied to $V_{\pi,\mathfrak{p}}$ predicts

$$\dim_{E_\mathfrak{p}} H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = \mathrm{ord}_{s=1/2} L(\pi, \mathrm{spin}, 1/2),$$

where the L-function is normalized so that $s = 1/2$ is the central value. Our first main result proves many cases of the Bloch-Kato conjecture for $V_{\pi,\mathfrak{p}}$ in rank zero.

Theorem A (Theorem 9.1.4)**.** *Suppose π is not CAP or endoscopic, and for some $\ell|N(\pi)$, π_ℓ has a local Jacquet-Langlands transfer to the compact inner form of* $\mathrm{GSp}_{4,\mathbb{Q}_\ell}$. *Let $\mathfrak{p}|p$ be a prime of E such that:*

1. *π_p is unramified.*
2. *The residual representation $\overline{\rho}_{\pi,\mathfrak{p}}$ is absolutely irreducible and generic (Definition 2.7.3).*
3. *There exists a prime $q \nmid N(\pi)$ such that $q^4 \not\equiv 1 \pmod{p}$, and $\overline{\rho}_{\pi,\mathfrak{p}}(\mathrm{Frob}_q)$ has eigenvalues*

$$\{q, 1, \alpha, q/\alpha\}$$

with $\alpha \notin \{\pm 1, \pm q, q^2, q^{-1}\}$.

Then

$$L(\pi, \mathrm{spin}, 1/2) \neq 0 \implies H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = 0.$$

Remarks.
- (i) The conditions (1) and (2) are always satisfied for cofinitely many $\mathfrak{p}$. Condition (3) is always satisfied for the primes $\mathfrak{p}$ lying over a set of rational primes of positive Dirichlet density; it is satisfied for all but finitely many $\mathfrak{p}$ under conditions listed in Theorem C.4.11.
- (ii) When $p > 5$, which is necessary for (3), π is CAP or endoscopic if and only if $V_{\pi,\mathfrak{p}}$ is reducible.
- (iii) For all but finitely many of the primes $\mathfrak{p}$ satisfying the conditions in Theorem A, we are able to strengthen the result to the vanishing of the dual Selmer group $H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}/T_{\pi,\mathfrak{p}})$, where $T_{\pi,\mathfrak{p}} \subset V_{\pi,\mathfrak{p}}$ is a Galois-stable lattice; see Definition 9.3.7 and Corollary 9.3.9.

For instance, from Theorem A we can deduce the following:

Corollary B (Corollary 9.3.10)**.** *Suppose π is not CAP or endoscopic, and there exists a prime $\ell|N(\pi)$ such that π_ℓ is of type IIa in the notation of* [95]. *Then*

$$L(\pi, \mathrm{spin}, 1/2) \neq 0 \implies H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}/T_{\pi,\mathfrak{p}}) = 0$$

for all but finitely many $\mathfrak{p}$.

In §9.4, we give applications of Theorem A to certain π arising by automorphic induction; this gives new results towards Bloch-Kato for Hilbert modular forms over real quadratic fields of non-parallel weight $(2,4)$ (including CM forms), and for twists of classical modular forms of weight 3 by certain Hecke characters of imaginary quadratic fields.

Our second main result concerns the rank one case of Bloch-Kato for $V_{\pi,\mathfrak{p}}$. To state it, let V be a quadratic space of signature $(3,2)$, and suppose π admits a Jacquet-Langlands transfer Π to the inner form $\mathrm{GSpin}(V)$ of GSp_4; for instance, if V is split, then $\mathrm{GSpin}(V) = \mathrm{GSp}_4$ and we can take $\Pi = \pi$. Let $K \subset \mathrm{GSpin}(V)(\mathbb{A}_{\mathbb{Q}})$ be a neat compact open subgroup such that $\Pi_f^K \neq 0$, and let $\mathrm{Sh}_K(V)$ be the Shimura variety for $\mathrm{GSpin}(V)$ at level K, which is a classical Siegel threefold when V is split. Let $\mathfrak{p}|p$ be a prime of E; then we have

$$H^i_{\text{ét}}(\mathrm{Sh}_K(V)_{\overline{\mathbb{Q}}}, E_{\mathfrak{p}})[\Pi_f] = \begin{cases} \Pi_f^K \otimes V_{\pi,\mathfrak{p}}(-2), & i = 3, \\ 0, & i \neq 3. \end{cases} \tag{0.1}$$

In particular, if $\mathrm{CH}^2(\mathrm{Sh}_K(V))_{\mathfrak{m}_{\pi_f}}$ denotes the Chow group of codimension-two algebraic cycles, localized at the maximal ideal of an appropriate Hecke algebra corresponding to the eigenvalues of π, we have an Abel-Jacobi map

$$\begin{aligned} \partial_{\mathrm{AJ},\Pi_f} : \mathrm{CH}^2(\mathrm{Sh}_K(V))_{\mathfrak{m}_{\pi_f}} &\to H^1_f(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V)_{\overline{\mathbb{Q}}}, E_{\mathfrak{p}}(2))[\Pi_f]) \\ &= H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) \otimes \Pi_f^K. \end{aligned} \tag{0.2}$$

We consider the codimension-two Kudla cycles

$$Z(T,\varphi) \in \mathrm{CH}^2(\mathrm{Sh}_K(V)), \tag{0.3}$$

which are indexed by the data of a 2×2 symmetric, positive-semidefinite matrix T and a test function $\varphi \in \mathcal{S}(V^2 \otimes \mathbb{A}_f, \mathbb{Z})$. When T is nondegenerate, $Z(T,\varphi)$ is a linear combination of Shimura curves over $\mathbb{Q}$, embedded into $\mathrm{Sh}_K(V)$ by $\mathrm{GSpin}(V)(\mathbb{A}_f)$-translates of group maps of the form

$$\mathrm{GSpin}(1,2) \hookrightarrow \mathrm{GSpin}(3,2);$$

conversely, any such embedded Shimura curve can be written as a Kudla cycle for some T and φ.

Theorem C (Theorem 12.3.1). *Suppose π is not CAP or endoscopic, and its base change to $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ does not arise by automorphic induction from an imaginary quadratic or quartic CM field. Let $\mathfrak{p}|p$ be a prime of E satisfying (1)-(3) from Theorem A. Then for any Kudla cycle $Z(T,\varphi) \in \mathrm{CH}^2(\mathrm{Sh}_K(V))$,*

$$\partial_{\mathrm{AJ},\Pi_f}(Z(T,\varphi)) \neq 0 \implies \dim_{E_{\mathfrak{p}}} H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = 1.$$

We now explain the relation of the Bloch-Kato Conjecture to Theorem C, in which L-values do not directly appear. According to the resolution by Bruinier and Westerholt-Raum of Kudla's modularity conjecture [15], the formal q-series

$$\Theta_\varphi := \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}} Z(T,\varphi) q^T \tag{0.4}$$

lies in $\mathrm{CH}^2(\mathrm{Sh}_K(V)) \otimes_{\mathbb{Z}} M_{5/2,2}$, where $M_{5/2,2}$ is the space of holomorphic Siegel modular forms of degree 2 and weight 5/2. Let $f \in M_{5/2,2}$ generate the automorphic representation of the metaplectic group $\mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}})$ corresponding to Π under the generalized Shimura-Waldspurger correspondence of Gan-Li [33]; then the Petersson inner product $\Theta_\varphi(f) := \langle \Theta_\varphi, f \rangle$ lies in $\mathrm{CH}^2(\mathrm{Sh}_K(V))_{\mathbb{C}}$. The still-conjectural[1] *arithmetic Rallis inner product formula* proposes that, up to some local factors depending on φ and f, the derivative $L'(\pi, \mathrm{spin}, 1/2)$ is related to the height of the cycle $\Theta_\varphi(f)$ (suitably interpreted). Assuming Beilinson's conjecture on the injectivity of the p-adic Abel-Jacobi map, Theorem C could then be reformulated as

$$L'(\pi, \mathrm{spin}, 1/2) \neq 0 \implies \dim_{E_{\mathfrak{p}}} H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = 1,$$

which is consistent with the Bloch-Kato conjecture.

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[1] But see the important progress towards this result by Li-Zhang [63], and the unitary case proved by Li-Liu [61, 62].

0.2. **Overview of the proofs.** Before giving more detailed sketches below, we briefly indicate the methods of proof of Theorems A and C. Pursuing a strategy initiated by Bertolini and Darmon for elliptic curves [7] – and later extended to many new contexts by other authors [24, 26, 66, 67, 68, 69, 120] – we bound the Selmer group by constructing ramified Galois cohomology classes through level-raising congruences and special cycles on Shimura varieties for ramified GSpin_5 groups. Proving the classes we construct are ramified is the most delicate part: by a calculation on the special fiber, the ramification is essentially measured by linear combinations of compact $\mathrm{GSpin}_3 - \mathrm{GSpin}_5$ periods for a Jacquet-Langlands transfer of π.[2] To access the underlying representation theory of these periods and relate them to L-values, we interpret them as the Fourier coefficients of certain theta lifts in $M_{5/2,2}$.

0.3. **Sketch of the proof in the rank zero case.** Let us explain the main ideas involved in the proof of Theorem A. Fix a $G_{\mathbb{Q}}$-stable lattice $T_{\pi,\mathfrak{p}} \subset V_{\pi,\mathfrak{p}}$, and let $T_{\pi,n} \coloneqq T_{\pi,\mathfrak{p}}/\mathfrak{p}^n T_{\pi,\mathfrak{p}}$ for $n \geq 1$. The mechanism for bounding the Selmer group is a collection of auxiliary Galois cohomology classes

$$\kappa_n(q) \in H^1(\mathbb{Q}, T_{\pi,n}),$$

indexed by primes q satisfying appropriate level-raising congruence conditions modulo $\mathfrak{p}^n$, and having the following two properties:

(1) The restriction $\mathrm{Res}_\ell \, \kappa_n(q)$ is unramified for all $\ell \nmid N(\pi)q$, and crystalline for $\ell = p$.
(2) Under the assumption $L(\pi, \mathrm{spin}, 1/2) \neq 0$, $\mathrm{Res}_q \, \kappa_n(q)$ is ramified (if n is sufficiently large).

Given such a system of classes, a standard argument using Poitou-Tate duality shows that $H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}})$ vanishes.

As indicated in §0.2, we obtain the classes $\kappa_n(q)$ by level-raising congruences from special cycles on Shimura varieties for ramified GSpin_5 groups. More precisely, fix the prime $\ell | N(\pi)$ such that π_ℓ has a local Jacquet-Langlands transfer (we will see later that this is crucial for the argument). Let $V_{q\ell}$ be the unique five-dimensional quadratic space of signature $(3, 2)$ and trivial discriminant that ramifies precisely at q and ℓ; one exists by the local-global classification of quadratic forms. On the corresponding GSpin Shimura variety $\mathrm{Sh}(V_{q\ell})$ we have the codimension-two Kudla cycles $Z(T, \varphi) \in \mathrm{CH}^2(\mathrm{Sh}(V_{q\ell}))$ as in §0.1. (For simplicity, we suppress the choice of level structure.) After localizing at the maximal ideal $\mathfrak{m}$ of the Hecke algebra corresponding to $\overline{\rho}_{\pi,\mathfrak{p}}$, the main result of [42] implies that $H^4(\mathrm{Sh}(V_{q\ell}), O_{\mathfrak{p}})_{\mathfrak{m}} = 0$, where $O \subset E$ is the ring of integers – this is the most crucial way that the condition (2) of Theorem A enters the argument. In particular, one has an Abel-Jacobi map

$$\partial_{\mathrm{AJ},\mathfrak{m}} : \mathrm{CH}^2(\mathrm{Sh}(V_{q\ell}))_{\mathfrak{m}} \to H^1(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell}), O_{\mathfrak{p}}(2))_{\mathfrak{m}}).$$

However, it is no longer true that $V_{\pi,\mathfrak{p}}$ appears in the étale cohomology $H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell})_{\overline{\mathbb{Q}}}, E_{\mathfrak{p}})$, because, as π_q is unramified, π does not have a Jacquet-Langlands transfer to $\mathrm{GSpin}(V_{q\ell})$.

Instead, we construct *level-raising maps* (see below)

$$\alpha_n : H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell}), O_{\mathfrak{p}}(2))_{\mathfrak{m}} \to T_{\pi,n}.$$

Then we obtain a family of Galois cohomology classes

$$\kappa_n(q, T, \varphi, \alpha_n) \coloneqq \alpha_n \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T, \varphi)) \in H^1(\mathbb{Q}, T_{\pi,n}),$$

for varying T and $\varphi \in \mathcal{S}(V_{q\ell}^2 \otimes \mathbb{A}_f, \mathbb{Z})$. Any $\kappa_n(q, T, \varphi, \alpha_n)$ will satisfy the property (1) above, as long as the level structure for $\mathrm{Sh}(V_{q\ell})$ is chosen to be hyperspecial outside $N(\pi)q$. The proof of (2) is far more subtle, as we explain below, and we are only able to show that *some* $\kappa_n(q, T, \varphi, \alpha_n)$ is ramified at q; in particular, the choice of T cannot be made explicit. Once we have any ramified class, however, we can use it as the class called $\kappa_n(q)$ above.

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[2] Or a congruent automorphic form to π, in the case of Theorem C.

0.3.1. *Ramification and the relation to L-values.* Let V_ℓ be the unique five-dimensional, positive-definite quadratic space of trivial discriminant that ramifies only at ℓ. By the assumption that π_ℓ has a local Jacquet-Langlands transfer, one can find a function

$$\beta_\pi : \mathrm{GSpin}(V_\ell)(\mathbb{Q})\backslash \mathrm{GSpin}(V_\ell)(\mathbb{A}_f)/K \to O \tag{0.5}$$

with the same spherical Hecke eigenvalues as π, where K is a sufficiently small compact open subgroup, with K_q hyperspecial. We abbreviate this double coset space as $\mathrm{Sh}(V_\ell)$. Because π has trivial central character, β_π descends to an automorphic form on $\mathrm{SO}(V_\ell)$; in particular, for any $\varphi \in \mathcal{S}(V_\ell^2 \otimes \mathbb{A}_f, \mathbb{Z})$, one can consider the classical theta lift

$$\Theta_\varphi(\beta_\pi) \in M_{5/2,2} \otimes_{\mathbb{Z}} O.$$

The classical Rallis inner product formula computes the Petersson norm of $\Theta_\varphi(\beta_\pi)$:

$$\langle \Theta_\varphi(\beta_\pi), \Theta_\varphi(\beta_\pi)\rangle \doteq \langle \beta_\pi, \beta_\pi\rangle \cdot L(\pi, \mathrm{spin}, 1/2) \tag{0.6}$$

up to local factors depending on φ and β_π.

On the other hand, the Fourier coefficients of $\Theta_\varphi(\beta_\pi)$ can be computed as GSpin_3-periods of β_π. In fact, for $T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$ and $\varphi \in \mathcal{S}(V_\ell^2 \otimes \mathbb{A}_f, \mathbb{Z})$, one can define $Z(T, \varphi) \in \mathbb{Z}[\mathrm{Sh}(V_\ell)]$ analogously to Kudla's cycles in (0.3), arising from group embeddings $\mathrm{GSpin}_3 \hookrightarrow \mathrm{GSpin}(V_\ell)$. Then we have:

$$\Theta_\varphi(\beta_\pi) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}} \beta_\pi(Z(T,\varphi))q^T.$$

In particular,

$$L(\pi, \mathrm{spin}, 1/2) \neq 0 \iff \exists\, T, \varphi \text{ s.t. } \beta_\pi(Z(T,\varphi)) \neq 0. \tag{0.7}$$

The connection to Galois cohomology classes comes from the following (idealized) identity, for a certain choice of α_n:

$$\mathrm{Res}_q\, \kappa_n(q, T, \varphi^q \otimes \varphi_q^{\mathrm{ram}}, \alpha_n) = \beta_\pi(Z(T, \varphi^q \otimes \varphi_q^{\mathrm{unr}})) \pmod{\mathfrak{p}^n}. \tag{0.8}$$

Here $\varphi^q \in \mathcal{S}(V_{q\ell}^2 \otimes \mathbb{A}_f^q, \mathbb{Z}) \simeq \mathcal{S}(V_\ell^2 \otimes \mathbb{A}_f^q, \mathbb{Z})$ is any test function, $\varphi_q^{\mathrm{ram}} \in \mathcal{S}(V_{q\ell}^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ is designed so that $Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}}) \in \mathrm{CH}^2(\mathrm{Sh}(V_{q\ell}))$ will have a reasonable integral model over $\mathbb{Z}_{(q)}$ – in fact, $Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}})$ is a linear combination of quaternionic Shimura curves ramified at q, with the usual maximal compact level structure – and $\varphi_q^{\mathrm{unr}} \in \mathcal{S}(V_\ell^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ is an explicit test function whose origin is described in the next paragraph. Finally, to make sense of the identity (0.8), we use that $H^1(I_{\mathbb{Q}_q}, T_{\pi,n})^{\mathrm{Frob}_q = 1}$ is free of rank one over $O/\mathfrak{p}^n$, by the congruence conditions imposed on q; so we are viewing both sides as elements of $O/\mathfrak{p}^n$.

The main tool for proving (0.8) is an analysis of the weight spectral sequence for a semistable integral model of $\mathrm{Sh}(V_{q\ell})$ over $\mathbb{Z}_{(q)}$; this is obtained by blowup from the canonical PEL model, which has ordinary quadratic singularities. The double coset space $\mathrm{Sh}(V_\ell)$ enters the picture as the indexing set for the irreducible components of the (two-dimensional) supersingular locus of the special fiber of $\mathrm{Sh}(V_{q\ell})$, and indeed φ_q^{unr} arises from considering the intersections of the special fiber of $Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}})$ with various strata in the supersingular locus. (Of course, for this discussion to be accurate, the level structures for $\mathrm{Sh}(V_{q\ell})$ and $\mathrm{Sh}(V_\ell)$ are chosen to be compatible away from q.)

However, even once we have proved (0.8), one major obstacle remains to proving that at least one of the classes $\mathrm{Res}_q\, \kappa_n(q, T, \varphi^q \otimes \varphi_q^{\mathrm{ram}}, \alpha_n)$ is ramified. From (0.7), one can deduce that there exists φ^q and T such that $\beta_\pi(Z(T, \varphi^q \otimes \varphi_q^{\mathrm{sph}})) \neq 0 \pmod{\mathfrak{p}^n}$ for sufficiently large n, where φ_q^{sph} is the indicator function of a self-dual lattice. Unfortunately, although φ_q^{unr} is invariant by a hyperspecial subgroup K_q of $\mathrm{GSpin}(V_\ell)(\mathbb{Q}_q)$, it is not equal to φ_q^{sph}. To get around this, we give a criterion on K_q-invariant functions $\varphi_q \in \mathcal{S}(V_\ell^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ under which we can prove:

$$\beta_\pi(Z(T, \varphi^q \otimes \varphi_q^{\mathrm{sph}})) \neq 0 \pmod{\mathfrak{p}^n} \implies \exists\, T' \text{ s.t. } \beta_\pi(Z(T', \varphi^q \otimes \varphi_q)) \neq 0 \pmod{\mathfrak{p}^n}. \tag{0.9}$$

The function φ_q^{unr} satisfies the criterion, so indeed we can use (0.7), (0.8), and (0.9) to obtain the local ramification of some class $\mathrm{Res}_q\,\kappa_n(q, T', \varphi, \alpha_n)$. The lack of control over T' (in particular its possible dependence on q) is the reason that we must consider the full family of classes $\kappa_n(q, T, \varphi, \alpha_n)$.

The proof of (0.9) uses that the theta lift map

$$\Theta_-(\beta_\pi) : \mathcal{S}(V_\ell^2 \otimes \mathbb{A}_f, \mathbb{Z}) \to M_{5/2,2} \otimes_{\mathbb{Z}} O$$

factors as a product of local maps, defined purely in terms of the local Weil representation. Thus (0.9) can be reduced to a local question about the mod-p Weil representation of $\mathrm{Mp}_4(\mathbb{Q}_q) \times \mathrm{SO}(V_\ell)(\mathbb{Q}_q)$ on $\mathcal{S}(V_\ell, \overline{\mathbb{F}}_p)$.

0.3.2. *Level-raising.* In the discussion above, we asserted the existence of a level raising map

$$\alpha_n : H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell}), O_{\mathfrak{p}}(2))_{\mathfrak{m}} \to T_{\pi,n} \tag{0.10}$$

such that the classes $\kappa_n(q, T, \varphi^q \otimes \varphi_q^{\mathrm{ram}}, \alpha_n)$ satisfy the identity (0.8). We now explain the construction of α_n in more detail. The weight spectral sequence for $\mathrm{Sh}(V_{q\ell})$ gives rise to a diagram of the following form:

$$\begin{tikzcd}
M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \arrow[r, hook] \arrow[d, two heads, "\xi"] & H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)^{\mathrm{Frob}_q=1} \arrow[ddl, dashed, "\alpha_{n,*}"] \\
O_{\mathfrak{p}}\left[\mathrm{Sh}(V_\ell)\right]_{\mathfrak{m}}/(\mathtt{T}_q^{\mathrm{lr}}) \arrow[d, "\beta_\pi"] & \\
O/\mathfrak{p}^n \simeq H^1(I_{\mathbb{Q}_q}, T_{\pi,n})^{\mathrm{Frob}_q=1}, &
\end{tikzcd} \tag{0.11}$$

where $\mathtt{T}_q^{\mathrm{lr}}$ is a certain spherical Hecke operator measuring the level-raising congruence at the prime q; M_{-1} refers to the monodromy filtration; and the equality in the bottom row uses the level-raising condition on q. Any level-raising map as in (0.10) induces a map $\alpha_{n,*}$ as in (0.11), not necessarily making the diagram commute.

Using an idea of Scholze [98] on "typic-ness" of Galois modules, we show that the data of a Hecke-equivariant $\alpha_{n,*}$ is actually *equivalent* to the data of α_n. Moreover, the identity (0.8) is more or less equivalent to the commutativity of the diagram (0.11). So our task is to find the Hecke-equivariant dashed arrow in (0.11) lifting $\beta_\pi \circ \xi$.

One approach to finding the lift in (0.11) would be Taylor-Wiles patching, in the spirit of [65]. This would show that $H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell})_{\overline{\mathbb{Q}}}, O)_{\mathfrak{m}}$ is free as a Hecke module, and as a byproduct that the top arrow in (0.11) is an isomorphism. However, the drawback of the patching argument is that it requires some strong "residual ramification" conditions on $\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}_\ell}}$, for primes $\ell | N(\pi)$; in general, one expects that the Hecke-module structure of $H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell})_{\overline{\mathbb{Q}}}, O)_{\mathfrak{m}}$ may be more complicated.

Instead of taking this route, our method still uses Galois deformation theory, but now only to obtain a rather coarse quantitative control on the possible congruences between $\beta_\pi \circ \xi$ and other Hecke eigensystems in $H^1(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}(V_{q\ell})_{\overline{\mathbb{Q}}}, O_{\mathfrak{p}}(2))_{\mathfrak{m}})$. Indeed, if there were no congruences at all, then (0.11) would just be a diagram of $O/\mathfrak{p}^n$-modules, and the lift $\alpha_{n,*}$ would exist for trivial reasons. In general, we are able to lift $\beta_\pi \circ \xi$ only after multiplying by a generator of $\mathfrak{p}^C$, where C is essentially the length of a certain adjoint Selmer group for $T_{\pi,n}$. We can control C using a theorem of Newton and Thorne [83, 110] that $H^1_f(\mathbb{Q}, \mathrm{ad}^0\,\rho_{\pi,\mathfrak{p}}) = 0$; since C is independent of n and q (and actually 0 for cofinitely many $\mathfrak{p}$) its appearance causes no harm in the argument.[3]

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[3] We remark that the theorem of Newton and Thorne still uses Taylor-Wiles patching, but in a more flexible context.

0.4. **Sketch of the proof in the rank one case.** Let V be the quadratic space from §0.1. For an auxiliary prime q_1 satisfying a level-raising congruence condition, we consider the "nearby" quadratic space V_{q_1}, which has trivial discriminant and signature $(5, 0)$, and is ramified precisely at q_1 and the primes of ramification for V.

The key point to prove Theorem C is to produce a Hecke-equivariant map of the form

$$\beta_{q_1} : \mathrm{Sh}(V_{q_1}) \to O/\mathfrak{p}^n \text{ s.t. } \exists\, T, \varphi \text{ with } \beta_{q_1}(Z(T, \varphi)) \neq 0, \tag{0.12}$$

where φ lies in $\mathcal{S}(V_{q_1}^2 \otimes \mathbb{A}_f, \mathbb{Z})$, $\mathrm{Sh}(V_{q_1})$ is a finite double coset space like the one in (0.5), the Hecke module structure on $O/\mathfrak{p}^n$ is given by π, and again the choice of level structure is suppressed for simplicity. Once we have β_{q_1}, a similar argument to the proof of Theorem A allows us to produce a Galois cohomology class $\kappa_n(q_1q_2) \in H^1(\mathbb{Q}, T_{\pi,n})$ which is ramified at q_2, crystalline at p, and unramified at ℓ for $\ell \nmid N(\pi)q_1q_2$. This class is then used as the input to the duality argument to bound $H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}})$.

Let $Z(T, \varphi) \in \mathrm{CH}^2(\mathrm{Sh}(V))$ be the class from Theorem C with $\partial_{\mathrm{AJ},\Pi_f}(Z(T, \varphi)) \neq 0$. The idea to find β_{q_1} is to study the special fiber of $\mathrm{Sh}(V)$ at q_1, a prime of good reduction. The supersingular locus is purely one-dimensional, indexed by a Shimura set $\mathrm{Sh}(V_{q_1})$. We have the Abel-Jacobi map on the special fiber:

$$\partial_{\mathrm{AJ},\mathfrak{m}} : \mathrm{CH}^2(\mathrm{Sh}(V)_{\mathbb{F}_{q_1}}) \to H^1(\mathbb{F}_{q_1}, H^3_{\text{ét}}(\mathrm{Sh}(V)_{\overline{\mathbb{Q}}}, O_{\mathfrak{p}}(2))_{\mathfrak{m}}). \tag{0.13}$$

Restricting (0.13) to supersingular cycles gives a map

$$\partial_{\mathrm{AJ},\mathfrak{m},ss} : \mathbb{Z}[\mathrm{Sh}(V_{q_1})] \to H^1(\mathbb{F}_{q_1}, H^3_{\text{ét}}(\mathrm{Sh}(V)_{\overline{\mathbb{Q}}}, O_{\mathfrak{p}}(2))_{\mathfrak{m}}). \tag{0.14}$$

Composing with a map $H^3_{\text{ét}}(\mathrm{Sh}(V)_{\overline{\mathbb{Q}}}, O_{\mathfrak{p}}(2))_{\mathfrak{m}} \to T_{\pi,\mathfrak{p}}$, cf. (0.1), and noting that $H^1(\mathbb{F}_{q_1}, T_{\pi,\mathfrak{p}})$ is free of rank one over $O/\mathfrak{p}^n$ when q_1 satisfies the level-raising congruence condition modulo $\mathfrak{p}^n$, we obtain our β_{q_1}.

Using the Chebotarev density theorem and the local-global compatibility of the Abel-Jacobi map, one can ensure $\partial_{\mathrm{AJ},\mathfrak{m}}(Z(T, \varphi)_{\mathbb{F}_{q_1}})$ is nonzero for a good choice of q_1. Now, if the special fiber $Z(T, \varphi)_{\mathbb{F}_{q_1}}$ were purely supersingular, then we could identify this local Abel-Jacobi class with the image of a special cycle in $\mathbb{Z}[\mathrm{Sh}(V_{q_1})]$ under β_{q_1}, and obtain the nonvanishing (0.12). (This is the strategy of [7], where the role of $Z(T, \varphi)$ is played by a Heegner point.) Unfortunately, such is not the case for general $Z(T, \varphi)$.[4]

Instead, we use an auxiliary GSpin_4 Shimura variety S such that $Z(T, \varphi) \subset S \subset \mathrm{Sh}(V)$. On the special fiber of S, the supersingular locus is one-dimensional, and – by [111] – any one-cycle is cohomologically equivalent to a linear combination of supersingular cycles, at least after the application of a certain GSpin_4-Hecke operator $\mathtt{T}_{q_1}$. We leverage this to rewrite $\partial_{\mathrm{AJ},\mathfrak{m}}(\mathtt{T}_{q_1} \cdot Z(T, \varphi)_{\mathbb{F}_{q_1}})$ as the image of some $Z(T, \varphi')$ under $\partial_{\mathrm{AJ},\mathfrak{m},ss}$, with $\varphi' \in \mathcal{S}(V_{q_1}^2 \otimes \mathbb{A}_f, \mathbb{Z})$.

Now it remains to choose q_1 so that $\partial_{\mathrm{AJ},\mathfrak{m}}(\mathtt{T}_{q_1} \cdot Z(T, \varphi)_{\mathbb{F}_{q_1}})$ is nontrivial. For this choice to be possible, we need some non-entanglement between $\rho_{\pi,\mathfrak{p}}$ and the Galois representations appearing in the cohomology of S, which are closely related to Hilbert modular forms. In particular, the support of $Z(T, \varphi)$ in the cohomology of S needs to avoid some problematic Hecke eigensystems associated to CM forms. To ensure this, we use a version of the modifying-the-test-function trick from (0.9), this time at (non-level-raising) auxiliary primes $\ell \neq q_1, q_2$. Since the trick uses representation theory, it relies on the modularity conjecture recalled after the statement of Theorem C. (The subtleties that appear in this part of the argument prevent us from bounding the dual Selmer group $H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}/T_{\pi,\mathfrak{p}})$ in the context of Theorem C, in contrast to the rank zero case.)

One final difficulty that arises in the proof of Theorem C is the application of the level-raising arguments in §0.3.2 to β_{q_1}. Since β_{q_1} does not lift to a characteristic-zero Hecke eigenfunction in general, we cannot directly apply the theorem of Newton and Thorne to produce the lift in the diagram (0.11). Instead, we use the relative deformation theory developed by Fakhruddin, Khare, and Patrikis, along with a certain control of level-raised adjoint Selmer groups, to show that either β_{q_1} lifts to an eigenfunction, or the Hecke eigensystem of π is congruent modulo $\mathfrak{p}^n$ to that of an automorphic representation π' ramified at q_1 *and* q_2; in either case, we leverage the congruence to solve the lifting problem in (0.11). It is at this step that the extra hypothesis in

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[4] One could imagine modifying φ_q to ensure a purely supersingular special fiber, and applying a version of (0.9) to ensure that $Z(T, \varphi)$ remains nontrivial under this change of test function. Somewhat to our surprise, this strategy fails: any appropriate choice of φ_q fails the criterion under which we prove (0.9).

Theorem C (that π does not arise from certain automorphic inductions) enters the argument; if the image of the Galois representation $\rho_{\pi,\mathfrak{p}}$ is too small, then we are unable to control the adjoint Selmer groups needed to apply Fakhruddin, Khare, and Patrikis' method.

0.5. **Comparison to other work.** When π_p is Borel-ordinary with respect to $\mathfrak{p}$, and under a slightly different large-image condition, Theorem A can be deduced from the main result of [71], which also covers more general weights. However, the existence of ordinary primes is still an open problem for general automorphic representations of GSp_4, so the ordinarity hypothesis is potentially a serious one.

As indicated in §0.2, this article fits into an extensive literature of bounding Selmer groups using level-raising congruences (the method of "bipartite Euler systems"). Compared to these works, one of the main underlying difficulties to prove Theorem A is the non-factorizability of $\mathrm{GSpin}_3 - \mathrm{GSpin}_5$ period integrals, which is ultimately responsible for the appearance of the exotic test function φ_q^{unr} (rather than the indictor function of a self-dual lattice or a translate of such by the spherical Hecke algebra) in the identity (0.8). Another way to formulate this obstruction is in terms of spherical functions and Hironaka's conjecture [45, 128]. To our knowledge, the only prior work on bipartite Euler systems facing this challenge was the recently appeared PhD thesis of Corato Zanarella [27]. Interestingly, rather than our change-of-test-functions technique, the issue in *op. cit.* is resolved using derived algebraic geometry; in our context, the analogous idea would be to work with $Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}})$ for more general φ_q^{ram}, even in the absence of a good integral model.

Another novel aspect in the present work is in finding the β_{q_1} in (0.12). To our knowledge, the idea of using intersection theory on the special fiber of an auxiliary Shimura variety S, in which the supersingular locus and the special cycle are of complementary dimension, is a new one.

Finally, we warn the reader that the "first and second reciprocity laws" (corresponding to finding $\kappa_n(q)$ and β_{q_1}, respectively, in the discussions above) in the table of contents are named only by analogy to [7]. In both cases, the full reciprocity law would constitute an equality, where we only prove an inequality, and that only up to a bounded error. The exact statements are Theorems 8.5.1 and 11.2.6.

0.6. **Comments on the endoscopic case.** An automorphic representation π as in the beginning of §0.1 is called *endoscopic* associated to a pair (π_1, π_2) of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ – necessarily arising from classical modular forms of weights 2 and 4, respectively, and trivial nebentypus characters – if the associated Galois representations satisfy

$$\rho_{\pi,\mathfrak{p}} = \rho_{\pi_1,\mathfrak{p}} \boxplus \rho_{\pi_2,\mathfrak{p}} \tag{0.15}$$

for one, or equivalently all, primes $\mathfrak{p}$ of E. Here we normalize $\rho_{\pi_i,\mathfrak{p}} : G_\mathbb{Q} \to \mathrm{GL}_2(E_\mathfrak{p})$ to have cyclotomic determinant; also let $V_{\pi_i,\mathfrak{p}}$ be the underlying Galois module. The set of π satisfying (0.15) is by definition the endoscopic L-packet $\Pi(\pi_1, \pi_2)$. If V is a quadratic space of signature $(3, 2)$ and $\Pi = \Pi_f \otimes \Pi_\infty$ is an automorphic representation of $\mathrm{GSpin}(V)(\mathbb{A}_\mathbb{Q})$ nearly equivalent to the members of $\Pi(\pi_1, \pi_2)$, then we have

$$H^i_{\text{ét}}(\mathrm{Sh}_K(V)_{\overline{\mathbb{Q}}}, E_\mathfrak{p})[\Pi_f] = \begin{cases} \Pi_f^K \otimes V_{\pi_1,\mathfrak{p}}(-2), & i = 3,\ \Pi_\infty \text{ generic}, \\ \Pi_f^K \otimes V_{\pi_2,\mathfrak{p}}(-2), & i = 3,\ \Pi_\infty \text{ holomorphic}, \\ 0 & i \neq 3. \end{cases} \tag{0.16}$$

Note here that Π_∞ is uniquely determined by Π_f via the Arthur multiplicity formula, which is known in this case [21]. In particular, we still have a well-defined map

$$\partial_{\mathrm{AJ},\Pi_f} : \mathrm{CH}^2(\mathrm{Sh}_K(V)) \to \Pi_f^K \otimes \left(H^1_f(\mathbb{Q}, V_{\pi_1,\mathfrak{p}} \oplus V_{\pi_2,\mathfrak{p}})\right). \tag{0.17}$$

The analogue of Theorem C is now:

Proposition D (Theorem 12.3.2)**.** *Suppose π_1 and π_2 are non-CM, and $\mathfrak{p}|p$ is a prime of E such that:*

(1) $\pi_{1,p}$ and $\pi_{2,p}$ are unramified.
(2) The residual representations $\overline{\rho}_{\pi_1,\mathfrak{p}}$ and $\overline{\rho}_{\pi_2,\mathfrak{p}}$ are both absolutely irreducible, and $\overline{\rho}_{\pi_1,\mathfrak{p}} \oplus \overline{\rho}_{\pi_2,\mathfrak{p}}$ is generic (Definition 2.7.3).

(3) *For both* $i = 1, 2$, *there exists a prime* $q \nmid N(\pi)$ *such that* $q^4 \not\equiv 1 \pmod{p}$, $\overline{\rho}_{\pi_i,\mathfrak{p}}(\mathrm{Frob}_q)$ *has eigenvalues* $\{1, q\}$, *and* $\overline{\rho}_{\pi_{3-i},\mathfrak{p}}(\mathrm{Frob}_q)$ *has eigenvalues* $\{\alpha, q/\alpha\}$ *with* $\alpha \notin \left\{\pm 1, \pm q, q^2, q^{-1}\right\}$.

Suppose as well that

$$(*) \qquad H^1_f(\mathbb{Q}, V_{\pi_1,\mathfrak{p}} \otimes V_{\pi_2,\mathfrak{p}}(-1)) = 0.$$

Then for any Kudla cycle $Z(T, \varphi) \in \mathrm{CH}^2(\mathrm{Sh}_K(V))$,

$$\partial_{\mathrm{AJ},\Pi_f}(Z(T, \varphi)) \neq 0 \implies \dim_{E_\mathfrak{p}} H^1_f(\mathbb{Q}, V_{\pi_1,\mathfrak{p}}) + \dim_{E_\mathfrak{p}} H^1_f(\mathbb{Q}, V_{\pi_2,\mathfrak{p}}) = 1.$$

Remarks. (i) The conditions (1)-(3) hold for cofinitely many $\mathfrak{p}$ (Lemma 4.1.5 and Proposition C.4.12).

(ii) The condition $(*)$ is always expected to hold, by a classical result of Shahidi on nonvanishing of Rankin-Selberg L-values [100]. Unfortunately, we remark that $(*)$ does *not* follow from the main result of [60], which applies only to Rankin-Selberg convolutions of modular forms with different central characters. The role of $(*)$ in the proof is to control congruences to non-endoscopic automorphic representations of GSpin_5 groups.

(iii) Consider the case when V is split, so that $\mathrm{Sh}_K(V)$ is a classical Siegel threefold, and π_1 is associated to an elliptic curve E over $\mathbb{Q}$. In particular, the Abel-Jacobi map (0.17) gives a way to construct classes in $H^1_f(\mathbb{Q}, T_pE)$ from special cycles on Siegel threefolds. Weissauer asked in [124] how such classes related to the classical theory of Heegner points, a question to which Proposition D provides a partial answer: like the Selmer classes coming from Heegner points, these Abel-Jacobi classes can be nonzero only if E has rank one (at least modulo the assumption on Rankin-Selberg convolutions).

(iv) In the text, we prove a stronger version of Proposition D that includes some CM cases.

Similarly, the methods used to Prove Theorem A can also be used for endoscopic representations under the condition $(*)$. This case is included in Theorem 9.2.4 for completeness, but note that the result gives nothing new beyond Kato's work [48].

0.7. **Organization of the paper.** In §1, we lay out the notational conventions for the article and cover preliminary materials on involutions, Clifford algebras, and Selmer groups. In §2, we collect the necessary background results related to automorphic representations, Galois representations, and Shimura varieties. The most important role of this section is to make some of the results of [96] (on global Jacquet-Langlands transfers for inner forms of GSp_4) unconditional on Conjecture 7.5 of *op. cit.* In §3, we define Kudla's cycles $Z(T, \varphi)$ and their analogues on compact GSpin groups, and explain their relation to classical theta lifts. In §4, we define the Galois cohomology classes and special periods used in the Euler system arguments. In §5, we use the mod-p theory of the Weil representation to prove the change-of-test-functions criterion explained above in (0.9); this is one of the most crucial (and technical) parts of the argument.

In §6, we turn to geometry, studying a ramified Rapoport-Zink space for GSpin_5. This section is based on [118], although we need more details on intersection theory for our applications to special cycles. In §7, these results are applied to study the special fiber of the ramified GSpin_5 Shimura variety, and compute the local part of the Abel-Jacobi image of special cycles. (Some of the results in this section are generalizations of those in [119].) In §8, we perform the level-raising and complete the program described in §0.3 above. In §9, we prove the main results in the rank zero case. In §10, we study special cycles on the special fibers of GSpin_4 and GSpin_5 Shimura varieties with good reduction. In §11 and §12, we prove our main result in the rank one case.

The paper contains three appendices: in the short Appendix A, we explain the relation of the cohomology of GSpin_4 Shimura varieties to Hilbert modular forms, which is well-known but for which we were unable to find a suitable reference. In Appendix B, we develop a general framework for deformation-theoretic characteristic-zero level raising of G-valued Galois representations, using ideas from [29]. These results are most important for the proof of the rank one case. In Appendix C, we prove some large-image results for the p-adic Galois representations that appear in the article, which are necessary for various Chebotarev arguments.

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1. PRELIMINARIES

1.1. **Notation.**

1.1.1. *Number fields and Galois groups.* Let $L \subset \overline{\mathbb{Q}}$ be a number field.

- We denote by O_L the ring of integers. If $\mathfrak{p} \subset O_L$ is a prime, then we write $O_{\mathfrak{p}}$ and $L_{\mathfrak{p}}$ for the respective completions, and $\varpi_{\mathfrak{p}}$ for a uniformizer of $O_{\mathfrak{p}}$.
- Let $G_L := \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ be the absolute Galois group. If S is a finite set of places of L, we denote by L^S the maximal unramified-outside-S extension of L, and set $G_{L,S} := \mathrm{Gal}(L^S/L) \subset G_L$. If $\mathfrak{p}$ is a prime of L, we also write $G_{L_{\mathfrak{p}}}$ for the absolute Galois group of $L_{\mathfrak{p}}$, with inertia subgroup $I_{L_{\mathfrak{p}}} \subset G_{L_{\mathfrak{p}}}$.
- For any p, $\chi_{p,\mathrm{cyc}} : G_L \to \mathbb{Z}_p^\times$ denotes the p-adic cyclotomic character. We normalize the definition of Hodge-Tate weights so that $\chi_{p,\mathrm{cyc}}|_{G_{\mathbb{Q}_p}}$ has weight one.

1.1.2. *Adèle groups and class field theory.*

- For a number field K, let $\mathbb{A}_K$ be the adèle ring; when $K = \mathbb{Q}$ we typically write $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. For a finite set S of places of $\mathbb{Q}$, we write

$$\mathbb{A}_S = \prod_{v \in S} \mathbb{Q}_v, \quad \mathbb{A}^S = \prod_{v \notin S}{}' \mathbb{Q}_v, \quad \mathbb{A}_f^S = \prod_{v \notin S \cup \{\infty\}}{}' \mathbb{Q}_v.$$

- If $\pi = \otimes_v' \pi_v$ is an irreducible admissible representation of $G(\mathbb{A})$ for some $\mathbb{Q}$-group G, then for a squarefree integer D, π_f^D denotes $\otimes'_{\ell \nmid D} \pi_\ell$, and π_D denotes $\otimes_{\ell | D} \pi_\ell$. The similar notations π_f^S, π_S hold for finite sets of primes S.
- For any prime p, let $\langle p \rangle \in \mathbb{A}_f^\times$ be the image of p under the natural inclusion $\mathbb{Q}_p^\times \hookrightarrow \mathbb{A}_f^\times$.
- We always normalize the reciprocity maps of class field theory to send uniformizers to geometric Frobenius.
- For any number field L and finite-order character $\chi : L^\times \backslash \mathbb{A}_L^\times \to k^\times$, with k a field, we write $\mathrm{rec}(\chi) : G_L \to k^\times$ for the pullback by the reciprocity map.

1.1.3. *Coefficient rings.* For coefficient rings, we will often use a discrete valuation ring O which is a finite flat extension of $\mathbb{Z}_p$, with uniformizer ϖ. In this context:

- We denote by CNL_O the category of complete local Noetherian O-algebras with residue field O/ϖ.
- For any torsion O-module M and any $m \in M$, $\mathrm{ord}_\varpi(m) \geq 0$ denotes the least integer such that $\varpi^{\mathrm{ord}_\varpi(m)} m = 0$.

1.1.4. *Symplectic groups.*

- For any n, let $\Omega_n = \begin{pmatrix} \mathbf{0}_n & I_n \\ -I_n & \mathbf{0}_n \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{Z})$, where $\mathbf{0}_n$ is the $n \times n$ matrix of zeros and I_n is the $n \times n$ identity matrix.
- Define the algebraic group GSp_{2n} over $\mathbb{Z}$ by

$$\mathrm{GSp}_{2n}(R) = \left\{ (g, \lambda) \in \mathrm{GL}_{2n}(R) \times R^\times \ : \ g\Omega g^t = \lambda\Omega \right\} \tag{1.1}$$

  for any ring R.
- We have the natural similitude character $\nu : \mathrm{GSp}_{2n} \to \mathbb{G}_m$, and the symplectic group $\mathrm{Sp}_{2n} \subset \mathrm{GSp}_{2n}$ is the kernel of ν.
- For any prime p, we write $\langle p \rangle \in \mathrm{GSp}_{2n}(\mathbb{A}_f)$ for the scalar matrix corresponding to $\langle p \rangle \in \mathbb{A}_f^\times$.

1.1.5. *GSpin groups.* Let V be a quadratic space over a field F.

- We denote the pairing $V \times V \to F$ by $(v, w) \mapsto v \cdot w$.
- The Clifford algebra of V is the associative F-algebra $C(V)$ generated by $v \in V$, subject to the relation
$$v^2 = (v \cdot v)1.$$
There is a natural $\mathbb{Z}/2\mathbb{Z}$ grading on $C(V)$, with respect to which the plus part is denoted $C^+(V)$.
- We denote by $*$ the natural involution on $C(V)$, determined by $v^* = v$. Then $\mathrm{GSpin}(V)$ is the algebraic group over F defined by
$$\mathrm{GSpin}(V)(R) = \left\{(g, \lambda) \in \left(C^+(V) \otimes R\right)^\times \times R^\times \, : \, gg^* = \lambda\right\}.$$
The natural similitude character is again denoted ν.
- If $F = \mathbb{Q}$, then for any prime p we again denote by $\langle p \rangle \in \mathrm{GSpin}(V)(\mathbb{A}_f)$ the scalar element $p \in C^+(V)^\times$.

1.1.6. *Quaternion algebras and quadratic spaces.*

- For a squarefree integer $D \geq 1$, let B_D be the quaternion algebra over $\mathbb{Q}$ which ramifies at the factors of D, and possibly at infinity. Let $*_D$ be the standard involution on B_D. We also denote by $*_D$ the involution on $M_n(B_D)$ given by the composite of $*_D$ and transposition.
- For all squarefree D, let
$$V_D := M_2(B_D)^{*_D=1,\mathrm{tr}=0}, \tag{1.2}$$
which is a 5-dimensional quadratic space of trivial discriminant, whose Hasse invariant coincides with that of B_D. The signature of V_D is $(5, 0)$ or $(3, 2)$ when B_D is ramified or split at infinity, respectively.
- We note that $\mathrm{GSpin}(V_D)$ is an inner form of $\mathrm{GSpin}(V_1) \cong \mathrm{GSp}_4$.

1.1.7. *Algebraic geometry.*

- For a closed subscheme X of a scheme Y, we denote by $\mathcal{N}_{X/Y}$ the normal sheaf.
- If X is a scheme, $X_{\mathrm{red}} \subset X$ denotes the maximal reduced subscheme.

1.1.8. *Miscellaneous.*

- For a squarefree integer $D \geq 1$, we denote by $\mathrm{div}(D)$ its set of prime factors, and let $\sigma(D) := \#\,\mathrm{div}(D)$.
- Suppose V is an F-vector space for a nonarchimedean local field F with ring of integers O_F. For two O_F-lattices $\Lambda \subset \Lambda'$ of V, the notation $\Lambda \subset_n \Lambda'$ means that Λ'/Λ has O_F-length n.
- For any group G, we let $Z_G \subset G$ denote the center.
- For a prime p, let $\breve{\mathbb{Z}}_p$ be the Witt vectors of $\overline{\mathbb{F}}_p$.

1.2. **Central simple algebras and involutions.**

1.2.1. Let M be a central simple algebra over a field F of characteristic zero. Throughout this paper an involution on M will be understood to mean an involution of the first kind, i.e. fixing F. Such involutions fall into two categories: main type or nebentype (also called symplectic and orthogonal type [51]). For example, the transpose involution on $M = M_2(F)$ is of nebentype; the standard involution on a quaternion algebra is of main type. For later use, we now recall some basic facts.

Proposition 1.2.2. *Let $(M_1, *_1)$ and $(M_2, *_2)$ be two central simple algebras equipped with involutions. The induced involution $*_1 \otimes *_2$ on M is of nebentype if and only if $*_1$ and $*_2$ are of the* same *type.*

Proof. This is [51, Proposition 2.23]. □

1.2.3. Suppose $F = \mathbb{R}$ or $\mathbb{Q}$, and recall that an element $\mu \in M$ is called *totally positive* if it has strictly positive eigenvalues as an endomorphism of M.

Proposition 1.2.4. *Let $M = M_n(\mathbb{R})$ or $M_n(\mathbb{H})$, and suppose M is equipped with a* positive *involution $*$, i.e. such that $\operatorname{tr}(a^*a) > 0$ for all $0 \neq a \in M$. Then the positive involutions on M are those of the form*

$$a \mapsto \mu a^* \mu^{-1},$$

where $\mu^* = \mu$ and either μ or $-\mu$ is totally positive. In particular, all positive involutions on M have the same type.

Proof. This is [115, Proposition 8.4.7 and Lemma 8.4.12]. ☐

1.2.5. The tensor product of two quaternion algebras is called a *biquaternion algebra* (BQA). Involutions of nebentype on rational BQAs are particularly simple:

Lemma 1.2.6. *Let M be a BQA over $\mathbb{Q}$, and let $*_1, *_2$ be two involutions of M of main type. If either $M \otimes \mathbb{R}$ is split, or $*_1, *_2$ are both positive, then there exists an element $g \in M^\times$ such that*

$$\operatorname{Int}(g) \circ *_1 = *_2 \circ \operatorname{Int}(g).$$

In particular, conjugation by g defines an isomorphism $(M, *_1) \simeq (M, *_2)$.

Here, $\operatorname{Int}(g)$ is the automorphism $a \mapsto gag^{-1}$ of M.

Proof. By the Skolem-Noether theorem, there exists $h \in M^\times$ such that $\operatorname{Int}(h) \circ *_1 = *_2$. The condition on g in the lemma is equivalent to $h = \lambda g^{*_1} g$, for some $\lambda \in \mathbb{Q}^\times$. By [51, Theorem 16.19], there exists such a g if and only if the Pfaffian norm of h (with respect to $*_1$) belongs to $(\mathbb{Q}^\times)^2 \operatorname{Nrd}(M^\times)$. If $M \otimes \mathbb{R}$ is split, then $\operatorname{Nrd}(M^\times) = \mathbb{Q}^\times$, so we are done. If $M \otimes \mathbb{R}$ is not split, then $\operatorname{Nrd}(M^\times) = \mathbb{Q}^\times_{>0}$, and it suffices to show that the equation $h = \lambda g^{*_1} g$ has a solution over $\mathbb{R}$. Assuming without loss of generality that $*_1$ is the standard involution on $M \otimes \mathbb{R} \simeq M_2(\mathbb{H})$, this can be checked directly using Proposition 1.2.4. ☐

1.2.7. Let F be a nonarchimedean local field with ring of integers O_F, and let B be the unique nonsplit quaternion algebra over O_F. We denote by O_B the unique maximal O_F-order in B, with uniformizer $\pi \in O_B$ and natural valuation ord_π.

Proposition 1.2.8. *Suppose $g \in \mathrm{GL}_n(B)$ satisfies*

$$gM_n(O_B)g^{-1} = M_n(O_B).$$

Then, up to rescaling by an element of $F^\times$, we have either $g \in \mathrm{GL}_n(O_B)$ or $g \in \pi\, \mathrm{GL}_n(O_B)$.

Proof. Suppose g is given by the matrix (g_{ij}) and g^{-1} is given by the matrix (h_{kl}). Let α_{jk} be the matrix with a 1 in the jk position and 0s elsewhere; since $g\alpha_{jk}g^{-1} \in M_n(O_B)$, we see that

$$g_{ij}h_{kl} \in O_B, \quad \text{for all } i, j, k, l. \tag{1.3}$$

Without loss of generality, by rescaling g, we may assume $g_{ij} \in O_B$ for all i, j but $\operatorname{ord}_\pi(g_{ij}) \leq 1$ for some i, j. If $h_{ij} \in O_B$ for all i, j, then $g \in \mathrm{GL}_n(O_B)$, so suppose without loss of generality that $\operatorname{ord}_\pi(h_{kl}) < 0$ for some k, l. It follows from (1.3) that $\operatorname{ord}_\pi(g_{ij}) \geq 1$ for all i, j, with equality holding for some i, j, and $\operatorname{ord}_\pi(h_{kl}) \geq -1$ for all k, l; in particular, $g \in \pi\, \mathrm{GL}_n(O_B)$. ☐

Motivated by Proposition 1.2.8, we make the following definition.

Definition 1.2.9. An involution $*$ of $M_n(B)$ stabilizing $M_n(O_B)$ is called of *unit type* if it is of the form $\alpha^* = g\overline{\alpha}^t g^{-1}$ for some $g \in \mathrm{GL}_n(O_B)$, and of *non-unit type* if it is of the form $\alpha^* = g\overline{\alpha}^t g^{-1}$ for some $g \in \pi\, \mathrm{GL}_n(O_B)$.

Similarly, if B is a quaternion algebra over $\mathbb{Q}$ and p is a prime such that $B \otimes \mathbb{Q}_p$ is not split, let $O_B \subset B$ be the unique maximal $\mathbb{Z}_{(p)}$-order. Then an involution $*$ of $M_n(B)$ stabilizing $M_n(O_B)$is called of unit or non-unit type according to whether the induced involution on $M_n(B) \otimes \mathbb{Q}_p$ is of unit or non-unit type.

Remark 1.2.10. By Proposition 1.2.8, an involution $*$ of $M_n(B)$ stabilizing $M_n(O_B)$ induces an involution on $M_n(O_B/\pi)$, which acts trivially on the center of $M_n(O_B/\pi)$ if and only if $*$ is of non-unit type.

1.2.11. The same proof as Proposition 1.2.8 also shows:

Proposition 1.2.12. *Let F be a nonarchimedean local field with ring of integers O_F. If $g \in \mathrm{GL}_n(F)$ satisfies*

$$gM_n(O_F)g^{-1} = M_n(O_F),$$

then we have $g \in F^\times \mathrm{GL}_n(O_F)$.

1.3. **PEL data.**

1.3.1. Recall the notion of a PEL datum $\mathcal{D} = (B, *, V, \psi)$ from [77, Chapter 8], and set $C = \mathrm{End}_B(V)$. Then C is equipped with an involution $c \mapsto c'$, the adjoint with respect to ψ, and the $\mathbb{Q}$-group $G_{\mathcal{D}}$ associated to $\mathcal{D}$ is defined by

$$G_{\mathcal{D}}(R) = \left\{(g, \lambda) \in (C \otimes R)^\times \times R^\times \ : \ gg' = \lambda\right\}.$$

To $\mathcal{D}$, there is associated the reflex field $E = E_{\mathcal{D}}$, and, for any compact open subgroup $K \subset G_{\mathcal{D}}(\mathbb{A}_f)$, a moduli functor M_K over E. Let us briefly recall the definition of M_K, for which more details can be found in [52, p. 390]. For a connected scheme $S \to \operatorname{Spec} E$, $M_K(S)$ is the set of isomorphism classes of tuples $(A, \iota, \lambda, \eta)$ where:

- A/S is an abelian scheme up to isogeny;
- $\iota : B \hookrightarrow \mathrm{End}^0(A/S)$ is an embedding satisfying the Kottwitz determinant condition derived from $\mathcal{D}$;
- $\lambda : A \to A^\vee$ is a quasi-polarization such that $\iota(b^*)^\vee \circ \lambda = \lambda \circ \iota(b)$ for all $b \in B$;
- η is a K-level structure for A, i.e., for any geometric point s of S, a $\pi_1(S, s)$-stable K-orbit of isomorphisms
  $$\eta : H_{1,\text{ét}}(A_s, \mathbb{A}_f) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}_f$$
  respecting the actions of B and the symplectic pairings on both sides up to a scalar.

If K is neat, then M_K is represented by a smooth quasi-projective scheme over E.

1.3.2. Let p be a prime. A self-dual p-integral refinement $\mathscr{D}$ of $\mathcal{D}$ is the additional data of a $*$-stable maximal $\mathbb{Z}_{(p)}$-order $O_B \subset B$ and a self-dual O_B-stable $\mathbb{Z}_{(p)}$-lattice $\Lambda \subset V$. (This is a special case of the notion in [93, §6].) For a compact open subgroup $K^p \subset G_{\mathcal{D}}(\mathbb{A}_f^p)$, the corresponding moduli problem $\mathcal{M}_{K^p}$ is defined as follows. For a connected scheme $S \to \operatorname{Spec} \mathcal{O}_E \otimes \mathbb{Z}_{(p)}$, $\mathcal{M}_{K^p}(S)$ is the set of isomorphism classes of tuples $(A, \iota, \lambda, \eta^p)$ where:

- A/S is an abelian scheme up to prime-to-p isogeny;
- $\iota : O_B \hookrightarrow \mathrm{End}(A/S) \otimes \mathbb{Z}_{(p)}$ is an embedding satisfying the Kottwitz condition;
- $\lambda : A \to A^\vee$ is a prime-to-p quasi-polarization such that $\iota(b^*)^\vee \circ \lambda = \lambda \circ \iota(b)$ for all $b \in O_B$;
- η^p is a K^p-level structure, i.e., for any geometric point s of S, a $\pi_1(S, s)$-stable K^p-orbit of isomorphisms
  $$\eta : H_{1,\text{ét}}(A_s, \mathbb{A}_f^p) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$
  respecting the actions of B and the symplectic pairings on both sides up to a scalar.

If K^p is neat, then $\mathcal{M}_{K^p}$ is represented by a quasi-projective scheme over $O_E \otimes \mathbb{Z}_{(p)}$. Its generic fiber is $M_{K^pK_p}$, where $K_p = \mathrm{Stab}_{G_{\mathcal{D}}(\mathbb{Q}_p)}(\Lambda)$.

Lemma 1.3.3. *Let S be a scheme and B a simple $\mathbb{Q}$-algebra with involution $*$ of the first kind. Suppose given an abelian scheme A/S with an embedding of $\mathbb{Q}$-algebras*

$$\iota : B \hookrightarrow \mathrm{End}^0(A/S),$$

and a quasi-polarization $\lambda : A \to A^\vee$ such that

$$\iota(b^*)^\vee \circ \lambda = \lambda \circ \iota(b), \quad \forall b \in B.$$

Then for any totally positive $g \in B^\times$ with $g^* = g$, $\lambda \circ g$ defines a quasi-polarization of A.

Proof. It suffices to prove the lemma when $S = \mathrm{Spec}\, k$ with k an algebraically closed field. In this case, by the discussion in [78, Chapter 21, Application III], it suffices to show that g has strictly positive eigenvalues on $\mathrm{End}^0(A)$. But this follows from the positivity of the eigenvalues on B, because $\mathrm{End}^0(A)$ is a semisimple $\mathbb{Q}$-algebra containing B, and any such is a product of copies of B as a B-module. ☐

The following corollary is immediate.

Corollary 1.3.4. *Let $\mathcal{D} = (B, *, V, \psi)$ be a PEL datum, where B is a simple $\mathbb{Q}$-algebra and $*$ is an involution of the first kind. Suppose given a totally positive $g \in B^\times$ such that $g^* = g$, and let $\mathcal{D}_g = (B, *_g, V, \psi_g)$, where $*_g := g \circ * \circ g^{-1}$ and*

$$\psi_g(x, y) := \psi(x, g^{-1}y), \quad \forall x, y \in V.$$

Then $G_{\mathcal{D}} = G_{\mathcal{D}_g}$, and, for all $K \subset G_{\mathcal{D}}(\mathbb{A}_f)$, there is a canonical isomorphism

$$M_{\mathcal{D},K} \xrightarrow{\sim} M_{\mathcal{D}_g,K}$$

defined by

$$(A, \iota, \lambda, \eta) \mapsto (A, \iota, \lambda \circ g, \eta).$$

If $\mathscr{D} = (O_B, *, \Lambda, \psi)$ is a self-dual p-integral refinement of $\mathcal{D}$ and g lies in $O_B^\times$, then $\mathscr{D}_g = (O_B, *_g, \Lambda, \psi_g)$ is a self-dual p-integral refinement of $\mathcal{D}_g$ and the above isomorphism extends to an isomorphism of integral models

$$\mathcal{M}_{\mathscr{D},K^p} \xrightarrow{\sim} \mathcal{M}_{\mathscr{D}_g,K^p}$$

for all compact open $K^p \subset G_{\mathcal{D}}(\mathbb{A}_f^p)$.

☐

We also deduce:

Corollary 1.3.5. *Let B be an indefinite quaternion algebra over $\mathbb{Q}$, with $O_B \subset B$ a maximal $\mathbb{Z}_{(p)}$-order, and fix an integer $n \geq 1$. Let $*$ be a positive involution on $M_n(O_B)$, of non-unit type if $B \otimes \mathbb{Q}_p$ is ramified. Then there exists an abelian scheme A over $\mathrm{Spec}\, \breve{\mathbb{Z}}_p$ of dimension $2n$ with supersingular reduction, a prime-to-p quasi-polarizon $\lambda : A \to A^\vee$, and an embedding $\iota : M_n(O_B) \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$, such that*

$$\iota(b^*)^\vee \circ \lambda = \lambda \circ \iota(b), \quad \forall b \in M_n(O_B).$$

Proof. Using Lemma 1.3.3 along with Propositions 1.2.8 and 1.2.12, we reduce to the case that $\alpha^* = \alpha^{\dagger t}$ for all $\alpha \in M_n(O_B)$, where $\dagger$ is a positive involution on O_B, of non-unit type if B ramifies at p. Thus it suffices to prove the corollary when $n = 1$, in which case it is well-known from the classical theory of Shimura curves; see the discussions in [12, Chapter III]. ☐

1.4. PEL data for $\mathrm{GSpin}_{3,2}$ groups.

1.4.1. In this article, we will consider PEL data that arise in the following way. Let $D \geq 1$ be squarefree with $\sigma(D)$ *even*, and let q be a prime, possibly with $q|D$. Suppose fixed a maximal $\mathbb{Z}_{(q)}$-order $O_D \subset B_D$, with a nebentype involution $*$, and an embedding $\breve{\mathbb{Z}}_q \hookrightarrow \mathbb{C}$.

Definition 1.4.2. (1) An $(O_D, *)$-triple is a triple $(A_0, \iota_0, \lambda_0)$, where:

   (a) A_0 is an abelian scheme over $\operatorname{Spec} \breve{\mathbb{Z}}_q$ of rank 4 with supersingular reduction;
   (b) ι_0 is an embedding $O_D \hookrightarrow \operatorname{End}(A_0) \otimes \mathbb{Z}_{(q)}$;
   (c) $\lambda_0 : A_0 \to A_0^\vee$ is a prime-to-q polarization, such that

$$\iota_0(\alpha^*)^\vee \circ \lambda_0 = \lambda_0 \circ \iota_0(\alpha), \ \ \forall \alpha \in O_D.$$

(2) Given an $(O_D, *)$-triple as above, we set $H := H_1(A_0(\mathbb{C}), \mathbb{Q})$, with its canonical symplectic form ψ. Let $\Lambda \subset H$ be the lattice $H_1(A_0(\mathbb{C}), \mathbb{Z}_{(q)})$. The PEL datum associated to $(A_0, \iota_0, \lambda_0)$ is defined by

$$\mathcal{D} = (B_D, *, H, \psi),$$

with self-dual q-integral refinement

$$\mathscr{D} = (O_D, *, \Lambda, \psi).$$

1.4.3. Let $D \cdot q := Dq/\gcd(D, q)$. Given an $(O_D, *)$-triple $(A_0, \iota_0, \lambda_0)$:

- H is a B_D-module, and $\operatorname{End}(H, B_D)$ is isomorphic to $M_2(B_D)$. The adjoint involution $\dagger$ on $\operatorname{End}(H, B_D)$ is of main type (by Proposition 1.2.2, because the adjoint involution on $\operatorname{End}(H) = B_D \otimes \operatorname{End}(H, B_D)$ is of main type).
- Set $\overline{A}_0 := (A_0)_{\overline{\mathbb{F}}_q}$, with its induced $\mathcal{O}_D$-action $\overline{\iota}_0$ and polarization $\overline{\lambda}_0$. Then $\operatorname{End}^0(\overline{A}_0, \overline{\iota}_0)$ is isomorphic to $M_2(B_{D\cdot q})$, and its Rosati involution $\dagger$ is positive, hence of main type by Proposition 1.2.4.

Definition 1.4.4.

(1) A q-adic *uniformization datum* $(A_0, \iota_0, \lambda_0, i_D, i_{D\cdot q})$ for $(O_D, *)$ is an $(O_D, *)$-triple $(A_0, \iota_0, \lambda_0)$ as above, along with a choice of isomorphisms of algebras with involution:

$$i_D : (\operatorname{End}(H, B_D), \dagger) \xrightarrow{\sim} (M_2(B_D), *_D),$$
$$i_{D\cdot q} : \left(\operatorname{End}^0(\overline{A}_0, \overline{\iota}_0), \dagger\right) \xrightarrow{\sim} (M_2(B_{D\cdot q}), *_{D\cdot q}).$$

(2) A q-adic *uniformization datum* $(*, A_0, \iota_0, \lambda_0, i_D, i_{D\cdot q})$ for V_D is a choice of positive nebentype involution $*$ on O_D – of unit type if $q|D$ – and a uniformization datum $(A_0, \iota_0, \lambda_0, i_D, i_{D\cdot q})$ for $(O_D, *)$.

Recall here that V_D was defined in (1.2).

Remark 1.4.5. (1) Given any $(O_D, *)$-triple $(A_0, \iota_0, \lambda_0)$, the choices in Definition 1.4.4(1) exist by Lemma 1.2.6.

(2) Given a q-adic uniformization datum $(*, A_0, \iota_0, \lambda_0, i_D, i_{D\cdot q})$ for V_D, we also obtain isomorphisms

$$\operatorname{End}(H, B_D)^{\dagger=1, \mathrm{tr}=0} \xrightarrow{\sim} V_D, \ \operatorname{End}^0(\overline{A}_0, \overline{\iota}_0)^{\dagger=1, \mathrm{tr}=0} \xrightarrow{\sim} V_{D\cdot q}.$$

The former determines an isomorphism

$$G_{\mathcal{D}} \xrightarrow{\sim} \operatorname{GSpin}(V_D).$$

Moreover, the action of $\operatorname{End}(\overline{A}_0, \overline{\iota}_0)$ on

$$H_{1,\text{ét}}(\overline{A}_0, \mathbb{A}_f^q) \cong H_1(A_0(\mathbb{C}), \mathbb{A}_f^q)$$

induces an isomorphism $V_{D\cdot q} \otimes_{\mathbb{Q}} \mathbb{A}_f^q \cong V_D \otimes_{\mathbb{Q}} \mathbb{A}_f^q$. In turn, this induces an isomorphism $\operatorname{GSpin}(V_D)(\mathbb{A}_f^q) \cong \operatorname{GSpin}(V_{D\cdot q})(\mathbb{A}_f^q)$.

1.5. **Local conditions and Selmer groups.**

1.5.1. In this subsection, O is the ring of integers of a finite extension $E/\mathbb{Q}_p$, and $\varpi \in O$ is a uniformizer.

Notation 1.5.2.

(1) Suppose M is a finite free O-module with an action of $G_{\mathbb{Q}_\ell}$, where ℓ may be equal to p. We consider the Bloch-Kato local conditions

$$H^1_f(\mathbb{Q}_\ell, M) := \ker\left(H^1(\mathbb{Q}_\ell, M) \to \frac{H^1(\mathbb{Q}_\ell, M \otimes \mathbb{Q}_p)}{H^1_f(\mathbb{Q}_\ell, M \otimes \mathbb{Q}_p)}\right).$$

(2) Globally, if M is a finite free O-module with $G_{\mathbb{Q}}$-action, let

$$H^1_f(\mathbb{Q}, M) := \ker\left(H^1(\mathbb{Q}, M) \to \prod_\ell \frac{H^1(\mathbb{Q}_\ell, M)}{H^1_f(\mathbb{Q}_\ell, M)}\right).$$

(3) Suppose $\ell \neq p$. For an unramified, finitely generated O-module M (either finite or infinite) with $G_{\mathbb{Q}_\ell}$-action, we let

$$H^1_f(\mathbb{Q}_\ell, M) := H^1_{\mathrm{unr}}(\mathbb{Q}_\ell, M).$$

Remark 1.5.3. Notations 1.5.2(1,3) are consistent because, when M is free over O and unramified as a $G_{\mathbb{Q}_\ell}$-module, the map $H^1(I_\ell, M) \to H^1(I_\ell, M \otimes \mathbb{Q}_p)$ is injective.

1.5.4. Fix integers $a \leq 0 \leq b$. Recall that a finite $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$-module M is said to be *torsion crystalline* with Hodge-Tate weights in $[a, b]$ if there exists a crystalline $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-module V with Hodge Tate weights in $[a, b]$, and two $G_{\mathbb{Q}_p}$-stable lattices $T_1 \subset T_2 \subset V$, such that $M = T_2/T_1$. A finitely generated $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$-module M is torsion crystalline with Hodge-Tate weights in $[a, b]$ if M/p^n is so for all $n \geq 1$.

If M is torsion crystalline with Hodge-Tate weights in $[a, b]$, let

$$H^1_{f,\mathrm{tors}}(\mathbb{Q}_p, M) \subset H^1(\mathbb{Q}_p, M)$$

be the subspace of cohomology classes such that the corresponding extension

$$0 \to M \to * \to \mathbb{Z}_p \to 0$$

is torsion crystalline with Hodge-Tate weights in $[a, b]$. If M is an $O[G_{\mathbb{Q}_p}]$-module, then $H^1_{f,\mathrm{tors}}(\mathbb{Q}_p, M)$ is an O-submodule of $H^1(\mathbb{Q}_p, M)$.

Proposition 1.5.5. *A finite free $O[G_{\mathbb{Q}_p}]$-module M is torsion crystalline with Hodge-Tate weights in $[a, b]$ if and only if $M \otimes \mathbb{Q}_p$ is crystalline with Hodge-Tate weights in $[a, b]$. In particular, if M is a finite free $O[G_{\mathbb{Q}_p}]$-module, then $H^1_f(\mathbb{Q}_p, M) = H^1_{f,\mathrm{tors}}(\mathbb{Q}_p, M)$.*

Proof. This is [64, §7.3].[5] □

Notation 1.5.6. We will from now on write $H^1_f(\mathbb{Q}_p, M)$ in place of $H^1_{f,\mathrm{tors}}(\mathbb{Q}_p, M)$ for any finitely generated O-module M with $G_{\mathbb{Q}_p}$-action; by Proposition 1.5.5, this is consistent with Notation 1.5.2(1).

1.6. **Duality.**

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[5] In fact, for the main results it suffices to assume $b - a < p - 1$, in which case Proposition 1.5.5 is due to Breuil [14, Proposition 6].

1.6.1. Let ℓ be a prime, possibly equal to p. If M is a finite free O-module with $G_{\mathbb{Q}_\ell}$-action, let $M^* = \mathrm{Hom}_O(M, O(1))$. Similarly, if M is a locally compact O-module with $G_{\mathbb{Q}_\ell}$-action, let $M^\vee = \mathrm{Hom}_O(M, E/O(1))$ be the Cartier dual. We have the local Tate pairings

$$\langle \cdot, \cdot \rangle_\ell : H^1(\mathbb{Q}_\ell, M) \times H^1(\mathbb{Q}_\ell, M^\vee) \to H^2(\mathbb{Q}_\ell, E/O(1)) = E/O \tag{1.4}$$

and

$$\langle \cdot, \cdot \rangle_\ell : H^1(\mathbb{Q}_\ell, M) \times H^1(\mathbb{Q}_\ell, M^*) \to H^2(\mathbb{Q}_\ell, O(1)) = O. \tag{1.5}$$

The former is perfect, and the latter is perfect modulo torsion. We recall the following standard fact.

Lemma 1.6.2. *Suppose M is a finite free O-module with $G_{\mathbb{Q}_\ell}$-action. Then the subspaces $H^1_f(\mathbb{Q}_\ell, M)$ and $H^1_f(\mathbb{Q}_\ell, M^*)$ pair to zero under the local Tate pairing (1.5).*

Proof. When $O = \mathbb{Z}_p$, the result follows from [9, Proposition 3.8].[6] To reduce to this case, note that M^* is canonically isomorphic to $M' := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$, and we have a commutative diagram:

$$\begin{tikzcd}
H^1_f(\mathbb{Q}_\ell, M) \times H^1_f(\mathbb{Q}_\ell, M^*) \arrow[r, "\langle \cdot,\cdot \rangle_p"] \arrow[d, "\sim"] & O \arrow[d, "\mathrm{tr}"] \\
H^1_f(\mathbb{Q}_\ell, M) \times H^1_f(\mathbb{Q}_\ell, M') \arrow[r, "\langle \cdot,\cdot \rangle_p = 0"] & \mathbb{Z}_p.
\end{tikzcd}$$

Since $H^1_f(\mathbb{Q}_\ell, M^*) \subset H^1(\mathbb{Q}_\ell, M^*)$ is O-stable and the trace pairing $O \times O \to \mathbb{Z}_p$ is nondegenerate, the lemma follows. ☐

Lemma 1.6.3. *Fix integers $a \leq 0 \leq b$, and let M be a finite free O-module with $[G_{\mathbb{Q}_p}]$-action which is torsion crystalline with Hodge-Tate weights in $[a, b]$. Then:*

- *(1) We have $\varprojlim H^1_f(\mathbb{Q}_p, M/\varpi^n) = H^1_f(\mathbb{Q}_p, M)$.*
- *(2) For all $n \geq 1$, there exists $m \geq n$ such that*

  $$\mathrm{im}\left(H^1_f(\mathbb{Q}_p, M/\varpi^m) \to H^1_f(\mathbb{Q}_p, M/\varpi^n)\right) = \mathrm{im}\left(H^1_f(\mathbb{Q}_p, M) \to H^1_f(\mathbb{Q}_p, M/\varpi^n)\right).$$
- *(3) For all $n \geq 1$, there exists $m \geq n$ such that the local Tate pairing*

  $$\langle \cdot, \cdot \rangle : H^1(\mathbb{Q}_p, M/\varpi^n) \times H^1(\mathbb{Q}_p, M^*/\varpi^n) \to O/\varpi^n$$

  vanishes when restricted to

  $$\mathrm{im}\left(H^1_f(\mathbb{Q}_p, M/\varpi^m) \to H^1_f(\mathbb{Q}_p, M/\varpi^n)\right) \times \mathrm{im}\left(H^1_f(\mathbb{Q}_p, M^*/\varpi^m) \to H^1_f(\mathbb{Q}_p, M^*/\varpi^n)\right).$$

Remark 1.6.4. When $2(b-a) \leq p-2$ and $O = \mathbb{Z}_p$, we may take $m = n$ in the final part by [84, Corollary 6.1].

Proof. The first part is clear from Proposition 1.5.5, and it follows that

$$\cap_{m \geq n} \mathrm{im}\left(H^1_f(\mathbb{Q}_p, M/\varpi^m) \to H^1_f(\mathbb{Q}_p, M/\varpi^n)\right) = \mathrm{im}\left(H^1_f(\mathbb{Q}_p, M) \to H^1_f(\mathbb{Q}_p, M/\varpi^n)\right).$$

Since $H^1_f(\mathbb{Q}_p, M/\varpi^n)$ is a finite O-module, it is clear that (2) holds.

For (3), take m sufficiently large to satisfy (2) for both M and M^*. The assertion then follows from Lemma 1.6.2 and the commutativity of the diagram:

---

[6] Although *loc. cit.* assumes $M \otimes \mathbb{Q}_p$ is de Rham when $\ell = p$, this is needed to show that $H^1_f(\mathbb{Q}_\ell, M)$ and $H^1_f(\mathbb{Q}_\ell, M^*)$ are *exact* annihilators; the proof shows that they annihilate each other in general.

$$\begin{tikzcd}
H^1(\mathbb{Q}_p, M) \times H^1(\mathbb{Q}_p, M^*) \arrow[r] \arrow[d] & O \arrow[d] \\
H^1(\mathbb{Q}_p, M/\varpi^n) \times H^1(\mathbb{Q}_p, M^*/\varpi^n) \arrow[r] & O/\varpi^n.
\end{tikzcd}$$

□

2. AUTOMORPHIC REPRESENTATIONS, HECKE ALGEBRAS, AND SHIMURA VARIETIES

2.1. **Hecke algebras and Galois representations.** Let G be a split, connected, reductive algebraic group over $\mathbb{Z}[S_0^{-1}]$ for a finite set of primes S_0, with Borel subgroup $B = TU \subset G$ and Weyl group W_G. For simplicity, we assume that the derived subgroup of G is simply connected.

Definition 2.1.1.

(1) For a prime $\ell \notin S_0$ and a ring R, let $\mathbb{T}_{G,\ell,R}$ denote the spherical Hecke algebra of compactly supported, R-valued, $G(\mathbb{Z}_\ell)$-biinvariant functions on $G(\mathbb{Q}_\ell)$. For a finite set $S \supset S_0$ and a ring R,

$$\mathbb{T}^S_{G,R} \coloneqq \bigotimes_{\ell\notin S}{}' \mathbb{T}_{G,\ell,R}.$$

(2) Let $P = MN \subset G$ be a parabolic subgroup. For all $\ell \notin S_0$, we define a natural map

$$S^G_M : \mathbb{T}_{G,\ell,R} \to \mathbb{T}_{M,\ell,R}$$

by

$$S^G_M(f)(m) = \int_{N(\mathbb{Q}_\ell)} f(mn)\mathrm{d}n,$$

where the Haar measure on $N(\mathbb{Q}_\ell)$ gives volume 1 to $N(\mathbb{Z}_\ell)$.

When $R = \mathbb{Z}$, we drop it from the notation.

Proposition 2.1.2. *We have a commutative diagram of functors*

$$\begin{tikzcd}
R[M(\mathbb{Q}_\ell)] - \mathrm{Mod} \arrow[r, "\Gamma_{M(\mathbb{Z}_\ell)}"] \arrow[d, "\natural-\mathrm{Ind}"] & \mathbb{T}_{M,\ell,R} - \mathrm{Mod} \arrow[d, "(S^G_M)^*"] \\
R[G(\mathbb{Q}_\ell)] - \mathrm{Mod} \arrow[r, "\Gamma_{G(\mathbb{Z}_\ell)}"] & \mathbb{T}_{G,\ell,R} - \mathrm{Mod},
\end{tikzcd} \tag{2.1}$$

where $\natural - \mathrm{Ind}$ *denotes the unnormalized parabolic induction.*

Proof. This follows from combining Lemmas 2.4, 2.7, and 2.9 of [82]. □

Notation 2.1.3. Let $\widehat{T} \subset \widehat{G}$ be the dual torus to T. We write $X^\bullet$ and $X_\bullet$ for the character and cocharacter groups of any split algebraic torus.

Definition 2.1.4. Suppose now that R is a $\mathbb{Z}[\ell^{1/2}, \ell^{-1/2}]$-algebra and recall the Satake transform

$$\begin{gathered} R[X^\bullet(\widehat{T})]^{W_G} \xrightarrow{\sim} \mathbb{T}_{G,\ell,R} \\ \lambda \mapsto [c_\lambda]. \end{gathered} \tag{2.2}$$

(1) If $\mathfrak{m} \subset \mathbb{T}_{G,\ell,R}$ is a maximal ideal with residue field k, the *Satake parameter* for $\mathfrak{m}$ is the unique element

$$\mathrm{Sat}_{G,\ell}(\mathfrak{m}) \in \widehat{T}(\overline{k})/W_G$$

such that, for any $\lambda \in R[X^\bullet(\widehat{T})]^{W_G}$, $\lambda(\mathrm{Sat}_{G,\ell}(\mathfrak{m})) = [c_\lambda] \pmod{\mathfrak{m}}$. A maximal ideal $\mathfrak{m} \subset \mathbb{T}^S_{G,R}$ defines a maximal ideal of each $\mathbb{T}_{G,\ell,R}$ with $\ell \notin S$; we denote by $\mathrm{Sat}_{G,\ell}(\mathfrak{m})$ the corresponding element of $\widehat{T}(\overline{k})/W_G$ for each $\ell \notin S$.

(2) Let $\rho_G \in X_\bullet(\widehat{T}) = X^\bullet(T)$ be the half-sum of positive roots. A *normalization* of the Satake transform for G (cf. [41, §8]) is a choice of element $\omega_G \in X_\bullet(Z_{\widehat{G}})$ such that

$$\omega_G \equiv \rho_G \pmod{2X_\bullet(\widehat{T})}.$$

2.1.5. For the rest of this subsection, we assume that R is a $\mathbb{Z}_{p^2}$-algebra, with a fixed choice of square root $\ell^{1/2} \in \mathbb{Z}_{p^2}$ for any $\ell \neq p$.

Definition 2.1.6. Let $S \supset S_0$ be a finite set of primes.

(1) Given a maximal ideal $\mathfrak{m} \subset \mathbb{T}^S_{G,R}$ with residue field k, a semisimple Galois representation

$$\overline{\rho}_\mathfrak{m} : G_\mathbb{Q} \to \widehat{G}(\overline{k})$$

is said to be *associated* to $\mathfrak{m}$ (with respect to a normalization ω_G of the Satake transform) if for all but finitely many $\ell \notin S$, $\overline{\rho}_\mathfrak{m}|_{G_{\mathbb{Q}_\ell}}$ is unramified and

$$\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_{\ell^{-1}})^{ss} \sim \omega_G(\ell^{-1/2}) \cdot \mathrm{Sat}_{G,\ell}(\mathfrak{m});$$

here $\sim$ denotes $\widehat{G}(\overline{k})$-conjugacy. If this holds for *all* $\ell \notin S \cup \{p\}$, then $\overline{\rho}_\mathfrak{m}$ is said to be *strongly associated* with $\mathfrak{m}$.

(2) If there exists a Galois representation (strongly) associated to $\mathfrak{m}$, then $\mathfrak{m}$ is said to be (strongly) of *Galois type*.

Note that whether $\mathfrak{m}$ is (strongly) of Galois type is independent of the choice of normalization; the corresponding representations $\overline{\rho}_\mathfrak{m}$ differ by the composite of χ_p^{cyc} and an algebraic cocharacter of $Z_{\widehat{G}}$.

Example 2.1.7. Suppose π is an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, unramified outside S, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. This determines a distinguished square root $\ell^{1/2} \in \overline{\mathbb{Q}}_p$ for all ℓ. Moreover, the Hecke action on $\iota^{-1}\pi^S$ defines a maximal ideal $\mathfrak{m}_\pi \subset \mathbb{T}^S_{\mathrm{GSp}_4,\overline{\mathbb{Q}}_p}$. If π is relevant in the sense of Definition 2.2.5 below, then the representation $\rho_{\pi,\iota}$ of Theorem 2.2.10 is strongly associated to $\mathfrak{m}_\pi$ with respect to the normalization given by the scalar subgroup $\mathbb{G}_m \hookrightarrow \mathrm{GSp}_4$.

Definition 2.1.8. Let $S \supset S_0$ be a finite set of primes. A maximal ideal $\mathfrak{m} \subset \mathbb{T}^S_{G,R}$ with residue field k is said to be *Eisenstein* if there exists an associated $\overline{\rho}_\mathfrak{m}$ (with respect to any normalization) which factors as

$$\overline{\rho}_\mathfrak{m} : G_\mathbb{Q} \to \widehat{M}(\overline{k}) \hookrightarrow \widehat{G}(\overline{k})$$

for some standard parabolic subgroup $P = MN \subsetneq G$.

Proposition 2.1.9. *Let $S \supset S_0$ be a finite set of primes. Suppose for some proper parabolic subgroup $P = MN \subsetneq G$, $\mathfrak{m}_M \subset \mathbb{T}^S_{M,R}$ is a maximal ideal of Galois type. Then $\mathfrak{m} = (S^G_M)^*\mathfrak{m}_M \subset \mathbb{T}^S_{G,R}$ is Eisenstein.*

Proof. Fix normalizations ω_M and ω_G of the Satake transform for M and G, respectively. Also let $d_P \in X_\bullet(Z_{\widehat{M}})$ be the dual of the character

$$\det(\mathrm{ad}(-)|N) : M \to \mathbb{G}_m.$$

By the well-known compatibility of the Satake transform with *normalized* parabolic induction, combined with Proposition 2.1.2, $\mathrm{Sat}_{G,\ell}(\mathfrak{m}) \in \widehat{T}(\overline{k})/W_G$ is represented by any representative of $\mathrm{Sat}_{M,\ell}(\mathfrak{m}_M) \cdot$

$d_P(\ell^{1/2}) \in \widehat{T}(\overline{k})/W_M$. It follows that

$$\overline{\rho}_{\mathfrak{m}} := \overline{\rho}_{\mathfrak{m}_M} \otimes (\chi_p^{\mathrm{cyc}})^{\alpha} : G_{\mathbb{Q}} \to \widehat{M}(\overline{k}) \hookrightarrow \widehat{G}(\overline{k})$$

is associated to $\mathfrak{m}$, where

$$\alpha := \frac{\omega_G - \omega_M - d_P}{2} \in X_\bullet(Z_{\widehat{M}}).$$

Hence $\mathfrak{m}$ is Eisenstein.

□

2.1.10. *The case of* $G = \mathrm{GSp}_4$. When $G = \mathrm{GSp}_4$, the Hecke algebra $\mathbb{T}_{G,\ell,R}$ can be identified with the polynomial algebra $R[T_{\ell,1}, T_{\ell,2}, \langle \ell \rangle]$, where the generators correspond to the following double coset operators:

$$T_{\ell,1} = \mathbb{1}\left(\mathrm{GSp}_4(\mathbb{Z}_\ell)\begin{pmatrix}\ell &&& \\ & 1 && \\ && \ell^{-1} & \\ &&& 1\end{pmatrix}\mathrm{GSp}_4(\mathbb{Z}_\ell)\right),$$

$$T_{\ell,2} = \mathbb{1}\left(\mathrm{GSp}_4(\mathbb{Z}_\ell)\begin{pmatrix}\ell &&& \\ & \ell && \\ && 1 & \\ &&& 1\end{pmatrix}\mathrm{GSp}_4(\mathbb{Z}_\ell)\right),$$

$$\langle \ell \rangle = \mathbb{1}\left(\mathrm{GSp}_4(\mathbb{Z}_\ell)\begin{pmatrix}\ell &&& \\ & \ell && \\ && \ell & \\ &&& \ell\end{pmatrix}\mathrm{GSp}_4(\mathbb{Z}_\ell)\right).$$

The Satake parameter of a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{\mathrm{GSp}_4,\ell,R}$ with residue field k can be identified with the data of an element $\nu \in k$ – the similitude factor of $\mathrm{Sat}_{\mathrm{GSp}_4,\ell}(\mathfrak{m})$ – together with the multi-set $\{\alpha, \beta, \nu/\alpha, \nu/\beta\}$ of elements of $\overline{k}$ – the eigenvalues of $\mathrm{Sat}_{\mathrm{GSp}_4,\ell}(\mathfrak{m})$ in the standard four-dimensional representation of $\widehat{\mathrm{GSp}_4} = \mathrm{GSp}_4$. We will abusively write the Satake parameter as $\{\alpha, \beta, \nu/\alpha, \nu/\beta\}$; in general, ν is not always determined by this unordered set.

The relation to Hecke eigenvalues is given explicitly in this case by

$$\begin{aligned} T_{\ell,1} &= \ell^2\,(\alpha\beta/\nu + \alpha/\beta + \beta/\alpha + \nu/\alpha\beta) + (\ell^2 - 1) \pmod{\mathfrak{m}} \\ T_{\ell,2} &= \ell^{3/2}\,(\alpha + \beta + \nu/\alpha + \nu/\beta) \pmod{\mathfrak{m}} \\ \langle \ell \rangle &= \nu \pmod{\mathfrak{m}}. \end{aligned} \tag{2.3}$$

When it is clear from context that $G = \mathrm{GSp}_4$, the subscript G in Hecke algebras and Satake parameters may be omitted from the notation.

2.2. **Automorphic forms and Galois representations.** Let F be a totally real field. If we fix a prime p and an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, then to each archimedean place $v|\infty$ of F, we can associate an embedding $\iota^* v : F \hookrightarrow \overline{\mathbb{Q}}_p$.

Theorem 2.2.1. *Suppose π is a unitary, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to a Hilbert modular form of weight $(2k_v)_{v|\infty}$, where v runs over archimedean places of F and $k_v \geq 1$ for all v. Then for every isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, with p a prime, there exists a Galois representation*

$$\rho_{\pi,\iota} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

with the following properties.

(1) $\rho_{\pi,\iota}|_{G_{F_v}}$ *is potentially semistable for all* $v|p$*, and for all nonarchimedean primes* v *of* F*:*

$$\iota WD(\rho_{\pi,\iota}|_{G_{F_v}})^{F-ss} \simeq \mathrm{rec}(\pi_v \otimes |\cdot|^{\frac{1}{2}}).$$

Moreover this Weil-Deligne representation is pure of weight -1*.*

(2) For each place $v|\infty$ *of* F*, the Hodge-Tate weights of* $\rho_{\pi,\iota}$ *with respect to the embedding* $\iota^* v : F \hookrightarrow \overline{\mathbb{Q}}_p$ *are* $1 - k_v$ *and* k_v*.*

(3) The similitude character of $\rho_{\pi,\iota}$ *is* $\iota^{-1}\,\mathrm{rec}(\omega_\pi)\chi_{p,\mathrm{cyc}}$*, where* ω_π *is the central character.*

Here rec is the usual local Langlands correspondence for GL_n, normalized to coincide with the reciprocity map of local class field theory when $n = 1$.

Proof. Note that the purity in (1) follows immediately from the claimed identity of Weil-Deligne representations and the Ramanujan conjecture for π [8].

The existence of $\rho_{\pi,\iota}$ with the property (1) for all $v \nmid p$ was proved by Carayol [19] under the assumption that either $[F : \mathbb{Q}]$ is odd, or π_v is square-integrable for some finite prime v of F. In general, $\rho_{\pi,\iota}$ was constructed by Taylor [107], and the proof of (1) for $v \nmid p$ follows the argument of [127, Theorem 2.1.3]. Finally, the property (1) for $v|p$, along with (2), were established in general by Skinner [103] (except that our normalizations of Hodge-Tate weights and reciprocity maps are inverted from *loc. cit.*). The property (3) is an immediate corollary of (1). □

Definition 2.2.2. An automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ has (strong) coefficient field $E_0 \subset \mathbb{C}$ if E_0 is a number field and, for all primes p and isomorphisms $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, $\rho_{\pi,\iota}$ is defined over the p-adic closure of $\iota^{-1}(E_0)$. In this case, it depends only on the prime $\mathfrak{p}$ of E_0 induced by $\mathbb{C}$, and we obtain a well-defined $\rho_{\pi,\mathfrak{p}} : G_F \to \mathrm{GL}_2(E_{0,\mathfrak{p}})$ such that $\rho_{\pi,\iota}$ is the extension of scalars of $\rho_{\pi,\mathfrak{p}}$.

Remark 2.2.3. The argument of [23, Proposition 3.2.5] shows that strong coeffient fields exist in the situation of Theorem 2.2.1.

Notation 2.2.4. Let π be as in Theorem 2.2.1, and let E_0 be a strong coefficient field of π. Fix a prime $\mathfrak{p}$ of E_0 with residue field $k(\mathfrak{p})$.

(1) Write $V_{\pi,\mathfrak{p}}$ for the underlying $E_{0,\mathfrak{p}}[G_F]$-module of $\rho_{\pi,\mathfrak{p}}$.

(2) Let $T_{\pi,\mathfrak{p}} \subset V_{\pi,\mathfrak{p}}$ be a G_F-stable $O_{E_0,\mathfrak{p}}$-lattice. Then the $k(\mathfrak{p})[G_F]$-module $\overline{T}_{\pi,\mathfrak{p}}$ depends only on $V_{\pi,\mathfrak{p}}$ up to semisimplification. We write

$$\overline{\rho}_{\pi,\mathfrak{p}} : G_F \to \mathrm{GL}_2(k(\mathfrak{p}))$$

for the corresponding semisimple Galois representation.

(3) When $\mathfrak{p}$ is clear from context, it may be dropped from all subscripts in the above notations.

We now turn to automorphic representations of GSp_4.

Definition 2.2.5. An automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$ will be called *relevant* if:

- π is cuspidal and not CAP, and has unitary central character.
- π_∞ belongs to the discrete series L-packet of weight $(3, 3)$.

2.2.6. If π is an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, then for all ℓ such that π_ℓ is unramified, recall that we write the Satake parameter of π_ℓ as a multiset of four complex numbers of the form $\{\alpha, \beta, \nu\alpha^{-1}, \nu\beta^{-1}\}$. If π has central character ω_π, then $\nu = \omega_\pi(\langle \ell \rangle) = \mathrm{rec}(\omega_\pi)(\mathrm{Frob}_\ell^{-1})$.

Definition 2.2.7. A cuspidal, non-CAP automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$ is *endoscopic* associated to an unordered pair (π_1, π_2) of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with the same central character if the following holds: for all but finitely many primes ℓ, if $\pi_{i,\ell}$ are both unramified with Satake parameters $\{\alpha_i, \nu/\alpha_i\}$, then π_ℓ is unramified with Satake parameter $\{\alpha_1, \alpha_2, \nu/\alpha_1, \nu/\alpha_2\}$.

2.2.8. In this case, π_1 and π_2 are necessarily distinct by [126, Lemma 5.2], and the central character of π is the common central character of the π_i. In the literature, the property in Definition 2.2.7 is often called being *weakly* endoscopic, and being endoscopic requires a stronger compatibility condition of local Langlands parameters at all finite places; however, the distinction is unimportant, cf. the results of [126, §5].

Lemma 2.2.9. *Suppose π is an endoscopic automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. Then π is relevant if and only if π_1 and π_2 are unitary with discrete series archimedean components of weights 2 and 4 in some order.*

Proof. Recall π is cuspidal and not CAP by definition. Also, π clearly has unitary central character if and only if π_1 and π_2 are unitary, so it suffices to consider the archimedean weights.

Consider the representation $\pi_{1,\infty} \boxtimes \pi_{2,\infty}$ of $\mathrm{GSO}_{2,2}(\mathbb{R}) = (\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}))/\mathbb{R}^\times$; by [126, Lemma 5.6], π_∞ belongs to the local L-packet attached to $\pi_{1,\infty} \boxtimes \pi_{2,\infty}$ via the known Langlands parametrization for real groups and the map of dual groups

$$^L\,\mathrm{GSO}_{2,2} = (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)(\mathbb{C}) \hookrightarrow \mathrm{GSp}_4(\mathbb{C}) = {}^L\,\mathrm{GSp}_{4,\mathbb{R}}\,.$$

We conclude that π_∞ belongs to the desired discrete series L-packet if and only if $\pi_{1,\infty}$ and $\pi_{2,\infty}$ are discrete series of weights 2 and 4 in some order.

□

Theorem 2.2.10. *Let π be a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, and fix a prime p along with an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Then there exists a semisimple Galois representation*

$$\rho_{\pi,\iota} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$$

with the following properties.

1. *$\rho_{\pi,\iota}|_{G_{\mathbb{Q}_p}}$ is potentially semistable, and for all primes ℓ,*
$$\iota WD(\rho_{\pi,\iota}|_{G_{\mathbb{Q}_\ell}})^{F-ss} \simeq \mathrm{rec}_{\mathrm{GT}}(\pi_\ell \otimes |\cdot|^{\frac{1}{2}}).$$
Moreover this Weil-Deligne representation is pure of weight -1.
2. *The Hodge-Tate weights of $\rho_{\pi,\iota}|_{G_{\mathbb{Q}_p}}$ are $\{-1, 0, 1, 2\}$.*
3. *The similitude character of $\rho_{\pi,\iota}$ is $\iota^{-1}\,\mathrm{rec}(\omega_\pi)\chi_{p,\mathrm{cyc}}$, where ω_π is the central character.*

Here $\mathrm{rec}_{\mathrm{GT}}$ is the Gan-Takeda local Langlands correspondence of [36], which associates to an irreducible admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ a $\mathrm{GSp}_4(\mathbb{C})$-valued Weil-Deligne representation.

Proof. Suppose first that π is endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. By Lemma 2.2.9 and Theorem 2.2.1, we can take

$$\rho_{\pi,\iota} := \rho_{\pi_1,\iota} \oplus \rho_{\pi_2,\iota}$$

under the natural embedding $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \to \mathrm{GSp}_4$; this satisfies (1) by Theorem 2.2.1(1) combined with [126, Corollary 5.1, Theorem 5.2(3)] and the construction of $\mathrm{rec}_{\mathrm{GT}}$ in [36]. Then (2) and (3) are satisfied by Theorem 2.2.1(2, 3).

Now suppose π is not endoscopic. By [96, Proposition 10.1] combined with [125, Theorem 1], we may assume without loss of generality that π is globally generic. Under the extra hypothesis that π_v is Steinberg for some $v \neq p$, and omitting the case $\ell = p$ of (1), the theorem then follows from the main result of [105], except that we have twisted the $\rho_{\pi,\iota}$ of *loc. cit.* by $(\chi_p^{\mathrm{cyc}})^2$. The extra hypothesis in [105] is needed only to appeal to the results of Taylor-Yoshida [109], and has since been removed by work of Caraiani [17]. Similarly, the proof in [105] of (1) for $\ell \neq p$ extends to the case $\ell = p$ with the additional input of [18]. □

Notation 2.2.11. In the setting of Theorem 2.2.10, we write $V_{\pi,\iota}$ for the underlying four-dimensional Galois module of $\rho_{\pi,\iota}$.

Lemma 2.2.12. *In the situation of Theorem 2.2.10, suppose $p > 3$. Then $V_{\pi,\iota}$ is reducible if and only if π is endoscopic.*

Proof. This is [123, Theorem 3.1]. ☐

Definition 2.2.13. A relevant automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$ has (strong) coefficient field $E_0 \subset \mathbb{C}$ if E_0 is a number field and, for all primes p and isomorphisms $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, $\rho_{\pi,\iota}$ is defined over the p-adic closure of $\iota^{-1}(E_0)$. In this case, it depends only on the prime $\mathfrak{p}$ of E_0 induced by ι, and we obtain a well-defined $\rho_{\pi,\mathfrak{p}} : G_F \to \mathrm{GSp}_4(E_{0,\mathfrak{p}})$ such that $\rho_{\pi,\iota}$ is the extension of scalars of $\rho_{\pi,\mathfrak{p}}$.

Lemma 2.2.14. *If π is a relevant automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$*, then a strong coefficient field E_0 exists for π.*

Proof. If π is endoscopic, this is clear from Remark 2.2.3, so assume otherwise.

Let $r : \mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$ be the natural embedding, and let $\overline{\iota^{-1}(E_0)}$ be the p-adic closure for any $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. By the argument of [23, Proposition 3.2.5], there exists a number field E_0 such that, for all $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, $r \circ \rho_{\pi,\iota}$ is defined over $\overline{\iota^{-1}(E_0)}$ and $\rho_{\pi,\iota}(g)$ has distinct eigenvalues in $\overline{\iota^{-1}(E_0)}^{\times}$ for some $g \in G_{\mathbb{Q}}$.

It suffices to check $\rho_{\pi,\iota}$ is defined over $\overline{\iota^{-1}(E_0)}$ whenever $p > 3$. For this, let G be the absolute Galois group of $\overline{\iota^{-1}(E_0)}$. For all $\sigma \in G$, we have

$$\rho_{\pi,\iota}^{\sigma} = h(\sigma)\rho_{\pi,\iota}h(\sigma)^{-1}$$

for some $h(\sigma) \in \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$. Using the absolute irreducibility from Lemma 2.2.12 and Schur's lemma, we conclude $h(\sigma) \in \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$, and $\sigma \mapsto h(\sigma)$ defines a cocycle $h \in H^1(G, \mathrm{PGSp}_4(\overline{\mathbb{Q}}_p))$. The class of h determines an inner form H of GSp_4 over $\overline{\iota^{-1}(E_0)}$ such that $\rho_{\pi,\iota}$ can be conjugated to lie in $H(\overline{\iota^{-1}(E_0)})$. However, since $\rho_{\pi,\iota}(g)$ has distinct eigenvalues in $\overline{\iota^{-1}(E_0)}$ for some $g \in G_{\mathbb{Q}}$, H must be split, and this completes the proof. ☐

2.2.15. Analogously to Notation 2.2.4, we make the following notations.

Notation 2.2.16. Let π be as in Theorem 2.2.10, and let E_0 be a strong coefficient field of π. Fix a prime $\mathfrak{p}$ of E_0 with residue field $k(\mathfrak{p})$.

1. Write $V_{\pi,\mathfrak{p}}$ for the four-dimensional underlying $E_{0,\mathfrak{p}}[G_{\mathbb{Q}}]$-module of $\rho_{\pi,\mathfrak{p}}$.
2. Let $T_{\pi,\mathfrak{p}} \subset V_{\pi,\mathfrak{p}}$ be any $G_{\mathbb{Q}}$-stable $O_{\mathfrak{p}}$-lattice; we define $\overline{T}_{\pi,\mathfrak{p}} := (T_{\pi,\mathfrak{p}}/\varpi_{\mathfrak{p}})^{ss}$, which depends only on $V_{\pi,\mathfrak{p}}$. We also write
$$\overline{\rho}_{\pi,\mathfrak{p}} : G_F \to \mathrm{GL}_4(k(\mathfrak{p}))$$
for the corresponding semisimple Galois representation.
3. When $\mathfrak{p}$ is clear from context, it may be dropped from all subscripts in the above notations.

Lemma 2.2.17. *Suppose π is a relevant automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$*. Then there exists a base change* $\mathrm{BC}(\pi)$ *to an automorphic representation of* $\mathrm{GL}_4(\mathbb{A})$*, such that for each place v,*

$$\mathrm{rec}(\mathrm{BC}(\pi)_v) = r \circ \mathrm{rec}_{\mathrm{GT}}(\pi_v),$$

where

$$r : {}^L\mathrm{GSp}_4 = \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}) = {}^L\mathrm{GL}_4$$

is the natural embedding of dual groups. Moreover $\mathrm{BC}(\pi)$ *is cuspidal if and only if π is non-endoscopic.*

Proof. If π is endoscopic associated to (π_1, π_2), then there exists a non-cuspidal base change, which is the isobaric sum $\pi_1 \boxplus \pi_2$; the compatibility with local Langlands parameters is by the same reasoning as the endoscopic case of Theorem 2.2.10.

In the non-endoscopic case, the lemma follows from [96, Proposition 10.1] and its proof; note that the endoscopic transfer to $\mathrm{GL}_4 \times \mathrm{GL}_1$ used in *loc. cit.* is compatible with the Gan-Takeda local Langlands paremeters by the main result of [21]. ☐

Lemma 2.2.18. *Suppose π is a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. If $\mathrm{BC}(\pi)$ is a symmetric cube lift of a non-CM automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A})$, then π_0 has discrete series archimedean component of weight 2.*

Proof. This follows from matching archimedean L-parameters using [49, Theorem B] and Lemma 2.2.17. ☐

Lemma 2.2.19. *Suppose π is a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. If $\mathrm{BC}(\pi)$ is the automorphic induction of a non-CM automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ with K real quadratic, then π_0 has discrete series archimedean components of weights 2 and 4, in some order. Moreover the central characters ω_{π_0} of π_0 and ω_π of π satisfy*

$$\omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}} = \omega_{\pi_0}.$$

Proof. The assertion on archimedean components follows from Lemma 2.2.17 and the compatibility of automorphic induction with local Langlands parameters [3, Chapter 3, Theorem 5.1]. To check the relation of central characters, fix some prime p along with an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Let π_0^{tw} denote the $\mathrm{Gal}(K/\mathbb{Q})$-twist. Then we have

$$V_{\pi,\iota}|_{G_K} = \rho_{\pi_0,\iota} \oplus \rho_{\pi_0^{\mathrm{tw}},\iota}.$$

Using the identities

$$V_{\pi,\iota} = V_{\pi,\iota}^\vee \otimes \iota^{-1}\operatorname{rec}(\omega_\pi) \otimes \chi_{p,\mathrm{cyc}}, \quad \rho_{\pi_0,\iota} = \rho_{\pi_0,\iota}^\vee \otimes \iota^{-1}\operatorname{rec}(\omega_{\pi_0}) \otimes \chi_{p,\mathrm{cyc}},$$

and likewise for π_0^{tw}, it follows that

$$\rho_{\pi_0,\iota} \otimes \iota^{-1}\left(\operatorname{rec}(\omega_\pi)|_{G_K}/\operatorname{rec}(\omega_{\pi_0})\right) \oplus \rho_{\pi_0^{\mathrm{tw}},\iota} \otimes \iota^{-1}\left(\operatorname{rec}(\omega_\pi)|_{G_K}/\operatorname{rec}(\omega_{\pi_0^{\mathrm{tw}}})\right) \cong \rho_{\pi_0,\iota} \oplus \rho_{\pi_0^{\mathrm{tw}},\iota}.$$

Since $\rho_{\pi_0,\iota}$ and $\rho_{\pi_0^{\mathrm{tw}},\iota}$ are absolutely irreducible by Theorem C.3.2, and have different Hodge-Tate weights by Theorem 2.2.1(2), this implies $\operatorname{rec}(\omega_\pi)|_{G_K} = \operatorname{rec}(\omega_{\pi_0})$, i.e. $\omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}} = \omega_{\pi_0}$. ☐

In the next lemma, if K is a quadratic field, we write $\mathrm{BC}_{K/\mathbb{Q}}$ for the base change of an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ to $\mathrm{GL}_n(\mathbb{A}_K)$, which exists by [3].

Lemma 2.2.20. *Suppose π is a relevant, non-endoscopic automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. If $\mathrm{BC}(\pi)$ is the automorphic induction of an automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ with K imaginary quadratic and π_0 is not itself an automorphic induction, then π_0 is of the form $\mathrm{BC}_{K/\mathbb{Q}}(\sigma) \otimes \chi$, where σ is the unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to a non-CM classical modular form of weight $k = 2$ or 3, and χ is a Hecke character of K.*

Proof. By hypothesis, $\mathrm{BC}_{K/\mathbb{Q}} \circ \mathrm{BC}(\pi)$ is an isobaric sum $\pi_0 \boxplus \pi_0^{\mathrm{tw}}$, where tw denotes the $\mathrm{Gal}(K/\mathbb{Q})$-twist.

Considering archimedean L-parameters and using Lemma 2.2.17 combined with [3, Chapter 3, Theorem 5.1], we see that the local L-parameter of π_0 at the unique archimedean place of K is of the form

$$z \mapsto \begin{pmatrix} (z/\overline{z})^{\epsilon_1 \frac{1}{2}} & \\ & (z/\overline{z})^{\epsilon_2 \frac{3}{2}} \end{pmatrix} \tag{2.4}$$

for $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Let $\omega_\pi : \mathbb{A}^\times \to \mathbb{C}^\times$ and ω_{π_0} be the central characters of π and π_0, respectively; then because $\mathrm{BC}(\pi) \cong \mathrm{BC}(\pi)^\vee \otimes \omega_\pi$, we have

$$\begin{aligned} \pi_0^\vee \boxplus (\pi_0^{\mathrm{tw}})^\vee &\cong \pi_0 \otimes \omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}} \boxplus \pi_0^{\mathrm{tw}} \otimes \omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}} \\ &\cong \pi_0 \otimes \omega_{\pi_0}^{-1} \boxplus \pi_0^{\mathrm{tw}} \otimes \omega_{\pi_0}^{-\mathrm{tw}}. \end{aligned}$$

The archimedean L-parameter (2.4) shows that $\pi_0^\vee \ncong \pi_0 \otimes \omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}}$, and hence

$$\pi_0^\vee \cong \pi_0^{\mathrm{tw}} \otimes \omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}} \cong \pi_0 \otimes \omega_{\pi_0}^{-1}. \tag{2.5}$$

On the other hand, computing the central character of $\mathrm{BC}(\pi)$, we have:

$$\omega_\pi^2 = \omega_{\pi_0}|_{\mathbb{A}^\times}. \tag{2.6}$$

In particular, $(\omega_\pi \circ \mathrm{Nm}_{K/\mathbb{Q}})/\omega_{\pi_0}$ is trivial when restricted to $\mathbb{A}^\times$, hence of the form χ/χ^{tw} for an automorphic character χ of $\mathbb{A}_K^\times$. Then $(\pi_0 \otimes \chi^{-1})$ is isomorphic to its $\mathrm{Gal}(K/\mathbb{Q})$-twist by (2.5), hence arises as the base change of a cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$ by [3, Chapter 3, Theorem 4.2]. Without loss of generality, we may assume σ is unitary; considering the archimedean local L-parameter of σ and again using [3, Chapter 3, Theorem 5.1], the archimedean component of σ is discrete series of weight 2 or 3. We have $\pi_0 = \mathrm{BC}_{K/\mathbb{Q}}(\sigma) \otimes \chi$ by construction. □

2.3. **Shimura varieties and Shimura sets.**

2.3.1. Let V be a quadratic space over $\mathbb{Q}$, and recall the algebraic group $\mathrm{GSpin}(V)$ from (1.1.5).

2.3.2. *Indefinite case.* Suppose $V \otimes \mathbb{R}$ has signature $(n, 2)$. If $V^- \subset V_\mathbb{R}$ is a negative definite 2-plane, one obtains a map

$$C^+(V^-) \simeq \mathbb{C} \to C^+(V_\mathbb{R}),$$

which induces a Shimura datum

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathrm{GSpin}(V)_\mathbb{R}.$$

For a neat compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$, the resulting Shimura variety $\mathrm{Sh}_K(V)$ is a smooth quasi-projective variety over $\mathbb{Q}$.

2.3.3. *Definite case.* If V is a positive definite quadratic space over $\mathbb{Q}$, then $\mathrm{GSpin}(V)(\mathbb{R})$ is compact. For a compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$, let $\mathrm{Sh}_K(V)$ denote the finite double coset space

$$\mathrm{Sh}_K(V) := \mathrm{GSpin}(V)(\mathbb{Q})\backslash \mathrm{GSpin}(V)(\mathbb{A}_f)/K.$$

2.3.4. *Hecke algebras.* Suppose $V = V_D$ is one of the quadratic spaces from (1.1.6), and let O be a coefficient ring. If $K = \prod_\ell K_\ell \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$ is a neat compact open subgroup and S is a finite set of primes of $\mathbb{Q}$ containing all those such that K_ℓ is not hyperspecial, then $\mathbb{T}_O^S = \mathbb{T}_{\mathrm{GSp}_4, O}^S$ acts on $H^*_{\text{ét}}(\mathrm{Sh}_K(V)_{\overline{\mathbb{Q}}}, O)$; the cohomology is interpreted as $O[\mathrm{Sh}_K(V)]$ in the definite case. We denote by $\mathbb{T}^S_{K, V_D, O}$ the quotient of $\mathbb{T}^S_O$ defined by this action, and may drop the subscript O when it is clear from context.

2.4. **Automorphic representations of** GSpin_5 **groups.**

2.4.1. Recall the five-dimensional quadratic spaces V_D from (1.1.6).

Definition 2.4.2. An automorphic representation π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ is *Eisenstein* if there exists a parabolic subgroup $P \subset \mathrm{GSp}_4$, with Levi factor L, and an automorphic representation σ of $L(\mathbb{A})$ such that π is nearly equivalent to a constituent of $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})} \sigma$.

The following generalizes Definitions 2.2.5 and 2.2.7.

Definition 2.4.3. For a squarefree integer $D \geq 1$, an automorphic representation π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ will be called *relevant* if:

(1) π is not Eisenstein (in particular is cuspidal), and has unitary central character.
(2) π_∞ is trivial if $\sigma(D)$ is odd (in which case $\mathrm{GSpin}(V_D)(\mathbb{R})$ is a compact group); or belongs to the discrete series L-packet of weight $(3,3)$ if $\sigma(D)$ is even (in which case $\mathrm{GSpin}(V_D)(\mathbb{R}) = \mathrm{GSp}_4(\mathbb{R})$).

A non-Eisenstein automorphic representation π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ is called *endoscopic* associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ if the condition in Definition 2.2.7 is satisfied for all but finitely many $\ell \nmid D$.

Lemma 2.4.4. *Fix a squarefree $D \geq 1$ such that $\sigma(D)$ is even, and suppose π is an automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ that is not Eisenstein. Then for any neat compact open subgroup $K = \prod K_\ell$ of $\mathrm{GSpin}(V_D)(\mathbb{A}_f)$:*

(1) The natural map induces an isomorphism

$$H^i_c(\mathrm{Sh}_K(V_D), \mathbb{C})[\pi_f] \xrightarrow{\sim} H^i(\mathrm{Sh}_K(V_D), \mathbb{C})[\pi_f].$$

(2) We have

$$H^i(\mathrm{Sh}_K(V_D), \mathbb{C})[\pi_f] = \bigoplus_{\pi'_\infty} m(\pi_f \otimes \pi'_\infty) \cdot \pi_f^K \otimes H^i(\mathfrak{gsp}_4, U(2); \pi'_\infty) \neq 0,$$

where $U(2) \subset \mathrm{GSp}_4(\mathbb{R})$ is the maximal compact subgroup, π'_∞ runs over cohomological representations of $\mathrm{GSp}_4(\mathbb{R})$, and $m(\pi_f \otimes \pi'_\infty)$ is the multiplicity in the discrete (equivalently cuspidal) automorphic spectrum of $\mathrm{GSpin}(V_D)(\mathbb{A})$.

(3) If π is relevant and $H^i(\mathrm{Sh}_K(V_D), \mathbb{C})[\pi_f] \neq 0$, then $i = 3$.

Proof. The first part is well-known, cf. [43, Chapter 9]. Then (2) is immediate from Matsushima's formula and the diagram in [108, p. 293]. We now show (3). Without loss of generality, we may assume that $\pi_f^K \neq 0$; in particular, because π is relevant, (2) implies that

$$H^3(\mathrm{Sh}_K(V_D), \mathbb{C})[\pi_f] \neq 0. \tag{2.7}$$

Now let S be a finite set of places of $\mathbb{Q}$ such that K_ℓ is hyperspecial for $\ell \notin S$, and fix a prime $p \notin S \cup \{2,3\}$ along with an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. It suffices to show

$$H^i_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)[\pi_f] \neq 0 \implies i = 3. \tag{2.8}$$

For this, we argue as in [108], with the additional input of the Fontaine-Mazur conjecture for GL_2. Set W^i to be the π^S-isotypic component[7] of $H^i_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$ with respect to the natural action of $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}$; then W^i is a $G_{\mathbb{Q}}$-module. We have the following facts:

(1) By Margulis' superrigidity theorem [73, Chapter IX, Corollary 7.15(iii)], $W^1 = 0$, and clearly $W^0 = W^6 = 0$ because π_f is not one-dimensional.
(2) For each $\ell \notin S \cup \{p\}$, Frob_ℓ satisfies the Eichler-Shimura relation on $H^*(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$ by [122]; in particular, if π_ℓ has Satake parameters $\{\alpha, \beta, \nu\alpha^{-1}, \nu\beta^{-1}\}$, then

$$(\mathrm{Frob}_\ell^{-1} - \ell^{3/2}\alpha)(\mathrm{Frob}_\ell^{-1} - \ell^{3/2}\beta)(\mathrm{Frob}_\ell^{-1} - \ell^{3/2}\nu\alpha^{-1})(\mathrm{Frob}_\ell^{-1} - \ell^{3/2}\nu\beta^{-1}) = 0 \text{ on each } W^i. \tag{2.9}$$

(3) By Poincaré duality and part (1) of the theorem there are perfect pairings

$$W^i \times W^{6-i} \to \mathrm{rec}(\omega_\pi)(-3)$$

for all i, where ω_π is the central character of π, cf. [108, p. 297].
(4) Again by (1) of the theorem, for all $\ell \notin S \cup \{p\}$, the eigenvalues of Frob_ℓ on W^i are Weil numbers of weight i.

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[7] Defined using ι. We elide ι from the discussion to ease the burden of notation.

Now suppose $W^i \neq 0$ for some $i \neq 3$. By (1), $i \neq 0, 1$, and then by (3) we conclude that $W^2 \neq 0$. If $\ell \notin S \cup \{p\}$ is a prime such that Frob_ℓ has n distinct eigenvalues on W^2, it has n distinct eigenvalues on W^4 as well by (3), and at least one eigenvalue on W^3 by (2.7). In particular, by (4) and the fact that Frob_ℓ has at most 4 total eigenvalues by (2), we conclude that $n = 1$ and Frob_ℓ also has at most two eigenvalues on W^3. Now the same argument as [108, Proposition 3] shows that there is a two-dimensional representation $R : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ – possibly reducible – with distinct Hodge-Tate weights such that $(W^3)^{ss} = R^{\oplus e}$ for some $e \geq 1$, and a character $\chi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ such that $(W^2)^{ss} = \chi^{\oplus d}$ and $(W^4)^{ss} = (\mathrm{rec}(\omega_\pi)\chi_{p,\mathrm{cyc}}^{-3}\chi^{-1})^{\oplus d}$ for some $d \geq 1$. Note that ω_π is even because the central character of π_∞ is trivial, so R is odd by (3) above. Hence by [86, Theorem 1.0.4], R is automorphic. Comparing with (2), we see that there exists an automorphic character ω of $\mathbb{A}^\times$ and an automorphic representation σ of $\mathrm{GL}_2(\mathbb{A})$ such that π is nearly equivalent to a constituent of the representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})} \sigma \boxtimes \omega,$$

where $P \subset \mathrm{GSp}_4$ is the Siegel parabolic subgroup with Levi factor $\mathrm{GL}_2 \boxtimes \mathrm{GL}_1$. Hence π is Eisenstein, which contradicts the assumption that π is relevant and completes the proof. ☐

We will require some information about Jacquet-Langlands transfers of relevant automorphic representations between the various $\mathrm{GSpin}(V_D)(\mathbb{A})$.

Definition 2.4.5. For a tempered irreducible admissible representation π_ℓ of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$, we say π_ℓ is *transferrable* if it does not belong to the types I, IIIa, VIa, VIb, VII, VIIIa, VIIIb, or IXa in the notation of [95].

In particular, if π_ℓ is unramified, it is not transferrable.

Theorem 2.4.6. *Let $D \geq 1$ be squarefree. Then:*

1. *For each relevant, non-endoscopic automorphic representation $\Pi = \otimes' \Pi_v$ of $\mathrm{GSpin}(V_D)(\mathbb{A})$, $\Pi_f^D := \otimes'_{\ell \nmid D} \Pi_\ell$ can be completed to a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$.*
2. *Conversely, suppose $\pi = \otimes' \pi_v$ is a relevant, non-endoscopic automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Then $\pi_f^D := \otimes'_{\ell \nmid D} \pi_\ell$ can be completed to a relevant automorphic representation Π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ if and only if π_ℓ is transferrable for all primes $\ell | D$.*
3. *Let $\pi = \otimes' \pi_\ell$ be a relevant, non-endoscopic automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, and assume π_ℓ is transferrable for all $\ell | D$. Then any relevant automorphic representation Π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ with $\Pi_f^D \cong \pi_f^D$ has automorphic multiplicity one. The set of all such Π is a Cartesian product of local L-packets for $v | D\infty$, where the nonarchimedean L-packets are determined only by the corresponding local factors of π. The archimedean L-packet is the discrete series packet of weight $(3, 3)$ when $\sigma(D)$ is even and the trivial representation when $\sigma(D)$ is odd.*

Proof. When $\sigma(D)$ is odd, so that $\mathrm{GSpin}(V_D)(\mathbb{R})$ is compact, this is [96, Theorem 11.4]. To prove these assertions when $\sigma(D)$ is even, we follow the sketch indicated in the discussion following *loc. cit.*

The following fact will be used repeatedly:

$$\begin{gathered}\text{If } \pi_\infty \text{ is an irreducible admissible representation of } \mathrm{GSp}_4(\mathbb{R}) \text{ with } H^3(\mathfrak{gsp}_4, U(2); \pi_\infty) \neq 0, \\ \text{then } \pi_\infty \text{ lies in the discrete series } L\text{-packet of weight } (3, 3).\end{gathered} \tag{2.10}$$

This fact follows from the calculations in [108, p. 293].

To prove (1), suppose first that $\Pi = \otimes' \Pi_v$ is a relevant automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$. By [96, Corollary 7.4] combined with Lemma 2.4.4(3), we conclude that Π_f^D can be completed to a cohomological automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$. Moreover π_f necessarily contributes to cohomology in degree 3 by [96, Proposition 8.2], and in particular π is relevant by (2.10) combined with Lemma 2.4.4(2), so this proves (1).

To prove (2) and (3), we need some preparation. Fix a relevant automorphic representation $\pi = \otimes' \pi_v$ of $\mathrm{GSp}_4(\mathbb{A})$. For each prime $\ell | D$ such that π_ℓ is transferrable, there is a local L-packet of irreducible admissible representations of $\mathrm{GSpin}(V_D)(\mathbb{Q}_\ell)$ determined by the character relations of Lemma 11.1 of *op. cit.* Let $\mathbb{A}_D = \prod_{\ell | D} \mathbb{Q}_\ell$; by an irreducible admissible representation of $\mathrm{GSpin}(V_D)(\mathbb{A}_D)$ we mean a direct product of irreducible admissible representations of $\mathrm{GSpin}(V_D)(\mathbb{Q}_\ell)$ for $\ell | D$.

If π_ℓ is transferrable for all $\ell | D$, let S be the set of irreducible admissible representations of $\mathrm{GSpin}(V_D)(\mathbb{A}_D)$ obtained by taking the Cartesian product of the local L-packets. If π_ℓ is not transferrable for some $\ell | D$, then let S be the empty set.

The comparison of cohomological trace formulas from the proof of [96, Theorem 11.4] shows the following identity: for all irreducible admissible representations Π_D of $\mathrm{GSpin}(V_D)(\mathbb{A}_D)$, we have

$$\frac{1}{2} \sum_{\pi_\infty} m(\pi^D \otimes \Pi_D \otimes \pi_\infty) \chi(\pi_\infty) = \begin{cases} 2, & \Pi_D \in S, \\ 0, & \Pi_D \notin S. \end{cases} \tag{2.11}$$

Here the notation is as follows:

- The sum over π_∞ runs over cohomological representations of $\mathrm{GSp}_4(\mathbb{R})$, and $\chi(\pi_\infty)$ is the negative Euler characteristic $\sum (-1)^{i+1} \dim H^i(\mathfrak{gsp}_4, U(2); \pi_\infty)$.
- π^D is the representation $\otimes'_{\ell \nmid D} \pi_\ell$ of $\mathrm{GSpin}(V_D)(\mathbb{A}^D)$.
- $m(\pi^D \otimes \Pi_D \otimes \pi_\infty)$ is the multiplicity in the discrete (equivalently cuspidal, since π is relevant) automorphic spectrum of $\mathrm{GSpin}(V_D)(\mathbb{A})$.

We can further manipulate the left-hand side of (2.11). Let π_∞^W and π_∞^H be the generic and holomorphic members, respectively, of the discrete series L-packet of weight $(3, 3)$. We then claim that

$$\frac{1}{2} \sum_{\pi_\infty} m(\pi^D \otimes \Pi_D \otimes \pi_\infty) \chi(\pi_\infty) = m(\pi^D \otimes \Pi_D \otimes \pi_\infty^W) + m(\pi^D \otimes \Pi_D \otimes \pi_\infty^H). \tag{2.12}$$

To prove (2.12), suppose first that the right-hand side is positive. Then any cohomological π_∞ with $\Pi = \pi_f^D \otimes \Pi_D \otimes \pi_\infty$ automorphic can have Lie algebra cohomology only in degree 3 by Lemma 2.4.4(2, 3). In particular, if $m(\pi^D \otimes \Pi_D \otimes \pi_\infty)\chi(\pi_\infty) \neq 0$ then π_∞ is either π_∞^W or π_∞^H by (2.10), so – combined with the fact that $\chi(\pi_\infty^W) = \chi(\pi_\infty^H) = 2$ by [108, p. 293] – we have shown (2.12) when the right-hand side is positive. On the other hand, suppose the right-hand side of (2.12) vanishes; then each summand on the left-hand side of (2.12) is non-positive by (2.10) because Lie algebra cohomology vanishes for $\mathrm{GSp}_4(\mathbb{R})$ outside degrees 0, 2, 3, 4, and 6. But the left-hand side of (2.12) is also non-negative by (2.11), so we conclude that it is zero, hence (2.12) again holds.

In particular, by (2.12) and (2.11) together, we have

$$m(\pi^D \otimes \Pi_D \otimes \pi_\infty^W) + m(\pi^D \otimes \Pi_D \otimes \pi_\infty^H) = \begin{cases} 2, & \Pi_D \in S, \\ 0, & \Pi_D \notin S. \end{cases} \tag{2.13}$$

From (2.13), it is clear that π^D can be completed to a relevant automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ if and only if S is nonempty, i.e. if and only if π_ℓ is transferrable for all $\ell | D$; this shows (2).

We also see from (2.13) that (3) is equivalent to the assertion that $m(\pi^D \otimes \Pi_D \otimes \pi_\infty^W) = m(\pi^D \otimes \Pi_D \otimes \pi_\infty^H) = 1$ for all $\Pi_D \in S$. Suppose for contradiction that $m(\pi^D \otimes \Pi_D \otimes \pi_\infty^W)$ and $m(\pi^D \otimes \Pi_D \otimes \pi_\infty^H)$ are 0 and 2, in some order. Let $K = \prod K_\ell \subset \mathrm{GSpin}(V_D)$ be a neat compact open subgroup such that $\Pi_f^K \neq 0$, and let $p > 3$ be a prime such that K_p is hyperspecial. Fix as well an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Then by the discussion in [108, p. 296], $H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)[\Pi_f^K]$ is nonzero and has exactly two distinct Hodge-Tate weights as a representation of $G_{\mathbb{Q}_p}$.

On the other hand, by (2) from the proof of Lemma 2.4.4 combined with the irreducibility of $V_{\pi,\iota}$ (Lemma 2.2.12), up to semisimplification $H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)[\Pi_f^K]$ is a sum of copies of $V_{\pi,\iota}$, which is a contradiction because $V_{\pi,\iota}$ has distinct Hodge-Tate weights by Theorem 2.2.10(2). This completes the proof of (3). □

Remark 2.4.7. One can similarly prove Theorem 2.4.6 in all regular weights, by considering cohomology of Siegel threefolds with coefficients in more general local systems; we omit the details for concision.

2.5. **Relevant endoscopic automorphic representations.**

2.5.1. Relevant endoscopic automorphic representations of $\mathrm{GSpin}(V_D)(\mathbb{A})$ can be constructed as follows. Fix unitary cuspidal automorphic representations π_1 and π_2 of $\mathrm{GL}_2(\mathbb{A})$ with the same central character, and whose archimedean components are discrete series of weights 4 and 2, in some order. Let D_1 and D_2 be squarefree positive integers, such that π_i admits a Jacquet-Langlands transfer $\pi_i^{D_i}$ to $B_{D_i}^\times(\mathbb{A})$ for $i = 1, 2$ (notation as in (1.1.6)).

Define $D_1 * D_2 := D_1D_2/\gcd(D_1, D_2)^2$, which is a squarefree positive integer. Then as in [21, §3.3], there is a global theta lift,

$$\Theta(\pi_1^{D_1} \boxtimes \pi_2^{D_2}) = \Theta(\pi_2^{D_2} \boxtimes \pi_1^{D_1}), \tag{2.14}$$

which is either zero or a cuspidal automorphic representation of $\mathrm{GSpin}(V_{D_1 * D_2})(\mathbb{A})$.

Theorem 2.5.2. *With notation as above, relabel π_1 and π_2 if necessary so that π_1 has weight 2 and π_2 has weight 4.*

1. *The theta lift $\Theta(\pi_1^{D_1} \boxtimes \pi_2^{D_2})$ is nonzero if and only if either $\sigma(D_2)$ or $\sigma(D_1 * D_2)$ is even. When nonzero, $\Theta(\pi_1^{D_1} \boxtimes \pi_2^{D_2})$ is always relevant and endoscopic associated to the pair (π_1, π_2), and these representations are all distinct.*
2. *Conversely, each relevant endoscopic automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ arises in this way, and appears with automorphic multiplicity one in the discrete (equivalently cuspidal) spectrum.*

Proof. This follows from the results of [21, §3], combined with the discussion of the local archimedean theta lift in §2.4 of *op. cit.*. ☐

Corollary 2.5.3. *For any relevant automorphic representation Π of $\mathrm{GSpin}(V_D)(\mathbb{A})$, there exists a relevant automorphic representation π of $\mathrm{GSp}_4(\mathbb{A})$ such that:*

1. *For each prime $\ell \nmid D$, $\Pi_\ell \cong \pi_\ell$.*
2. *For each prime $\ell | D$, π_ℓ is transferrable, and Π_ℓ lies in the corresponding local packet of representations of $\mathrm{GSpin}(V_D)(\mathbb{Q}_\ell)$ from Theorem 2.4.6(3).*

Proof. If Π is not endoscopic, this follows from Theorem 2.4.6. If Π is endoscopic, (1) is immediate from Theorem 2.5.2. For (2), it follows because the Langlands correspondences of [36, 37] are constructed to be compatible with theta lifting, and the correspondence of local representations in [96, Table 3] respects Langlands parameters by Lemma 11.1 of *op. cit.* ☐

Remark 2.5.4. By Corollary 2.5.3, we can associate to each relevant automorphic representation Π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ a compatible system of Galois representations $\rho_{\Pi,\iota}$ as in Theorem 2.2.10.

2.6. **Local representations with paramodular fixed vectors.**

Notation 2.6.1. For all primes q (whether or not $q|D$), the *paramodular subgroup* of $\mathrm{GSpin}(V_D)(\mathbb{Q}_q)$ is a maximal compact subgroup described in [104, p. 918]. To avoid confusion, we denote this subgroup by K_q^{Pa} when $q \nmid D$ and by K_q^{ram} when $q|D$.

Lemma 2.6.2. *Let π be a relevant automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$, and suppose $q \nmid D$ is a prime such that π_q has a K_q^{Pa}-fixed vector. Then:*

1. *π_q is either spherical or of type IIa in the notation of* [95].

(2) *π is the unique completion of $\pi_f^q \otimes \pi_\infty$ to an automorphic representation of* $\mathrm{GSpin}(V_D)(\mathbb{A})$.

(3) *If π_q is of type IIa, then for any $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ with $q \neq p$, the action of $I_{\mathbb{Q}_q}$ on $V_{\pi,\iota}$ is unipotent with monodromy of rank one. Moreover, the corresponding local packet of representations of* $\mathrm{GSpin}(V_{Dq})(\mathbb{Q}_q)$ *in Theorem 2.4.6(3) is a single representation with a unique K_q^{ram}-fixed vector.*

Proof. By Corollary 2.5.3, $\pi_f^D = \otimes_{\ell \nmid D} \pi_\ell$ can be completed to a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. In particular, π_q is tempered (by Theorem 2.2.10(1)), so (1) follows from [95, Tables A.1, A.13].

If $\pi_f^q \otimes \pi_q' \otimes \pi_\infty$ is automorphic for some π_q', then $\pi_f^{Dq} \otimes \pi_q'$ can be completed to an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ by the same reasoning as for π_f^D. Hence by Theorem 2.2.10(1), π_q' and π_q belong to the same Gan-Takeda local L-packet. But the L-packets of type I and IIa are singletons, so this shows (2).

Finally, suppose π_q is the type IIa representation denoted $\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ in *loc. cit.* Because π_q has a K_q^{Pa}-fixed vector, χ and σ are unramified characters of $\mathbb{Q}_q^\times$. This implies the assertions on $\rho_{\pi,\iota}|_{I_{\mathbb{Q}_q}}$ in (3) by Theorem 2.2.10(1) and the explicit local Langlands paremeters found in [95, Table A.7]. The final claim in (3) follows from [96, Table 3] combined with [104, Theorem B]. □

2.7. **Generic maximal ideals and cohomology of GSpin_5 Shimura varieties.**

2.7.1. For this subsection, fix a coefficient field $E \subset \overline{\mathbb{Q}}_p$ with E a finite extension of $\mathbb{Q}_p$, and let $O \subset E$ be the ring of integers with uniformizer ϖ. Also fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, a squarefree $D \geq 1$, a neat compact open subgroup $K = \prod_\ell K_\ell \subset \mathrm{GSpin}(V_D)(\mathbb{A}_f)$, and a set S of places of $\mathbb{Q}$ containing all ℓ such that K_ℓ is not hyperspecial.

Lemma 2.7.2. *Suppose $\mathfrak{m} \subset \mathbb{T}_O^S$ is non-Eisenstein and $\sigma(D)$ is even. Then for all i, the natural maps induce isomorphisms:*

$$\begin{aligned} H_c^i(\mathrm{Sh}_K(V_D), O)_\mathfrak{m} &\xrightarrow{\sim} H^i(\mathrm{Sh}_K(V_D), O)_\mathfrak{m}, \\ H_c^i(\mathrm{Sh}_K(V_D), \overline{\mathbb{F}}_p)_\mathfrak{m} &\xrightarrow{\sim} H^i(\mathrm{Sh}_K(V_D), \overline{\mathbb{F}}_p)_\mathfrak{m}. \end{aligned} \tag{2.15}$$

Proof. We show the second isomorphism; the first follows formally. Let $P = MN \subset \mathrm{GSpin}(V_D)$ be a parabolic subgroup. Then we may fix an identification $\mathrm{GSpin}(V_D)(\mathbb{A}^S) \simeq \mathrm{GSp}_4(\mathbb{A}^S)$ such that $P(\mathbb{A}^S) = P_1(\mathbb{A}^S)$ for a parabolic subgroup $P_1 = M_1 N_1 \subset \mathrm{GSp}_4$.

The Levi factor M is abstractly isomorphic to either $B_D^\times \times \mathrm{GL}_1$ or GL_1^3 (the latter occurring only if $D = 1$). To any compact open subgroup $K_M \subset M(\mathbb{A}_f)$ we can associate a locally symmetric space $S_{K_M}(M) = M(\mathbb{Q}) \backslash M(\mathbb{A}) / K_M \cdot K_\infty$, where $K_\infty \subset M(\mathbb{R})$ is the product of the center and a maximal compact subgroup. In particular, each connected component of $S_{K_M}(M)$ is either a Shimura curve or an isolated point.

Using the Borel-Serre compactification of $\mathrm{Sh}_K(V_D)$ and the argument of [82, §4], it suffices to show the following: for any parabolic subgroup $P = MN \subset \mathrm{GSpin}(V_D)$ as above and any compact open subgroup $K_M = \prod_\ell K_{M,\ell} \subset M(\mathbb{A}_f)$ with $K_{M,\ell}$ hyperspecial for $\ell \notin S$, the support of the $\mathbb{T}^S_{\mathrm{GSp}_4, \overline{\mathbb{F}}_p}$-module $(S_M^{\mathrm{GSp}_4})^* H^i(S_{K_M}(M), \overline{\mathbb{F}}_p)$ is Eisenstein for all i. But this follows from Proposition 2.1.9 because, as $M = B_D^\times \times \mathrm{GL}_1$ or GL_1^3, every maximal ideal of $\mathbb{T}^S_{M, \overline{\mathbb{F}}_p}$ in the support of $H^i(S_{K_M}(M), \overline{\mathbb{F}}_p)$ is clearly of Galois type. □

Definition 2.7.3.

(1) A maximal ideal $\mathfrak{m} \subset \mathbb{T}_{\ell,O}$ is called *generic* if $p \nmid 2\ell(\ell^4 - 1)$, and the Satake parameter $\{\alpha, \beta, \nu/\alpha, \nu/\beta\}$ of $\mathfrak{m}$ is multiplicity-free with no two elements having ratio ℓ.

(2) A maximal ideal $\mathfrak{m} \subset \mathbb{T}_O^S$ is called *generic* if there exist infinitely many $\ell \notin S$ such that the induced maximal ideal of $\mathbb{T}_{\ell,O}$ is generic. For any quotient $\mathbb{T}$ of $\mathbb{T}_O^S$, a maximal ideal $\mathfrak{m} \subset \mathbb{T}$ is called generic if its pullback to $\mathbb{T}_O^S$ is so.

(3) A Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$, unramified outside a finite set S, is called *generic* if there exists a prime $\ell \notin S$ with $p \nmid 2\ell(\ell^4 - 1)$, such that $\overline{\rho}(\mathrm{Frob}_\ell)$ has distinct eigenvalues, no two having ratio ℓ.

Remark 2.7.4. If $\mathfrak{m} \subset \mathbb{T}^S_O$ has an associated Galois representation $\overline{\rho}_{\mathfrak{m}}$, clearly $\mathfrak{m}$ is generic if and only if $\overline{\rho}_{\mathfrak{m}}$ is so.

Theorem 2.7.5. *Suppose $\mathfrak{m} \subset \mathbb{T}^S_O$ is a generic maximal ideal and $\sigma(D)$ is even. Then:*

(1) For all $i < 3$, we have

$$H^i(\mathrm{Sh}_K(V_D), \overline{\mathbb{F}}_p)_{\mathfrak{m}} = H^{6-i}_c(\mathrm{Sh}_K(V_D), \overline{\mathbb{F}}_{\mathfrak{p}})_{\mathfrak{m}} = 0.$$

(2) If $\mathfrak{m}$ is also non-Eisenstein, then

$$H^i_c(\mathrm{Sh}_K(V_D), O)_{\mathfrak{m}} = H^i(\mathrm{Sh}_K(V_D), O)_{\mathfrak{m}}$$

is ϖ-torsion-free for all i, and vanishes unless $i = 3$.

Proof. Part (1) is immediate from [42, Theorem 1.16] and our definition of genericity. Using Lemma 2.7.2, (2) is a standard consequence of (1). ☐

Lemma 2.7.6. *Suppose $\mathfrak{m} \subset \mathbb{T}^S_{K,V_D,O}$ is non-Eisenstein and generic, and suppose π is an automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ such that $\pi_f^K \neq 0$, and the action of $\mathbb{T}^S_O$ on $\iota^{-1}\pi_f^K$ factors through $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$. Then π is not Eisenstein (in the sense of Definition 2.4.2).*

With a bit more care, the genericity assumption can be dropped; we leave the details to the reader.

Proof. Suppose first that $D = 1$, so π is an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$, and let $\mathfrak{m}_\pi \subset \mathbb{T}^S_{\mathrm{GSp}_4, \overline{\mathbb{Q}}_p}$ be the maximal ideal determined by the Hecke action on $\iota^{-1}\pi^S$. Then by hypothesis, $\mathfrak{m}_\pi$ is in the support of the Hecke module $H^3(\mathrm{Sh}_K(V_1), \overline{\mathbb{Q}}_p)_{\mathfrak{m}}$. By Lemma 2.7.2 combined with the diagram in [108, p. 293], we conclude that π^S can be completed to an automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A})$ that appears in the discrete spectrum. Suppose for contradiction that π is Eisenstein; then π' is either CAP or a residual representation. In either case, it follows that the Satake parameter of π_ℓ contains a pair of the form $\{\alpha, \alpha\ell\}$ for all $\ell \notin S$: when π' is CAP, this uses [88, Theorems 2.5, 2.6] and [106, Theorem C], and when π' is residual it uses well-known results on the irreducibility of principal series representations for GSp_4. But these Satake parameters are inconsistent with the genericity of $\mathfrak{m}$, so π cannot be Eisenstein, as desired.

We now handle the case of general D; the argument is a mild refinement of the trace formula method used in [96]. We abbreviate $G_D \coloneqq \mathrm{GSpin}(V_D)$ and fix a minimal parabolic subgroup $P_0 \subset G_D$ (which will be all of G_D if $\sigma(D)$ is odd). Let $f_D \in C^\infty_c(G_D(\mathbb{A}_f), \mathbb{C})$ be a test function with regular support, and define

$$T(f_D) = \sum_{P_0 \subset P = MN \subset G_D} (-1)^{\mathrm{rank}(M) - \mathrm{rank}(G_D)} \sum_{w \in W^P} (-1)^{\ell(w)} \,\mathrm{tr}\left(\overline{f}^P_D \cdot \chi^G_P, \varinjlim_{K_M} H^*(S_{K_M}(M), V_{w\cdot\rho_{G_D} - \rho_M}) \right). \tag{2.16}$$

Here W^P is the set of minimal-length coset representatives for the Weyl group of G_D modulo that of M, tr is the supertrace, $\overline{f}^P$ and $\chi^{G_D}_P$ are defined as in [126, §2.6], $S_{K_M}(M)$ is the symmetric space from the proof of Lemma 2.7.2, ρ_{G_D} and ρ_M are the half-sums of positive roots, and $V_{w\cdot\rho_{G_D} - \rho_M}$ is the complex local system on $S_{K_M}(M)$ of weight $w \cdot \rho_{G_D} - \rho_M$. Although $w \cdot \rho_{G_D} - \rho_M$ might not be integral, we interpret $V_{w\cdot\rho_{G_D} - \rho_M}$ as the twist of $V_{w\cdot\rho_{G_D} - \rho_{G_D}}$ by the real character $\delta_P^{1/2}$ of $M(\mathbb{R})$, i.e.

$$H^*(S_{K_M}(M), V_{w\cdot\rho_{G_D} - \rho_M}) = \delta_P^{-1/2} \otimes H^*(S_{K_M}(M), V_{w\cdot\rho_{G_D} - \rho_{G_D}}) \tag{2.17}$$

as Hecke modules.

For test functions $f_1 \in C_c^\infty(G_1(\mathbb{A}_f), \mathbb{C})$ with regular support, we define $T(f_1)$ analogously to (2.16). If f_D and $f_1 \in C_c^\infty(G_1(\mathbb{A}_f), \mathbb{C})$ are matching functions in the sense of [96, §5], then it follows from combining [126, Lemma 2.10] with [96, Theorem 5.3] that

$$T(f_D) = T(f_1)c_D \tag{2.18}$$

for a nonzero constant c_D depending only on D.

Shrinking K_S if necessary, we can fix a K_S-biinvariant test function $f_{D,S} \in C_c^\infty(G_D(\mathbb{A}_S))$ with regular support such that

$$\operatorname{tr}\left(f_{D,S}, H^*(\mathrm{Sh}_K(V_D), \mathbb{C})[\pi^S]\right) \neq 0; \tag{2.19}$$

when $\sigma(D)$ is even, we use Theorem 2.7.5(2) to ensure that contributions from π in different degrees do not cancel.Let $f_{1,S} \in C_c^\infty(G_1(\mathbb{A}_S))$ be a matching function, and let $K_{1,S} \subset G_1(\mathbb{A}_S)$ be a compact open subgroup such that $f_{1,S}$ is $K_{1,S}$-biinvariant. We also fix an isomorphism $G_D(\mathbb{A}_f^S) \simeq G_1(\mathbb{A}_f^S)$, and let $K_1 = K^S K_{1,S} \subset G_1(\mathbb{A}_f)$, which is a compact open subgroup. Let

$$\Pi(D) = \bigoplus_{P_0 \subset P = MN \subset G_D} \bigoplus_{w \in W^P} \operatorname{Ind}_{P(\mathbb{A}_f)}^{G_D(\mathbb{A}_f)} \varinjlim_{K_M} H^*(S_{K_M}(M), V_{w\cdot\rho_{G_D} - \rho_M}),$$

and let $\mathcal{P}_D$ be the set of irreducible $G_D(\mathbb{A}_f)$-constituents occurring in $\Pi(D)$ with a K-fixed vector. Likewise, we define $\Pi(1)$ and $\mathcal{P}_1$, where now we consider constituents with K_1-fixed vectors, and set $\mathcal{P} = \mathcal{P}_D \sqcup \mathcal{P}_1$.

Then $\mathcal{P}$ contains finitely many near equivalence classes, cf. [126, p. 45]. We can therefore fix a finite set T of primes of $\mathbb{Q}$, disjoint from S, so that two representations $\sigma, \tau \in \mathcal{P}$ are nearly equivalent if and only if $\sigma_T \cong \tau_T$. Now fix a $K_T = \prod_{\ell \in T} K_\ell$-biinvariant test function $f_T \in C_c^\infty(G_D(\mathbb{A}_T), \mathbb{C}) = C_c^\infty(G_1(\mathbb{A}_T), \mathbb{C})$, such that for all $\tau \in \mathcal{P}$, $\operatorname{tr}(f_T|\tau_T) = 0$ unless $\tau_T \cong \pi_T$, and $\operatorname{tr}(f_T|\pi_T) = 1$.

Also fix an auxiliary prime $v_0 \notin S \cup T$, and for a large constant $C > 0$ to be chosen later, let $f_{v_0,C} \in \mathbb{T}_{\mathrm{GSp}_4, v_0, \mathbb{C}}$ be a test function satisfying the conclusion of [96, Lemma 3.9] for the representation π_{v_0}; in particular, $\operatorname{tr}(f_{v_0,C}|\pi_{v_0}) = 1$.

We consider the global test function

$$f_D = f_{D,S} f_T f_{v_0,C} f^{S\cup T\cup\{v_0\}} \in C_c^\infty(G_D(\mathbb{A}_f)),$$

where $f^{S\cup T\cup\{v_0\}}$ is the indicator function of $K^{S\cup T\cup\{v_0\}}$. The matching function is

$$f_1 = f_{1,S} f_T f_{v_0,C} f^{S\cup T\cup\{v_0\}} \in C_c^\infty(G_1(\mathbb{A}_f)).$$

Claim. For all $P_0 \subset P = MN \subsetneq G_1$ and all $w \in W^P$, we have

$$\operatorname{tr}\left(\overline{f}_1^P \cdot \chi_P^G,\ \delta_P^{-1/2} \varinjlim_{K_M} H^*(S_{K_M}(M), V_{w\cdot\rho_{G_1} - \rho_{G_1}})\right) = 0,$$

and likewise for all $P_0 \subset P = MN \subsetneq G_D$.

Proof of claim. To ease notation, we prove the claim for G_1; the proof for G_D is identical. Let τ be an irreducible constituent of $\varinjlim H^*(S_{K_M}(M), V_{w\cdot\rho_{G_1} - \rho_{G_1}})$. It suffices to show that

$$\operatorname{tr}\left(\overline{f}_1^P \cdot \chi_P^{G_1}, \tau \otimes \delta_P^{-1/2}\right) = 0.$$

Now by the argument of [96, Proposition 3.10], if C is chosen sufficiently large, then

$$\operatorname{tr}\left(\overline{f}_1^P \cdot \chi_P^G, \tau \otimes \delta_P^{-1/2}\right) = \operatorname{tr}\left(\overline{f_{1,S} f_T f^{S\cup T\cup\{v_0\}}}^P \cdot f_{v_0}^P, \tau \otimes \delta_P^{-1/2}\right)$$

for an auxiliary spherical test function $f_{v_0}^P$ on $M(\mathbb{Q}_{v_0})$. In particular, because $\overline{f}^P$ is invariant under $K_1 \cap M(\mathbb{A}_f)$, we may assume without loss of generality that τ contains a fixed vector for $K_1 \cap M(\mathbb{A}_f)$. Moreover it suffices to show

$$\operatorname{tr}\left(\overline{f}_T^P, \tau_T \otimes \delta_P^{-1/2}\right) = 0.$$

By [96, Proposition 3.5(2)], the latter trace coincides with

$$\mathrm{tr}\left(f_T, \natural - \mathrm{Ind}_{M(\mathbb{A}_T)}^{G_1(\mathbb{A}_T)} \tau\right). \tag{2.20}$$

We will show (2.20) vanishes. Indeed, since each constituent of $\natural - \mathrm{Ind}_{M(\mathbb{A}_f)}^{G_1(\mathbb{A}_f)} \tau$ lies in $\mathcal{P}_1$, if (2.20) is nonzero then π is nearly equivalent to a constituent of $\natural - \mathrm{Ind}_{M(\mathbb{A}_f)}^{G_1(\mathbb{A}_f)} \tau$ by the choice of f_T. Now, the maximal ideal $\mathfrak{m}_\tau \subset \mathbb{T}^S_{M,\overline{\mathbb{Q}}_p}$ defined by the Hecke action on $\iota^{-1}\tau$ is of Galois type because τ is cohomological and $M = \mathrm{GL}_1^3$ or $\mathrm{GL}_2 \times \mathrm{GL}_1$. Hence by Propositions 2.1.2 and 2.1.9, the maximal ideal $\mathfrak{m}_\pi \subset \mathbb{T}^S_{\mathrm{GSp}_4,\overline{\mathbb{Q}}_p}$ defined by $\iota^{-1}\pi$ is Eisenstein, which one can easily check contradicts the hypothesis that $\mathfrak{m}$ is non-Eisenstein. □

In particular, the claim combined with (2.19) and the choice of f_T shows that $T(f_D) \neq 0$, so $T(f_1) \neq 0$ by (2.18). Using the claim again, we see that

$$\mathrm{tr}\left(f_1, H^*(\mathrm{Sh}_{K_1}(V_1), \mathbb{C})\right) \neq 0.$$

By Franke's theorem [32] and our choice of f_T, we conclude there exists an automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A})$ which is nearly equivalent to π and unramified outside S, such that the Hecke eigensystem of π'^S appears in $H^*(\mathrm{Sh}_{K_1}(V_1), \mathbb{C})$. Expand S to a larger set S' such that $\pi^{S'} \cong \pi'^{S'}$. Then the maximal ideal $\mathfrak{m}^{S'} \subset \mathbb{T}^{S'}_{\mathrm{GSp}_4,O}$ formed by restricting $\mathfrak{m}$ descends to $\mathbb{T}^{S'}_{K_1,V_1,O}$. By the case $D = 1$ of the lemma for $\mathfrak{m}^{S'}$ and π', π' is not Eisenstein, hence π is not either. □

Corollary 2.7.7. *Suppose $\sigma(D)$ is even, and $\mathfrak{m} \subset \mathbb{T}^S_{K,V_D,O}$ is a generic, non-Eisenstein maximal ideal. Also fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Then*

$$H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)_{\mathfrak{m}} = \bigoplus_{\pi_f} \iota^{-1}\pi_f^K \otimes \rho_{\pi_f},$$

where:

- *π_f runs over the finite parts of relevant automorphic representations π of $\mathrm{GSpin}(V_D)(\mathbb{A}_f)$ such that the $\mathbb{T}^S_O$-action on π_f^K factors through $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$.*
- *If π is not endoscopic, then $\rho_{\pi_f} = \rho_{\pi,\iota}(-2)$, cf. Remark 2.5.4.*
- *If π is endoscopic associated to a pair π_1, π_2 of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with discrete series archimedean components of weights 2 and 4, respectively, we can write $\pi = \Theta(\pi_1^{D_1} \boxtimes \pi_2^{D_2})$ by Theorem 2.5.2. Then*

$$\rho_{\pi_f} = \begin{cases} \rho_{\pi_1,\iota}(-2), & \sigma(D_1) \text{ even}, \\ \rho_{\pi_2,\iota}(-2), & \sigma(D_1) \text{ odd}. \end{cases}$$

Proof. It follows from Lemmas 2.7.6 and 2.4.4(2) that

$$H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)_{\mathfrak{m}} \cong \bigoplus_{\pi_f} \iota^{-1}\pi_f^K \otimes \rho_{\pi_f}$$

as Hecke modules, where π_f runs over the finite parts of non-Eisenstein automorphic representations of $\mathrm{GSpin}(V_D)(\mathbb{A}_f)$ with Hecke action factoring through $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$, and ρ_{π_f} is some Galois representation with

$$\dim \rho_{\pi_f} = \sum_{\pi'_\infty} m(\pi_f \otimes \pi'_\infty) \dim H^3(\mathfrak{gsp}_4, U(2); \pi'_\infty).$$

In particular, by (2.10), the only π_f with $\rho_{\pi_f} \neq 0$ are the finite parts of *relevant* automorphic representations π.

We first consider the non-endoscopic case. As in [108, p. 296], we see from Theorem 2.4.6(3) that ρ_{π_f} is four-dimensional with Hodge-Tate weights $\{0, -1, -2, -3\}$ if π is not endoscopic. Since for all but finitely many $\ell \notin S$, Frob_ℓ satisfies the Eichler-Shimura relation (2.9) on ρ_{π_f}, we conclude that $\rho_{\pi_f} = \rho_{\pi,\iota}(-2)$. It remains to consider the endoscopic case, when $\pi = \Theta(\pi_1^{D_1} \boxtimes \pi_2^{D_2})$. Then π is the unique completion of π_f to an automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ by Theorem 2.5.2. Moreover, by the local archimedean theta lift described in [44, Proposition 4.3.1], π_∞ is the generic or holomorphic member of the discrete series L-packet in the case that $\sigma(D_1)$ is even or odd, respectively. Hence we conclude as above that ρ_{π_f} is two-dimensional, with Hodge-Tate weights $\{-1,-2\}$ or $\{0,-3\}$ when $\sigma(D_1)$ is even or odd, respectively. Since we still have the Eichler-Shimura relation (2.9), it follows that ρ_{π_f} is either $\rho_{\pi_1,\iota}(-2)$ or $\rho_{\pi_2,\iota}(-2)$ depending on the Hodge-Tate weights, and the corollary follows. ☐

Corollary 2.7.8. *Let $\mathfrak{m} \subset \mathbb{T}^S_{K,V_D,O}$ be a generic and non-Eisenstein maximal ideal, and let $\mathcal{T}$ be the set of relevant automorphic representations π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ such that $\pi_f^K \neq 0$ and the Hecke action on $\iota^{-1}\pi_f^K$ factors through $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$. Then we have a natural embedding of $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$-algebras*

$$\mathbb{T}^S_{K,V_D,O,\mathfrak{m}} \hookrightarrow \bigoplus_{\pi\in\mathcal{T}} \overline{\mathbb{Q}}_p(\pi),$$

where $\overline{\mathbb{Q}}_p(\pi)$ is $\overline{\mathbb{Q}}_p$ with Hecke action through the eigenvalues on $\iota^{-1}\pi_f^K$.

Proof. If $\sigma(D)$ is odd, this is immediate from Lemma 2.7.6. If $\sigma(D)$ is even, it follows from Theorem 2.7.5(2) combined with Corollary 2.7.7. ☐

3. Special cycles and theta lifts

3.1. Special cycles.

3.1.1. In this subsection, we explain the construction of special cycles $Z(T,\varphi)$, due to Kudla in the indefinite case [54].

Construction 3.1.2. Let V be a quadratic space over $\mathbb{Q}$ of signature $(m,2)$ or $(m,0)$.

(1) If $V_0 \subset V$ is a positive definite subspace, then for any $g \in \mathrm{GSpin}(V_0)(\mathbb{A}_f)\backslash \mathrm{GSpin}(V)(\mathbb{A}_f)/K$, we obtain a canonical finite morphism

$$\mathrm{Sh}_{K_{0,g}}(V_0^\perp) \xrightarrow{\cdot g} \mathrm{Sh}_K(V), \tag{3.1}$$

with $K_{0,g} \coloneqq gKg^{-1} \cap \mathrm{GSpin}(V_0^\perp)(\mathbb{A}_f)$.

(i) If V has signature $(m,2)$ and $\dim V_0 = n$, then we write

$$Z(g,V_0,V)_K \in \mathrm{CH}^n(\mathrm{Sh}_K(V))$$

for the pushforward of the fundamental class on $\mathrm{Sh}_{K_{0,g}}(V_0^\perp)$ under (3.1).

(ii) If V has signature $(m,0)$, then we write

$$Z(g,V_0,V)_K \in \mathbb{Z}\,[\mathrm{Sh}_K(V)]$$

for the pushforward of the constant function 1 on $\mathrm{Sh}_{K_{0,g}}(V_0^\perp)$ under (3.1).

(2) For any $T \in \mathrm{Sym}_n(\mathbb{Q})_{\geq 0}$, let

$$\Omega_{T,V} = \{(x_1,\ldots,x_n) \in V^n \,:\, x_i \cdot x_j = T_{ij}\ \forall\, 1 \leq i,j \leq n\},$$

viewed as an affine algebraic variety over $\mathbb{Q}$.

(3) Now suppose given a neat compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$, along with a test function

$$\varphi \in \mathcal{S}(V^n \otimes \mathbb{A}_f, R)^K$$

for some $n \leq m$ and some ring R. (The action of K is the natural one, factoring through the map to $\mathrm{SO}(V)(\mathbb{A}_f)$.) For any $T \in \mathrm{Sym}_n(\mathbb{Q})_{\geq 0}$, if $\Omega_{T,V}(\mathbb{Q}) = \emptyset$, then the special cycle $Z(T,\varphi)_K$, in $\mathrm{CH}^n(\mathrm{Sh}_K(V),R) \coloneqq \mathrm{CH}^n(\mathrm{Sh}_K(V)) \otimes_{\mathbb{Z}} R$ or $R\,[\mathrm{Sh}_K(V)]$, is defined to vanish. Otherwise:

(i) If V is positive definite, then fix a base point $(x_1, \ldots, x_n) \in \Omega_{T,V}(\mathbb{Q})$, and let

$$V_0 = \mathrm{Span}_{\mathbb{Q}} \{x_1, \ldots, x_n\} \subset V.$$

Then we define

$$Z(T, \varphi)_K = \sum_{g \in \mathrm{GSpin}(V_0^{\perp})(\mathbb{A}_f) \backslash \mathrm{GSpin}(V)(\mathbb{A}_f)/K} \varphi(g^{-1}x_1, \ldots, g^{-1}x_n) Z(g, V_0, V)_K \in R[\mathrm{Sh}_K(V)]. \tag{3.2}$$

(ii) If V has signature $(m, 2)$ and T is positive definite, we define

$$Z(T, \varphi)_K \in \mathrm{CH}^n(\mathrm{Sh}_K(V), R)$$

by the same formula (3.2). If T is not positive definite (and still $\Omega_{T,V}(\mathbb{Q}) \neq 0$), we define $Z(T, \varphi)_K \in \mathrm{CH}^n(\mathrm{Sh}_K(V), R)$ by the recipe in [54, p. 61]; the details will not be needed.

Remark 3.1.3. By transitivity of the $\mathrm{GSpin}(V)(\mathbb{Q})$-action on $\Omega_{T,V}(\mathbb{Q})$, $Z(T, \varphi)_K$ is independent of the choice of base point for $\Omega_{T,V}(\mathbb{Q})$.

Proposition 3.1.4. *For neat compact open subgroups $K' \subset K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$, if $\mathrm{pr}_{K,K'} : \mathrm{Sh}_{K'}(V) \to \mathrm{Sh}_K(V)$ is the natural map, then $\mathrm{pr}^*_{K,K'} Z(T, \varphi)_K = Z(T, \varphi)_{K'}$.*

Proof. This is [54, Proposition 5.10] in the indefinite case; the definite case is proved in the same way. □

Notation 3.1.5. For any compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$, and any ring R, we define $\mathrm{SC}^n_K(V, R)$ to be the R-span of the special cycles $Z(T, \varphi)_K$ for $T \in \mathrm{Sym}_n(\mathbb{Q})_{\geq 0}$ and $\varphi \in \mathcal{S}(V^n \otimes \mathbb{A}_f, R)$. When $R = \mathbb{Z}$ it may be dropped from the notation.

Remark 3.1.6. Note that $\mathrm{SC}^n_K(V, \mathbb{Z})$ contains all of the special cycles $z = Z(g, V_0, V)_K$ from (3.1.1) with $\dim V_0 = n$. Indeed, choose a basis $\{e_1, \ldots, e_n\}$ for V_0, and set $T_{ij} := e_i \cdot e_j$. Then one can choose $\varphi \in \mathcal{S}(V \otimes \mathbb{A}_f, \mathbb{Z})^K$ such that $\varphi|_{\Omega_{T,V}(\mathbb{A}_f)}$ is the indicator function of $K \cdot g^{-1}(e_1, \ldots, e_n)$, and it follows that $Z(T, \varphi)_K = Z(g, V_0, V)_K$.

3.1.7. We will later need the following proposition to understand the double coset space appearing in (3.2).

Proposition 3.1.8. *Suppose $K_\ell \subset \mathrm{SO}(V)(\mathbb{Q}_\ell)$ is the stabilizer of a self-dual lattice $L \subset V \otimes \mathbb{Q}_\ell$, and $V = V_0 \oplus V_1$ be an orthogonal decomposition of V. Then the natural map*

$$\begin{aligned} \mathrm{SO}(V_0)(\mathbb{Q}_\ell) \backslash \mathrm{SO}(V)(\mathbb{Q}_\ell)/K_\ell &\to \{\textit{lattices } L_1 \subset V_1 \otimes \mathbb{Q}_\ell\} \\ g &\mapsto g \cdot L \cap V_1 \end{aligned}$$

is injective. If $\dim(V_1) < \dim(V_0)$, then its image consists of all L_1 on which the pairing is $\mathbb{Z}_\ell$-valued.

Proof. This follows from (the proof of) [26, Propositions 3.1.5, 3.1.6]. □

3.2. **Symplectic and metaplectic groups.**

In this section, we set up basic notions for symplectic and metaplectic groups, mostly following the exposition of [34].

Notation 3.2.1. (1) For an integer $n \geq 1$, we define the standard symplectic lattice W_{2n} with basis $e_1, \ldots, e_n, e_1^*, \ldots, e_n^*$ and pairing determined by

$$\langle e_i, e_j \rangle = 0, \;\; \langle e_i^*, e_j^* \rangle = 0, \;\; \langle e_i, e_j^* \rangle = \delta_{ij}. \tag{3.3}$$

The symplectic group Sp_{2n} as defined in (1.1.4) is the isometry group of W_{2n}.

(2) The Siegel parabolic subgroup $P = MN \subset \mathrm{Sp}_{2n}$ is the stabilizer of $\mathrm{Span}_{\mathbb{Z}}\{e_1, \dots, e_n\} \subset W_{2n}$. We identify N with Sym_n, the space of $n \times n$ symmetric matrices, and M with GL_n via the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{-t} \end{pmatrix} \in \mathrm{Sp}_{2n}, \quad g \in \mathrm{GL}_n .$$

(3) If k is a local field of characteristic zero, the metaplectic group $\mathrm{Mp}_{2n}(k)$ is defined as the unique non-split central extension of $\mathrm{Sp}_{2n}(k)$ by μ_2:

$$0 \to \mu_2 \to \mathrm{Mp}_{2n}(k) \xrightarrow{g \mapsto \overline{g}} \mathrm{Sp}_{2n}(k) \to 0.$$

The double cover $\mathrm{Mp}_{2n}(k) \to \mathrm{Sp}_{2n}(k)$ splits uniquely over $N(k)$, with $P = MN$ the Siegel parabolic as above. The preimage $\widetilde{P}(k)$ of $P(k)$ therefore has a Levi decomposition

$$\widetilde{P}(k) = \widetilde{M}(k)N(k)$$

with $\widetilde{M}(k)$ a nonsplit double cover of $M(k)$.

(4) If k is non-archimedean with ring of integers $\mathcal{O}$ and the residue characteristic of $\mathcal{O}$ is odd, let $\mathrm{Mp}_{2n}(\mathcal{O}) \subset \mathrm{Mp}_{2n}(k)$ be the unique lifting of $\mathrm{Sp}_{2n}(\mathcal{O})$ [35, §6].

(5) Let $U(n) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{R})$ be the embedding defined by

$$A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

We fix the Cartan decomposition

$$\mathfrak{sp}_{2n,\mathbb{C}} = \mathfrak{u}(n)_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

such that $\mathfrak{p}^+$ is isomorphic to the symmetric square of the defining representation of $U(n)$.

(6) Let $\widetilde{U}(n) \subset \mathrm{Mp}_{2n}(\mathbb{R})$ be the preimage of $U(n)$. If $j^{1/2}(g, z)$ is the half-integral weight automorphy factor of [101, p. 25], then we let $\det^{1/2} : \widetilde{U}(n) \to \mathbb{C}^\times$ be the restriction of $j^{1/2}(g, i)$, which is a square root of the determinant character. We set $\det^k := (\det^{1/2})^{2k}$ for all $k \in \frac{1}{2}\mathbb{Z}$.

(7) Globally, let

$$\mathrm{Mp}_{2n}(\mathbb{A}) = \prod_v{}' \mathrm{Mp}_{2n}(\mathbb{Q}_v)$$

be the restricted product with respect to the subgroups $\mathrm{Mp}_{2n}(\mathbb{Z}_v) \subset \mathrm{Mp}_{2n}(\mathbb{Q}_v)$ for $v \neq 2, \infty$. The inclusion $\mathrm{Sp}_{2n}(\mathbb{Q}) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{A})$ lifts naturally to

$$\mathrm{Sp}_{2n}(\mathbb{Q}) \hookrightarrow \mathrm{Mp}_{2n}(\mathbb{A}),$$

by which we will view $\mathrm{Sp}_{2n}(\mathbb{Q})$ as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$.

Definition 3.2.2. For $k \in \frac{1}{2}\mathbb{Z}$, define the space M_k^{2n} of adelic Siegel modular forms of degree $2n$ and weight k, consisting of smooth functions

$$f : \mathrm{Sp}_{2n}(\mathbb{Q}) \backslash \mathrm{Mp}_{2n}(\mathbb{A}) \to \mathbb{C}$$

such that:

(1) $f(gz) = \det^k(z) f(g)$ for any $z \in \widetilde{U}(n) \subset \mathrm{Mp}_{2n}(\mathbb{R})$.
(2) $X \cdot f(g) = 0$ for any $X \in \mathfrak{p}^- \subset \mathfrak{sp}_{2n,\mathbb{R}}$.

Remark 3.2.3. Note that M_k^{2n} is naturally an $\mathrm{Mp}_{2n}(\mathbb{A}_f)$-module.

Notation 3.2.4.

(1) Let ψ be the additive character of $\mathbb{Q}\backslash\mathbb{A}$ which is unramified at all finite places and satisfies
$$\psi(x_\infty) = e^{2\pi i x_\infty}$$
for $x_\infty \in \mathbb{R} \subset \mathbb{A}$.

(2) For $T \in \mathrm{Sym}_n(\mathbb{Q})$, define
$$\psi_T := \psi \circ \left(\frac{1}{2}\operatorname{tr}(T-)\right),$$
a unitary character of $N(\mathbb{Q})\backslash N(\mathbb{A})$ (where N is the unipotent radical of the Siegel parabolic).

From basic Fourier analysis, one deduces the following proposition.

Proposition 3.2.5. *Let $f \in M_k^{2n}$. Then we have an identity of functions on $N(\mathbb{A}) \subset \mathrm{Mp}_{2n}(\mathbb{A})$:*
$$f = \sum_{T \in \mathrm{Sym}_n(\mathbb{Q})} a_T(f) q_{\mathbb{A}}^T,$$
where $q_{\mathbb{A}}^T := e^{-\operatorname{tr}(T)} \cdot \psi_T$ and
$$a_T(f) := \frac{e^{\operatorname{tr}(T)}}{\mathrm{Vol}([N])} \int_{[N]} f(n)\psi_T^{-1}(n)\mathrm{d}n.$$

Definition 3.2.6. For any subring $R \subset \mathbb{C}$, define
$$M_{k,R}^{2n} = \left\{ f \in M_k^{2n} \ : \ a_T(f) \in R \text{ for all } T \in \mathrm{Sym}_n(\mathbb{Q}) \right\}.$$

3.3. **Weil representation.**

3.3.1. Let k be a local field, and let V be a quadratic space over k of *odd* dimension and trivial discriminant. Also fix an even integer $2n \geq 2$. For any nontrivial additive character ψ of k, the Weil representation ω_ψ of $\mathrm{SO}(V)(k) \times \mathrm{Mp}_{2n}(k)$ is realized on the complex Schwartz space $\mathcal{S}(V^n, \mathbb{C})$, and is determined by the following formulas.

$$\begin{cases} \omega_\psi(g,1)\varphi(x) = \varphi(g^{-1}x), & g \in \mathrm{SO}(V)(k). \\ \omega_\psi(1,u)\varphi(x) = \psi(\frac{1}{2}u(x)\cdot x)\varphi(x), & u \in N(k) \cong \mathrm{Sym}_n(k). \\ \omega_\psi(1,m)\varphi(x) = \chi_\psi(m)|\det(\overline{m})|^{\frac{\dim V}{2}}\varphi(\overline{m}^t x), & m \in \widetilde{M}(k). \\ \omega_\psi(1,w)\varphi(x) = \gamma_w \int_{V^n} \varphi(y)\psi(x\cdot y)\mathrm{d}y. \end{cases} \tag{3.4}$$

Here, the notation is as follows:

- $P = MN$ is the Siegel parabolic.
- χ_ψ is a μ_8-valued genuine character of $\widetilde{M}(k)$ described in [34, p. 1661].
- $w \in \mathrm{Mp}_{2n}(k)$ is a certain Weyl element such that $\overline{w}P\overline{w}^{-1} = P^{\mathrm{op}}$.
- γ_w is a certain eighth root of unity.
- The pairing on V^n in the second and fourth equations is
$$(x_1, \ldots, x_n)\cdot(y_1, \ldots, y_n) = \sum x_i \cdot y_i.$$
- $\mathrm{d}x$ in the fourth equation is a self-dual Haar measure.

3.3.2. If k is non-archimedean of residue characteristic ℓ and $R \subset \mathbb{C}$ is a $\mathbb{Z}\left[\frac{1}{\ell}\right]$-algebra containing all eighth and ℓth power roots of unity, then the same formulas give a well-defined action of $\mathrm{SO}(V)(k) \times \mathrm{Mp}_{2n}(k)$ on the space $\mathcal{S}(V^n, R)$ of R-valued Schwartz functions.

3.3.3. Globally, if V is a quadratic space over $\mathbb{Q}$, we have Weil representations $\mathcal{S}(V^n \otimes \mathbb{Q}_v, \mathbb{C})$ for all places v, defined using the localizations of the fixed global additive character ψ of $\mathbb{Q}\backslash\mathbb{A}$. Similarly, we have the Weil representations $\mathcal{S}(V^n \otimes \mathbb{A}, \mathbb{C})$, $\mathcal{S}(V^n \otimes \mathbb{A}_f, \mathbb{C})$, etc., defined as restricted tensor products.

3.4. **Classical theta lifting.**

Notation 3.4.1. For a Schwartz function $\varphi \in \mathcal{S}(V^n \otimes \mathbb{A}, \mathbb{C})$ and a cuspidal automorphic form α on $\mathrm{SO}(V)(\mathbb{A})$, we write $\theta_\varphi(\alpha)$ for the automorphic form on $\mathrm{Mp}_{2n}(\mathbb{A})$ defined by

$$\theta_\varphi(\alpha)(h) = \int_{[\mathrm{SO}(V)]} \alpha(g) \sum_{x \in V^n} \omega_\psi(g,h)\varphi(x)\mathrm{d}g. \tag{3.5}$$

The normalization depends on a choice of Haar measure.

3.4.2. Suppose V is positive definite and let $K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$ be a neat compact open subgroup. For $\alpha : \mathrm{Sh}_K(V) \to \mathbb{C}$, define an automorphic form $\overline{\alpha}$ on $\mathrm{GSpin}(V)(\mathbb{A})$, which descends to $\mathrm{SO}(V)(\mathbb{A})$, by

$$\overline{\alpha}(g_f g_\infty) = \frac{1}{\mathrm{Vol}(\widehat{\mathbb{Z}}^\times)} \int_{\widehat{\mathbb{Z}}^\times} \alpha(g_f z)\mathrm{d}z, \ \ \forall g = g_f g_\infty \in \mathrm{GSpin}(V)(\mathbb{A}).$$

Lemma 3.4.3. *Let $\varphi_\infty \in \mathcal{S}(V^n \otimes \mathbb{R}, \mathbb{C})$ be the Gaussian*

$$\varphi_\infty(x) = e^{-x\cdot x}.$$

Then for $\alpha : \mathrm{Sh}_K(V) \to \mathbb{C}$ and

$$\varphi_f \in \mathcal{S}(V^n \otimes \mathbb{A}_f, \mathbb{C})^K,$$

$\theta_{\varphi_f \otimes \varphi_\infty}(\overline{\alpha})$ lies in $M^{2n}_{\frac{\dim(V)}{2}}$ and

$$a_T\left(\theta_{\varphi_f\otimes\varphi_\infty}(\overline{\alpha})\right) = C_K \cdot \alpha\left(Z(T,\varphi_f)_K\right),$$

where the constant is

$$C_K = \frac{\mathrm{Vol}\left(\mathrm{SO}(V)(\mathbb{R}) \cdot \mathrm{Im}\left(K \to \mathrm{SO}(V)(\mathbb{A}_f)\right)\right)}{[K \cdot \widehat{\mathbb{Z}}^\times : K]}.$$

Proof. In the Fock model of the $\left(\mathfrak{sp}_{2n\dim(V)}, \widetilde{U}(n\dim(V))\right)$-module associated to $\mathcal{S}(V^n \otimes \mathbb{R}, \mathbb{C})$, φ_∞ has degree zero, hence is annihilated by $\mathfrak{p}^- \subset \mathfrak{sp}_{2n}$; cf. [46, (2.2)]. By comparing degrees with [1, Proposition 2.1(2)], we also conclude that $\widetilde{U}(n)$ acts on $\theta_{\varphi_f\otimes\varphi_\infty}(\overline{\alpha})$ by $\det^{\frac{\dim(V)}{2}}$, which proves $\theta_{\varphi_f\otimes\varphi_\infty}(\overline{\alpha}) \in M^{2n}_{\frac{\dim(V)}{2}}$.

It remains to compute the Fourier coefficients. We calculate:

$$\begin{aligned} a_T(\theta_{\varphi_f\otimes\varphi_\infty}(\overline{\alpha})) &= \frac{e^{\mathrm{tr}\,T}}{\mathrm{Vol}([N])} \int_{[N]} \psi_T^{-1}(u) \int_{[\mathrm{SO}(V)]} \overline{\alpha}(g) \sum_{x \in V^n(\mathbb{Q})} \omega_\psi(g,u)\varphi_f \otimes \varphi_\infty(x)\mathrm{d}g\mathrm{d}u \\ &= \frac{e^{\mathrm{tr}\,T}}{\mathrm{Vol}([N])} \int_{[\mathrm{SO}(V)]} \overline{\alpha}(g) \sum_{x\in V^n(\mathbb{Q})} \int_{[N]} \psi_T^{-1}(u)\psi\left(\frac{1}{2}u(x)\cdot x\right)\omega_\psi(g,1)\varphi_f\otimes\varphi_\infty(x)\mathrm{d}g\mathrm{d}u \\ &= \int_{[\mathrm{SO}(V)]} \overline{\alpha}(g) \sum_{x \in \Omega_{T,V}(\mathbb{Q})} \omega_\psi(g,1)\varphi_f(x)\mathrm{d}g. \end{aligned}$$

Fix a base point $x = (x_1, \dots, x_m) \in \Omega_{T,V}(\mathbb{Q})$ and let $V_0 = \mathrm{Span}_{\mathbb{Q}}\{x_1,\dots,x_m\}$. Then, since $\Omega_{T,V}(\mathbb{Q})$ is a single $\mathrm{SO}(V)(\mathbb{Q})$-orbit, we may rewrite the final equation as

$$a_T(\theta_{\varphi_f\otimes\varphi_\infty}(\alpha)) = \int_{\mathrm{SO}(V_0^\perp)(\mathbb{A})\backslash\,\mathrm{SO}(V)(\mathbb{A})} \varphi_f(g^{-1}x) \int_{[\mathrm{SO}(V_0^\perp)]} \overline{\alpha}(hg)\mathrm{d}h\mathrm{d}g.$$

This coincides with the claimed formula by definition of $\alpha(Z(T,\varphi_f)_K)$. ☐

3.4.4. As a special case, suppose $V = \mathbb{Q}$ is the one-dimensional space with quadratic form $a \mapsto a^2$. Then $\mathrm{SO}(V)$ is the trivial group and we take the unit Haar measure on $\mathrm{SO}(V)(\mathbb{A}) = \{\mathrm{id}\}$. For each prime ℓ, let

$$\varphi_\ell \in \mathcal{S}(V^n \otimes \mathbb{Q}_\ell, \mathbb{Z})$$

be the indicator function of the lattice

$$\mathbb{Z}_\ell^n \subset \mathbb{Q}_\ell^n = V^n \otimes \mathbb{Q}_\ell,$$

and let $\varphi_f = \otimes_\ell \varphi_\ell$. Let $\mathbb{1}$ be the unit automorphic form on $\mathrm{SO}(V)(\mathbb{A})$. We write

$$\theta_{\frac{1}{2}} := \theta_{\varphi_f \otimes \varphi_\infty}(\mathbb{1}). \tag{3.6}$$

Lemma 3.4.5. *Let ℓ be an odd prime, and suppose $R \subset \mathbb{C}$ is a $\mathbb{Z}\left[\frac{1}{\ell}\right]$-algebra containing all eighth and ℓth power roots of unity. Then for any $g \in \mathrm{Mp}_{2n}(\mathbb{Q}_\ell)$, $g \cdot \theta_{\frac{1}{2}}$ lies in $M^{2n}_{\frac{1}{2},R}$, and the constant term $a_0(g \cdot \theta_{\frac{1}{2}})$ of its Fourier expansion lies in $R^\times$.*

Proof. By the obvious equivariance of the theta lift, we have

$$g \cdot \theta_{\frac{1}{2}} = \theta_{\omega_\psi(1,g)\varphi_f \otimes \varphi_\infty}(\mathbb{1}).$$

Since

$$\mathcal{S}(V^n \otimes \mathbb{Q}_\ell, R) \subset \mathcal{S}(V^n, \mathbb{C})$$

is $\mathrm{Mp}_{2n}(\mathbb{Q}_\ell)$-stable, Lemma 3.4.3 shows

$$g \cdot \theta_{\frac{1}{2}} \in M^{2n}_{\frac{1}{2},R}.$$

Also, $\theta_{\frac{1}{2}}$ is $\mathrm{Mp}_{2n}(\mathbb{Z}_\ell)$-invariant by construction.

By the Iwasawa decomposition

$$\mathrm{Sp}_{2n}(\mathbb{Q}_\ell) = P(\mathbb{Q}_\ell) \cdot \mathrm{Sp}_{2n}(\mathbb{Z}_\ell),$$

it suffices to show $a_0(g \cdot \theta_{\frac{1}{2}}) \in R^\times$ for $g \in \widetilde{P}(\mathbb{Q}_\ell)$. Now, by Lemma 3.4.3 again,

$$a_0(g \cdot \theta_{\frac{1}{2}}) = \omega_\psi(1,g) \cdot \varphi_f(0),$$

and it follows from the explicit formulas in (3.3.1) that

$$\omega_\psi(1,g) \cdot \varphi_f(0) \in R^\times$$

for all $g \in \widetilde{P}(\mathbb{Q}_\ell)$; this proves the lemma. □

Proposition 3.4.6. *Let $R \subset \mathbb{C}$ be a $\mathbb{Z}\left[\frac{1}{\ell}\right]$-algebra containing all eighth and ℓth power roots of unity. Then for all $k \in \frac{1}{2}\mathbb{Z}$,*

$$M^{2n}_{k,R} \subset M^{2n}_k$$

is stable under the action of $\mathrm{Mp}_{2n}(\mathbb{Q}_\ell)$.

Proof. If $k \in \mathbb{Z}$ is integral, this is a consequence of the q-expansion principle for classical Siegel modular forms [30, Chapter 5, Proposition 1.8]. In general, for any $f \in M^{2n}_{k,R}$ and $f' \in M^{2n}_{k',R}$, the product $ff' \in M^{2n}_{k+k',R}$ has formal q-expansion:

$$(ff')|_{N(\mathbb{A})} = \sum_{T \in \mathrm{Sym}_n(\mathbb{Q})} \sum_{S \in \mathrm{Sym}_n(\mathbb{Q})} a_T(f) a_S(f') q_{\mathbb{A}}^{S+T}. \tag{3.7}$$

(This expression make sense because $a_T(f)$ and $a_S(f')$ are each supported on positive semi-definite matrices with bounded denominators in their entries, see [101, Proposition 1.1].)

We apply this to $f \in M^{2n}_{k,R}$, with $k \in \frac{1}{2} + \mathbb{Z}$, and $f' = \theta_{\frac{1}{2}}$. Since $k + \frac{1}{2} \in \mathbb{Z}$, we have

$$g(f\theta_{\frac{1}{2}}) = g(f)g(\theta_{\frac{1}{2}}) \in M^{2n}_{k+\frac{1}{2},R}$$

for any $g \in \mathrm{Mp}_{2n}(\mathbb{Q}_\ell)$.

Now, by Lemma 3.4.5 above, the q-expansion of $g(\theta_{\frac{1}{2}})$ has an inverse power series

$$\sum_{T \in \mathrm{Sym}_n(\mathbb{Q})} b_T q_{\mathbb{A}}^T$$

with $b_T \in R$. Hence, by the uniqueness of q-expansions, the Fourier coefficients of $g(f)$ lie in R as well. $\square$

3.5. **Formal theta lifts.**

3.5.1. Suppose V has signature $(m, 0)$ or $(m, 2)$ for some $m \geq 1$, and let

$$K = \prod K_\ell \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$$

be a neat compact open subgroup. For any subring $R \subset \mathbb{C}$, we define $\mathrm{Test}_K(V, R)$ to be

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathrm{Sh}_K(V)], R) = \mathrm{Hom}_R(R[\mathrm{Sh}_K(V)], R)$$

in the positive definite case, or

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{CH}^*(\mathrm{Sh}_K(V)), R) = \mathrm{Hom}_R(\mathrm{CH}^*(\mathrm{Sh}_K(V), R), R)$$

in the indefinite case.

3.5.2. Now fix $1 \leq n \leq m$. For any K-invariant Schwartz function $\varphi \in \mathcal{S}(V^n \otimes \mathbb{A}_f, R)$ and any $\alpha \in \mathrm{Test}_K(V, R)$, we define the formal theta lift

$$\Theta(\alpha, \varphi)_K := \sum_{T \in \mathrm{Sym}_n(\mathbb{Q})_{\geq 0}} \alpha\left(Z(T, \varphi)_K\right) q_{\mathbb{A}}^T.$$

The subscript K will be omitted when there is no risk of confusion.

3.5.3. Let ℓ be a prime such that K_ℓ has pro-order invertible in R, and let

$$\mathrm{Test}_{K^\ell}(V, R) := \varinjlim_{K_\ell'} \mathrm{Test}_{K^\ell K_\ell'}(V),$$

where the transition maps are induced by pushforward. Note that $\mathrm{Test}_{K^\ell}(V, R)$ has a natural action of $\mathrm{GSpin}(V)(\mathbb{Q}_\ell)$, dual to the one described in [129, p. 41]. For $\alpha \in \mathrm{Test}_{K^\ell}(V)$ and a K^ℓ-invariant Schwartz function $\varphi \in \mathcal{S}(V^n \otimes \mathbb{A}_f, R)$, we define the renormalized formal theta lift

$$\Theta_{K^\ell}(\alpha, \varphi) := \frac{\Theta(\alpha, \varphi)_{K^\ell K_\ell'}}{[K_\ell : K_\ell']}, \tag{3.8}$$

for any $K_\ell' \subset K_\ell$ fixing both α and φ. Because the cycles $Z(T, \varphi)_{K^\ell K_\ell'}$ are compatible under pullback, $\Theta_{K^\ell}(\alpha, \varphi)$ does not depend on the choice of K_ℓ'.

Proposition 3.5.4. *Suppose R is a $\mathbb{Z}[1/\ell]$-algebra containing all eighth and ℓth power roots of unity, and the pro-order of K_ℓ is invertible in R. Then Θ_{K^ℓ} defines a $\mathrm{GSpin}(V)(\mathbb{Q}_\ell) \times \mathrm{Mp}_{2n}(\mathbb{Q}_\ell)$-equivariant map*

$$\mathrm{Test}_{K^\ell}(V, R) \otimes \mathcal{S}(V^n \otimes \mathbb{A}_f, R)^{K^\ell} \to M^{2n}_{\frac{\dim(V)}{2}, R}.$$

Proof. Note that both modularity and equivariance can be checked after extending scalars, so without loss of generality suppose $R = \mathbb{C}$. In the definite case, the proposition is a formal consequence of Lemma 3.4.3 above. In the indefinite case, the modularity of the formal theta lift is [15, Theorem 6.2] and the equivariance is [54, Corollary 5.11] combined with [129, Corollary 2.12]. $\square$

4. CONSTRUCTION OF GALOIS COHOMOLOGY CLASSES AND SPECIAL PERIODS

Before beginning this section, we establish the notation that will be in force for most of the rest of the paper.

Notation 4.0.1.

(1) Fix a relevant automorphic representation π of GSp_4 with trivial central character. Fix as well a strong coefficient field E_0 for π (Definition 2.2.13). If π is endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, then we also assume without loss of generality that E_0 is a strong coefficient field for π_1 and π_2.[8]

(2) We fix a finite set S of finite primes, containing 2 and all ℓ such that π_ℓ is ramified.

(3) For all primes $\mathfrak{p}$ of E_0, let $O_\mathfrak{p} \subset E_{0,\mathfrak{p}}$ be the ring of integers, and let $\varpi_\mathfrak{p} \in O_\mathfrak{p}$ be a uniformizer, with residue field $k_\mathfrak{p} = O_\mathfrak{p}/\varpi_\mathfrak{p}$. We drop the subscript $\mathfrak{p}$ when there is no risk of confusion. For a finite set of primes $S' \supset S$, we write $\mathfrak{m}^{S'}_{\pi,\mathfrak{p}} \subset \mathbb{T}^{S'}_{O_\mathfrak{p}}$ for the maximal ideal defined by the Hecke eigenvalues of π, and drop the decorations when they are clear from context.

(4) Notation 2.2.16 remains in force.

Remark 4.0.2. We assume $2 \in S$ so that 2 is not admissible under Definition 4.2.1 below, and to prove the (convenient but not essential) Lemma 4.4.7.

4.1. Assumptions on $\mathfrak{p}$.

We now define some assumptions on primes $\mathfrak{p}$ of E_0. Let p denote the residue characteristic.

Assumption 4.1.1.

(1) p does not lie in S.

(2) There exists a rational prime $\ell \notin S \cup \{p\}$ such that $\ell^4 \not\equiv 1 \pmod{p}$ and $\overline{\rho}_{\pi,\mathfrak{p}}(\mathrm{Frob}_\ell)$ has distinct eigenvalues, no two having ratio ℓ.

The final assumption depends on whether π is endoscopic.

(3)
- If π is not endoscopic, then $\overline{\rho}_{\pi,\mathfrak{p}}$ is absolutely irreducible.
- If π is endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, then $\overline{\rho}_{\pi_1,\mathfrak{p}}$ and $\overline{\rho}_{\pi_2,\mathfrak{p}}$ are both absolutely irreducible.

Remark 4.1.2. Assumption 4.1.1(2) implies that $p > 5$.

Notation 4.1.3. Under Assumption 4.1.1(3):

(1) Let $T_{\pi,\mathfrak{p}}$ be an $O_\mathfrak{p}[G_\mathbb{Q}]$-module such that $T_{\pi,\mathfrak{p}} \otimes \mathbb{Q}_p = V_{\pi,\mathfrak{p}}$; Assumption 4.1.1(3) implies that $T_{\pi,\mathfrak{p}}$ is unique up to isomorphism.

(2) For all $n \geq 1$, we write $T_{\pi,\mathfrak{p},n} := T_{\pi,\mathfrak{p}}/\varpi_\mathfrak{p}^n T_{\pi,\mathfrak{p}}$. Also let $\overline{T}_{\pi,\mathfrak{p}} := T_{\pi,\mathfrak{p},1}$.

(3) When $\mathfrak{p}$ is clear from context, it may be dropped from the above notations.

Remark 4.1.4. Under Assumption 4.1.1(3), $T_{\pi,\mathfrak{p}}$ is isomorphic to its $O_\mathfrak{p}$-dual; we use this to view $\rho_{\pi,\mathfrak{p}}$ as valued in $\mathrm{GSp}_4(O_\mathfrak{p})$, and $\overline{\rho}_{\pi,\mathfrak{p}}$ as valued in $\mathrm{GSp}_4(k_\mathfrak{p})$.

Lemma 4.1.5. *Assumption 4.1.1 holds for all but finitely many primes $\mathfrak{p}$ of E_0.*

Proof. That Assumption 4.1.1(3) holds for all but finitely many primes $\mathfrak{p}$ follows from [123, Theorem 1.2(i)] in the non-endoscopic case; in the endoscopic case, it follows from [94, Theorem 2.1]. It is also obvious that Assumption 4.1.1(1) holds for cofinitely many $\mathfrak{p}$. We consider Assumption 4.1.1(2).

---

[8] In fact, it is not difficult to check that this last assumption is automatic. The main point is to use Hodge-Tate theory to verify that, for all $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, $\rho_{\pi_1,\iota}$ and $\rho_{\pi_2,\iota}$ cannot differ by a Galois twist.

First fix an arbitrary $\mathfrak{p}$ with residue characteristic p. The image of $\rho_{\pi,\mathfrak{p}}$ contains an element with distinct eigenvalues by [99, Theorem 1] and Theorem 2.2.10(2). Hence by the Chebotarev density theorem, there exists a prime $\ell \notin S \cup \{p\}$ such that Frob_ℓ has distinct eigenvalues on $\rho_{\pi,\mathfrak{p}}$. The Satake parameter of π_ℓ is therefore multiplicity-free and of the form $\left\{\alpha, \beta, \alpha^{-1}, \beta^{-1}\right\}$ with $|\alpha| = |\beta| = 1$. Let $E_1 = E_0(\alpha, \beta)$, which is a finite extension because the Hecke eigenvalues of π_ℓ lie in E_0. For all but finitely many primes $\mathfrak{p}_1 \nmid \ell$ of E_1, we will have

$$\begin{aligned} \alpha^2 &\not\equiv 1, \ell, \ell^{-1} \pmod{\mathfrak{p}_1} \\ \beta^2 &\not\equiv 1, \ell, \ell^{-1} \pmod{\mathfrak{p}_1} \\ \alpha\beta &\not\equiv 1, \ell, \ell^{-1} \pmod{\mathfrak{p}_1} \\ \alpha/\beta &\not\equiv 1, \ell, \ell^{-1} \pmod{\mathfrak{p}_1}, \\ \ell^4 &\not\equiv 1, \pmod{\mathfrak{p}_1}. \end{aligned}$$

For such a $\mathfrak{p}_1$, let $\mathfrak{p}' = \mathfrak{p}_1|_{E_0}$ and let p' be the residue characteristic of $\mathfrak{p}'$. Then we have $\ell^4 \not\equiv 1 \pmod{p'}$ and the eigenvalues of Frob_ℓ on $\overline{\rho}_{\pi,\mathfrak{p}'}$ are distinct and not of ratio ℓ, i.e. Assumption 4.1.1(2) holds for $\mathfrak{p}'$. ☐

Lemma 4.1.6. *Assume $\mathfrak{p}$ satisfies Assumption 4.1.1(3). Then:*

(1) $H^1(\mathbb{Q}, T_\pi)$ is ϖ-torsion-free.

(2) Suppose given $c \in H^1(\mathbb{Q}, T_\pi)$ and $a \geq 1$ such that $c \notin \varpi^a H^1(\mathbb{Q}, T_\pi)$. Then for all $n \geq 1$, the image $c_n \in H^1(\mathbb{Q}, T_{\pi,n})$ satisfies

$$\mathrm{ord}_\varpi c_n > n - a.$$

Proof. The assumption implies that $H^0(\mathbb{Q}, \overline{T}_\pi) = 0$. The long exact sequence in Galois cohomology associated to

$$0 \to T_\pi \xrightarrow{\varpi} T_\pi \to \overline{T}_\pi \to 0$$

therefore gives (1). For (2), a similar argument to (1) shows that the map $H^1(\mathbb{Q}, T_{\pi,a}) \xrightarrow{\varpi^{n-a}} H^1(\mathbb{Q}, T_{\pi,n})$ is injective, so the kernel of

$$\varpi^{n-a} : H^1(\mathbb{Q}, T_{\pi,n}) \to H^1(\mathbb{Q}, T_{\pi,n})$$

coincides with the kernel of $H^1(\mathbb{Q}, T_{\pi,n}) \to H^1(\mathbb{Q}, T_{\pi,a})$. Hence it suffices to show that $c_a \neq 0$, which is clear from the assumption $c \notin \varpi^a H^1(\mathbb{Q}, T_\pi)$ and the long exact sequence in Galois cohomology associated to

$$0 \to T_\pi \xrightarrow{\varpi^a} T_\pi \to T_{\pi,a} \to 0.$$

☐

Lemma 4.1.7. *Suppose $\mathfrak{p}$ satisfies Assumption 4.1.1. Then $\mathfrak{m}_{\pi,\mathfrak{p}} \subset \mathbb{T}_O^S$ is non-Eisenstein and generic.*

Proof. Recall from Remark 4.1.4 that $\overline{\rho}_\pi$ is valued in $\mathrm{GSp}_4(k)$. Then $\mathfrak{m}_{\pi,\mathfrak{p}}$ is clearly of Galois type associated to $\overline{\rho}_\pi$. The genericity of $\mathfrak{m}_{\pi,\mathfrak{p}}$ (Definition 2.7.3) therefore follows from Assumption 4.1.1(2). From Assumption 4.1.1(3), it follows that $\overline{T}_{\pi,\mathfrak{p}} \otimes \overline{\mathbb{F}}_p$ contains no Galois-stable line. So if $\mathfrak{m}_{\pi,\mathfrak{p}}$ were Eisenstein, $\overline{\rho}_{\pi,\mathfrak{p}}$ would have to factor through a Siegel parabolic subgroup. In particular, then $\overline{T}_{\pi,\mathfrak{p}} \otimes \overline{\mathbb{F}}_p = \rho_0 \oplus \rho_0 \otimes \det \rho_0^{-1} \otimes \omega_p$, where $\rho_0 : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is some irreducible representation, and ω_p is the mod-p cyclotomic character. This is clearly impossible by Assumption 4.1.1(3) and Lemma 4.1.8 below. ☐

Lemma 4.1.8. *Let $\mathfrak{p}|p$ be a prime of E_0 such that $p > 5$ and π_p is unramified. If π is endoscopic associated to (π_1, π_2), then $\overline{\rho}_{\pi_1}$ and $\overline{\rho}_{\pi_2}$ are not isomorphic.*

Proof. Recall that ρ_{π_1} and ρ_{π_2} have Hodge-Tate weights $\{-1, 2\}$ and $\{0, 1\}$ up to reording, by Lemma 2.2.9 and Theorem 2.2.1(2). The lemma then follows from Fontaine-Laffaille theory [31, Théorème 6.1]. ☐

4.2. **Admissible primes.** In this section, $\mathfrak{p}|p$ is a prime of E_0.

Definition 4.2.1.

(1) We say a prime $q \notin S \cup \{p\}$ is *admissible* for $\rho_\pi = \rho_{\pi,\mathfrak{p}}$ if $q^4 \not\equiv 1 \pmod{p}$ and the generalized eigenvalues of $\overline{\rho}_\pi(\mathrm{Frob}_q)$ are of the form $\{q, \alpha, 1, q/\alpha\}$, with $\alpha \neq \pm q, \pm 1, q^2, q^{-1}$.

(2) If q is admissible, define $n(q) \geq 1$ to be the greatest integer such that $\det(\mathrm{Frob}_q - q|V_\pi) \equiv 0 \pmod{\varpi^{n(q)}}$.

(3) We say q is *n-admissible* if it is admissible and $n(q) \geq n$.

(4) If $Q \geq 1$ is squarefree, we say Q is *admissible* (resp. *n-admissible*) if all primes $q|Q$ are so.

(5) Analogously, an element $g \in G_{\mathbb{Q}}$ is called *admissible* for ρ_π if $\nu_g := \chi_{p,\mathrm{cyc}}(g)$ satisfies $\nu_g^4 \not\equiv 1 \pmod{p}$, and g acts on V_π with generalized eigenvalues $\{\nu_g, 1, \alpha, \nu_g/\alpha\}$ for some
$$\alpha \not\equiv \pm\nu_g, \pm 1, \nu_g^2, \nu_g^{-1} \pmod{\varpi}.$$

Lemma 4.2.2. *Suppose $\mathfrak{p}$ satisfies Assumption 4.1.1(3). Then a prime q is n-admissible for $\rho_\pi = \rho_{\pi,\mathfrak{p}}$ if and only if there exists a $G_{\mathbb{Q}_q}$-stable decomposition*
$$T_{\pi,n} = M_{0,n} \oplus M_{1,n}$$
such that:

(1) $M_{0,n}$ and $M_{1,n}$ are each free of rank two over O/ϖ^n, and $\mathrm{Frob}_q|_{M_{0,n}} = \begin{pmatrix} q & \\ & 1 \end{pmatrix}$ in some basis.

(2) $\mathrm{Frob}_q^2 - 1$, $\mathrm{Frob}_q^2 - q^2$, $\mathrm{Frob}_q - q^2$, and $\mathrm{Frob}_q - q^{-1}$ all act invertibly on $M_{1,n}$.

Proof. Immediate from Definition 4.2.1. ☐

Lemma 4.2.3. *Let $\mathfrak{p}$ be a prime of E_0. The following are equivalent:*

(1) There exist admissible primes for ρ_π.

(2) For all n, there exist n-admissible primes for ρ_π.

(3) There exists an admissible element $g \in G_{\mathbb{Q}}$ for ρ_π.

Proof. Clearly (2) implies (1), and (3) implies (2) by the Chebotarev density theorem. The proof that (1) implies (2) follows the same argument of [69, Lemma 2.7.1], and (2) implies (3) by compactness. ☐

4.2.4. Now suppose π is endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$.

Definition 4.2.5.

(1) A prime $q \notin S \cup \{p\}$ is called *BD-admissible* for $\rho_{\pi_i} = \rho_{\pi_i,\mathfrak{p}}$, with $i = 1$ or 2, if $q^2 \not\equiv 1 \pmod{p}$ and the generalized eigenvalues of $\overline{\rho}_{\pi_i}(\mathrm{Frob}_q)$ are $\{1, q\}$.

(2) If q is BD-admissible for ρ_{π_i}, define $n(q) \geq 1$ to be the greatest integer such that $\det(\mathrm{Frob}_q - q|V_{\pi_i}) =\equiv 0 \pmod{\varpi^{n(q)}}$.

(3) We say q is *n-BD-admissible* if it is BD-admissible and $n(q) \geq 1$.

(4) If $Q \geq 1$ is squarefree, we say Q is BD-admissible (resp n-BD-admissible) for ρ_{π_i} if all $q|Q$ are so.

(5) Likewise, an element $g \in G_{\mathbb{Q}}$ is called BD-admissible for ρ_{π_i} if $\chi_{p,\mathrm{cyc}}(g)^2 \not\equiv 1 \pmod{p}$ and g acts on V_{π_i} with eigenvalues $\chi_{p,\mathrm{cyc}}(g)$ and 1.

Remark 4.2.6.

(1) Definition 4.2.5 is adapted from [7, p. 18], but there it is allowed that the eigenvalues of Frob_q on $\overline{\rho}_{\pi_i}$ are -1 and $-q$.

(2) If q is n-admissible for ρ_π, then it is n-BD-admissible for exactly one of ρ_{π_1} and ρ_{π_2}; and if $g \in G_{\mathbb{Q}}$ is admissible for ρ_π, then it is BD-admissible for exactly one of ρ_{π_1} and ρ_{π_2}. In particular, if $Q \geq 1$

is admissible for ρ_π, there is a unique factorization $Q = Q_1 \cdot Q_2$ with $Q_1, Q_2 \geq 1$, such that all $q|Q_i$ are BD-admissible for ρ_{π_i}.

4.2.7. For any prime $q \notin S \cup \{p\}$, recall that $H^1_f(\mathbb{Q}_q, T_{\pi,n}) = H^1_{\mathrm{unr}}(\mathbb{Q}_q, T_{\pi,n})$, and set $H^1_{/f}(\mathbb{Q}_q, T_{\pi,n}) = H^1(\mathbb{Q}_q, T_{\pi,n})/H^1_f(\mathbb{Q}_q, T_{\pi,n})$.

Proposition 4.2.8. *Suppose q is n-admissible. Then $H^1_f(\mathbb{Q}_q, T_{\pi,n})$ and $H^1_{/f}(\mathbb{Q}_q, T_{\pi,n})$ are each free of rank one over $\mathcal{O}/\varpi^n$, and local Poitou-Tate duality induces a perfect pairing*

$$H^1_f(\mathbb{Q}_q, T_{\pi,n}) \times H^1_{/f}(\mathbb{Q}_q, T_{\pi,n}) \to O/\varpi^n.$$

Proof. First note that the induced pairing is perfect because $H^1_f(\mathbb{Q}_q, T_{\pi,n})$ is self-annihilating under the Tate pairing, and one can check $\lg_O H^1(\mathbb{Q}_q, T_{\pi,n}) = 2 \lg_O H^1_f(\mathbb{Q}_q, T_{\pi,n})$ using the local Euler characteristic formula and local duality. So it suffices to prove that

$$H^1_f(\mathbb{Q}_q, T_{\pi,n}) = T_{\pi,n}/(\mathrm{Frob}_q - 1)T_{\pi,n}$$

is free of rank one over O/ϖ^n. Indeed, this is immediate from Lemma 4.2.2. ☐

Notation 4.2.9. If q is n-admissible and $S' \supset S$ is a finite set with $q \notin S'$, then by Proposition 4.2.8 we have the localization and residue maps:

$$\mathrm{loc}_q : H^1(\mathbb{Q}^{S'}/\mathbb{Q}, T_{\pi,n}) \to H^1_f(\mathbb{Q}_q, T_{\pi,n}) \simeq O/\varpi^n$$
$$\partial_q : H^1(\mathbb{Q}, T_{\pi,n}) \to H^1_{/f}(\mathbb{Q}_q, T_{\pi,n}) \simeq O/\varpi^n$$

4.3. **Level structures and test vectors.** Fix a prime $\mathfrak{p}$ of E_0 of residue characteristic p.

Definition 4.3.1. For any squarefree $D \geq 1$, an *S-level structure* for $\mathrm{GSpin}(V_D)$ is a compact open subgroup $K = \prod K_\ell \subset \mathrm{GSpin}(V_D)(\mathbb{A}_f)$ such that:

1. K is neat in the sense of [89, §0.1].
2. For all $\ell \notin S \cup \mathrm{div}(D)$, K_ℓ is hyperspecial.

An *S-tidy level structure* is an S-level structure satisfying:

3. If $K_Z \subset \mathbb{A}_f^\times$ is the intersection of K with the center of $\mathrm{GSpin}(V_D)(\mathbb{A}_f)$, then the finite group $\mathbb{Q}^\times \backslash \mathbb{A}_f^\times / K_Z$ has order coprime to p.

The reason for the final condition of Definition 4.3.1 is the following convenient lemma:

Lemma 4.3.2. *Suppose K is an S-tidy level structure for $\mathrm{GSpin}(V_D)$. Then for all finite sets $S' \supset S$ and all $\ell \notin S \cup \mathrm{div}(D)$, we have*

$$\langle \ell \rangle = 1 \textit{ on } H^*(\mathrm{Sh}_K(V_D), O)_{\mathfrak{m}^{S'}_{\pi,\mathfrak{p}}}.$$

Proof. By Definition 4.3.1(3), after replacing O by a finite extension we can write

$$H^*(\mathrm{Sh}_K(V_D), O) = \bigoplus_\chi H^*(\mathrm{Sh}_K(V_D), O)_\chi,$$

where χ runs over O-valued characters of the finite group $\mathbb{Q}^\times \backslash \mathbb{A}_f^\times / K_Z$ and

$$\langle \ell \rangle = \chi(\ell) \text{ on } H^*(\mathrm{Sh}_K(V_D), O)_\chi.$$

The characters χ are distinct modulo ϖ, and the lemma follows because π has trivial central character. ☐

Definition 4.3.3. Let $D \geq 1$ be squarefree, let K be an S-level structure for $\mathrm{GSpin}(V_D)$, and let $R = O$ or O/ϖ^n, viewed as a $\mathbb{T}_O^S$-module via the Hecke eigenvalues of π. We define $\mathrm{Test}_K(V_D, \pi, R)$ as follows.

(1) If $\sigma(D)$ is *even* and $\mathfrak{p}$ satisfies Assumption 4.1.1, then

$$\mathrm{Test}_K(V_D, \pi, R) = \mathrm{Hom}_{\mathbb{T}_O^{S\cup\mathrm{div}(D)}[G_{\mathbb{Q}}]}\left(H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2)), T_\pi \otimes_O R\right).$$

(2) If $\sigma(D)$ is *odd*, then

$$\mathrm{Test}_K(V_D, \pi, R) = \mathrm{Hom}_{\mathbb{T}_O^{S\cup\mathrm{div}(D)}}\left(O\left[\mathrm{Sh}_K(V_D)\right], R\right).$$

Remark 4.3.4. The definition of $\mathrm{Test}_K(V_D, \pi, R)$ depends only on K, and not on S; one can check this using Theorem 2.4.6(3), and Corollary 2.7.8 when $\sigma(D)$ is even.

4.4. **Constructions.** Fix a prime $\mathfrak{p}$ of E_0 of residue characteristic p.

Construction 4.4.1. Let $D \geq 1$ be squarefree, and let $Q \geq 1$ be admissible and coprime to D, such that $\sigma(DQ)$ is *odd*. For any S-level structure K for $\mathrm{GSpin}(V_{DQ})$:

(1) We define

$$\lambda_n^D(Q; K) \subset O/\varpi^n$$

to be the submodule spanned by the elements $\alpha(z)$, where:
   - ○ α lies in $\mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$.
   - ○ z lies in $\mathrm{SC}^2_K(V_{DQ}, O)$ (Notation 3.1.5).

(2) If $\varphi \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{A}_f, O)^K$ is a test function, then we define

$$\lambda_n^D(Q, \varphi; K) \subset \lambda_n^D(Q; K)$$

to be the submodule spanned by elements $\alpha(Z(T, \varphi)_K)$, where:
   - ○ α lies in $\mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$.
   - ○ T lies in $\mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$, and $Z(T, \varphi)_K$ was defined in Construction 3.1.2.

(3) If $Q = 1$, then we define $\lambda^D(1; K) \subset O$ and $\lambda^D(1, \varphi; K) \subset \lambda^D(1; K)$ analogously, where now α ranges over $\mathrm{Test}_K(V_D, \pi, O)$.

(4) We write $\lambda_n^D(Q) \subset O/\varpi^n$ for the submodule spanned by $\lambda_n^D(Q; K)$ as K varies, and likewise $\lambda^D(1)$.

In all of the above constructions, we include a subscript $\mathfrak{p}$ only when it is necessary for clarity.

4.4.2. Let $D \geq 1$ be squarefree with $\sigma(D)$ *even*, and suppose $\mathfrak{p}$ satisfies Assumption 4.1.1. Let $\mathfrak{m} := \mathfrak{m}_{\pi,\mathfrak{p}}^{S\cup\mathrm{div}(D)} \subset \mathbb{T}_O^{S\cup\mathrm{div}(D)}$. It follows from Lemma 4.1.7 and Theorem 2.7.5(2) that the étale realization map

$$\mathrm{CH}^2(\mathrm{Sh}_K(V_D), O)_{\mathfrak{m}} \to H^4(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}^{G_{\mathbb{Q}}}$$

is trivial. We therefore obtain a well-defined Abel-Jacobi map

$$\partial_{\mathrm{AJ},\mathfrak{m}} : \mathrm{CH}^2(\mathrm{Sh}_K(V_D), O)_{\mathfrak{m}} \to H^1(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}).$$

For any $\alpha \in \mathrm{Test}_K(V_D, \pi, R)$ with $R = O$ or O/ϖ^n, we obtain an induced map

$$\mathrm{CH}^2(\mathrm{Sh}_K(V_D), O)_{\mathfrak{m}} \xrightarrow{\partial_{\mathrm{AJ},\mathfrak{m}}} H^1(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}) \xrightarrow{\alpha_*} H^1(\mathbb{Q}, T_\pi \otimes_O R). \tag{4.1}$$

Construction 4.4.3. Let $D \geq 1$ be squarefree, and let $Q \geq 1$ be admissible and coprime to D, such that $\sigma(DQ)$ is *even*. Suppose $\mathfrak{p}$ satisfies Assumption 4.1.1, and let K be an S-level structure for $\mathrm{GSpin}(V_{DQ})$.

(1) We define

$$\kappa_n^D(Q; K) \subset H^1(\mathbb{Q}, T_{\pi,n})$$

to be the submodule spanned by $\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}}(z)$, where:
   - ○ α lies in $\mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$.

- ∘ z lies in $\mathrm{SC}^2_K(V_{DQ}, O)$ (Notation 3.1.5).

(2) For any $\varphi \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{A}_f, O)^K$, we define

$$\kappa_n^D(Q, \varphi; K) \subset \kappa_n^D(Q; K)$$

to be the submodule spanned by elements $\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T, \varphi)_K)$, where:

- ∘ α lies in $\mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$.
- ∘ T lies in $\mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$, and $Z(T, \varphi)_K$ was defined in Construction 3.1.2.

(3) If $Q = 1$, then we define $\kappa^D(1; K) \subset H^1(\mathbb{Q}, T_\pi)$ and $\kappa^D(1, \varphi; K) \subset \kappa^D(1; K)$ analogously, with now $\alpha \in \mathrm{Test}_K(V_D, \pi, O)$.

(4) We write $\kappa_n^D(Q) \subset H^1(\mathbb{Q}, T_{\pi,n})$ for the submodule spanned by $\kappa_n^D(Q; K)$ as K varies, and likewise $\kappa^D(1)$.

In all of the above constructions, we include a subscript $\mathfrak{p}$ only when it is necessary for clarity.

Remark 4.4.4. The only reason to distinguish between D and Q in Constructions 4.4.1 and 4.4.3 is to define $\lambda^D(1)$ and $\kappa^D(1)$ when $\sigma(D)$ is odd and even, respectively; moreover, one can check using Corollary 2.5.3 that $\lambda^D(1)$ or $\kappa^D(1)$ is trivial unless π_ℓ is transferrable for all $\ell | D$.

Now we prove some basic properties of Constructions 4.4.1 and 4.4.3.

Proposition 4.4.5. *Suppose $L(1/2, \pi, \mathrm{spin}) \neq 0$. If $D > 1$ is squarefree with $\sigma(D)$ odd and π_f^D can be completed to an automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$, then for any prime $\mathfrak{p}$ of E_0, $\lambda^D(1)_\mathfrak{p} \neq 0$. Moreover, for all but finitely many $\mathfrak{p}$, we have*

$$\lambda^D(1)_\mathfrak{p} \not\equiv 0 \pmod{\varpi_\mathfrak{p}}.$$

Proof. Let Π be any automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ with $\Pi_f^D \simeq \pi_f^D$. Because π has trivial central character, Π descends to an automorphic representation of $\mathrm{SO}(V_D)(\mathbb{A})$. By [39, Theorem 1.1], the global theta lift of Π to $\mathrm{Mp}_4(\mathbb{A})$ is nonzero; i.e., the map

$$\begin{aligned} \mathcal{S}(V_D^2 \otimes \mathbb{A}, \mathbb{C}) \otimes \Pi &\to \mathrm{Fun}(\mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Mp}_4(\mathbb{A}), \mathbb{C}) \\ \varphi \otimes \alpha &\mapsto \theta_\varphi(\alpha) \end{aligned} \tag{4.2}$$

is not identically zero (Notation 3.4.1). Also, the image of (4.2) lies in the space of cusp forms by the global tower property of the theta lift [91]: otherwise, Π would occur in the restriction to $\mathrm{SO}(V_D)(\mathbb{A})$ of the theta lift of an automorphic representation of $\mathrm{Mp}_2(\mathbb{A})$, which is ruled out by the Shimura-Waldspurger correspondence and the relevance of π. In particular, by the local-global compatibility of the theta correspondence (see the proof of [91, Theorem I.2.2]), if $\varphi = \otimes \varphi_v \in \mathcal{S}(V_D^2 \otimes \mathbb{A}, \mathbb{C})$ and $\alpha = \otimes \alpha_v \in \Pi$ are factorizable, then

$$\theta_\varphi(\alpha) \neq 0 \iff \langle \rho_v(\varphi_v), \alpha_v \rangle \neq 0 \; \forall v, \tag{4.3}$$

where by definition

$$\rho_v : \mathcal{S}(V_D^2 \otimes \mathbb{Q}_v, \mathbb{C}) \to (\Pi_v)^\vee \boxtimes \Theta_v(\Pi_v)$$

is the maximal $(\Pi_v)^\vee$-isotypic quotient of the Weil representation of $\mathrm{SO}(V_D)(\mathbb{Q}_v) \times \mathrm{Mp}_4(\mathbb{Q}_v)$ on $\mathcal{S}(V_D^2 \otimes \mathbb{Q}_v, \mathbb{C})$.

When $v = \infty$, then $\alpha_\infty \in \Pi_\infty$ is unique up to scalar, and we take $\varphi_\infty \in \mathcal{S}(V_D^2 \otimes \mathbb{R}, \mathbb{C})$ to be the Gaussian from Lemma 3.4.3. Then α_∞ and φ_∞ satisfy the local condition in (4.3) by the theory of joint harmonics [1, Proposition 2.1(2), §5]. In particular, (4.3) implies that we can fix an S-level structure K and data

$$\alpha \in \mathrm{Hom}_{\mathrm{set}}(\mathrm{Sh}_K(V_D), O_{E_0}) \cap \Pi \subset C_c^\infty(\mathrm{GSpin}(V_D)(\mathbb{A}), \mathbb{C}), \quad \varphi_f \in \mathcal{S}(V_D^2 \otimes \mathbb{A}_f, O_{E_0})^K$$

such that

$$\theta_{\varphi_f \otimes \varphi_\infty}(\alpha) \neq 0.$$

By Lemma 3.4.3, this means that

$$0 \neq \alpha(Z(T, \varphi_f)_K) \in O_{E_0}$$

for some $T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$. Now we note that α has the Hecke eigenvalues of π away from S by construction. In particular, for all primes $\mathfrak{p}$ of E_0, α lies in $\mathrm{Test}_K(V_D, \pi, O_{\mathfrak{p}})$, and

$$0 \neq \alpha(Z(T, \varphi_f)_K) \in \lambda^D(1, \varphi_f)_{\mathfrak{p}} \subset \lambda^D(1)_{\mathfrak{p}}.$$

Since $\alpha(Z(T, \varphi_f)_K) \not\equiv 0 \pmod{\mathfrak{p}}$ for all but finitely many $\mathfrak{p}$, this completes the proof. □

Proposition 4.4.6. *Suppose $D \geq 1$ is squarefree and $\mathfrak{p}$ satisfies Assumption 4.1.1.*

(1) For all admissible Q coprime to D with $\sigma(DQ)$ even and all $\ell \notin S \cup \mathrm{div}(DQ)$, we have

$$\mathrm{Res}_\ell\, \kappa_n^D(Q) \subset H^1_f(\mathbb{Q}_\ell, T_{\pi,n}).$$

(2) If $\sigma(D)$ is even, then we have

$$\kappa^D(1) \subset H^1_f(\mathbb{Q}, T_\pi).$$

Proof. Write $Q = 1$ in case (2). We claim that, for all S-level structures K and all $z \in \mathrm{CH}^2(\mathrm{Sh}_K(V_{DQ}))$,

$$\partial_{\mathrm{AJ},\mathfrak{m}}(z) \in H^1_f(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}). \tag{4.4}$$

Note here that $H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}$ is p-torsion-free by Lemma 4.1.7 and Theorem 2.7.5(2), so the Bloch-Kato Selmer group is defined as in Notation 1.5.2(1). Observe as well that (4.4) implies the proposition: indeed, it clearly implies (2) by the functoriality of the Bloch-Kato local conditions; and it also implies (1) by Remark 1.5.3 and Proposition 1.5.5.

Now note that, for all primes ℓ, $H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, E_{0,\mathfrak{p}}(2))_{\mathfrak{m}}$ is pure of weight one as a $G_{\mathbb{Q}_\ell}$-module by Corollary 2.7.7 combined with Theorem 2.2.10(1); hence

$$H^1\left(\mathbb{Q}_\ell, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, E_{0,\mathfrak{p}}(2))_{\mathfrak{m}}\right) = 0$$

for all $\ell \neq p$ and

$$H^1_g\left(\mathbb{Q}_p, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, E_{0,\mathfrak{p}}(2))_{\mathfrak{m}}\right) = H^1_f\left(\mathbb{Q}_p, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, E_{0,\mathfrak{p}}(2))_{\mathfrak{m}}\right).$$

Thus (4.4) follows from [81, Theorem 5.9] combined with the proof of [79, Theorem 3.1(ii)]. □

Lemma 4.4.7. *Let $D \geq 1$ be squarefree, and let $Q \geq 1$ be admissible and coprime to D. Then:*

(1) If $\sigma(DQ)$ is even and $\mathfrak{p}$ satisfies Assumption 4.1.1, then $\kappa_n^D(Q)$ is generated by $\kappa_n^D(Q; K)$ as K ranges over S-tidy level structures for $\mathrm{GSpin}(V_{DQ})$.

(2) If $\sigma(DQ)$ is odd and $p > 5$, then $\lambda_n^D(Q)$ is generated by $\lambda_n^D(Q; K)$ as K ranges over S-tidy level structures for $\mathrm{GSpin}(V_{DQ})$.

Proof. We prove the first statement, as the two are similar. Let K be an S-level structure for $\mathrm{GSpin}(V_{DQ})$, and note that $K_2 = K_{\ell=2}$ has pro-order prime to p.[9] After fixing a sufficiently small compact open subgroup $K_2' \subset K_2$, we can ensure that $K' := K_2' \cdot \prod_{\ell \neq 2}(K_\ell \mathbb{Z}_\ell^\times)$ is neat; it is then clearly S-tidy as well. Let $\mathrm{pr}_{K, K' \cap K} : \mathrm{Sh}_{K' \cap K}(V_{DQ}) \to \mathrm{Sh}_K(V_{DQ})$ and $\mathrm{pr}_{K', K' \cap K} : \mathrm{Sh}_{K' \cap K}(V_{DQ}) \to \mathrm{Sh}_{K'}(V_{DQ})$ be the natural maps. For $R = O$ or O/ϖ^n and $\alpha_K \in \mathrm{Test}_K(V_{DQ}, \pi, R)$, we set

$$\alpha_{K'} := \frac{\alpha_K \circ \mathrm{pr}_{K, K' \cap K, *} \circ \mathrm{pr}^*_{K', K' \cap K}}{[K : K' \cap K]} \in \mathrm{Test}_{K'}(V_{DQ}, \pi, R).$$

---

[9] To see this, one first observes that K_2 stabilizes some lattice $L \subset V_{DQ} \otimes \mathbb{Q}_2$ such that $2L \subset L^\vee \subset L$, where $L^\vee$ is the $\mathbb{Z}_2$-linear dual. One obtains a natural map $f : K_2 \to \mathrm{SO}(L^\vee/2L) \times \mathrm{SO}(L/L^\vee)$, where $L^\vee/2L$ and $L/L^\vee$ are naturally nondegenerate symmetric spaces over $\mathbb{F}_2$ of dimension at most 5. Since $p > 5$ by Remark 4.1.2, the image of f then has order prime to p, and the kernel of f is clearly pro-2.

If $\varphi \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{A}_f, O)$ is fixed by K, it is also fixed by K', and we have $\alpha_{K',*} \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T,\varphi)_{K'}) = \alpha_{K,*} \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T,\varphi)_K)$ by Proposition 3.1.4; the lemma follows. □

Finally, we introduce some notation that will be used in the endoscopic case.

Notation 4.4.8. Suppose π is endoscopic associated to a pair (π_1, π_2). Then for any squarefree $D \geq 1$ with $\sigma(D)$ *even*, any S-level structure K for $\mathrm{GSpin}(V_D)$, any $\varphi \in \mathcal{S}(V_D^2 \otimes \mathbb{A}_f, O)^K$, and $j = 1$ or 2, we define $\kappa^D(1,\varphi;K)^{(j)} \subset H^1_f(\mathbb{Q}, T_{\pi_j})$ as the image of $\kappa^D(1)$ under the natural projection $H^1_f(\mathbb{Q}, T_\pi) \to H^1_f(\mathbb{Q}, T_{\pi_j})$. We similarly write $\kappa^D(1;K)^{(j)}$ and $\kappa^D(1)^{(j)}$. It is easy to check that

$$\kappa^D(1,\varphi;K) = \kappa^D(1,\varphi;K)^{(1)} \oplus \kappa^D(1,\varphi;K)^{(2)},$$

etc. As usual, a subscript $\mathfrak{p}$ is included when necessary for clarity.

5. NONVANISHING CRITERIA FOR CHANGING TEST FUNCTIONS

5.1. **Setup and notation.**

5.1.1. Fix a prime q, and let $k = \mathbb{C}$ or $\overline{\mathbb{F}}_p$, for an odd prime $p \neq q$. In the latter case we also assume fixed an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, and assume throughout this section that

$$q - 1 \neq 0 \text{ in } k. \tag{5.1}$$

We denote by $|\cdot|^{1/2} : \mathbb{Q}_q^\times \to k^\times$ the unramified character such that $|q|^{1/2} = q^{-1/2} \in k^\times$, using the isomorphism ι when $k = \overline{\mathbb{F}}_p$.

5.1.2. Let V_m be the split quadratic space of dimension $2m+1$ over $\mathbb{Q}_q$, with basis $v_0, v_1, \ldots, v_m, v_1^*, \ldots, v_m^*$ and pairing given by:

$$v_i \cdot v_j^* = \delta_{ij}, \;\; v_i^* \cdot v_j^* = 0, \;\; v_i \cdot v_j = \delta_{i0}\delta_{j0}.$$

Then $L := \mathrm{Span}_{\mathbb{Z}_q}\{v_0, v_1, \ldots, v_m, v_1^*, \ldots, v_m^*\}$ is a self-dual lattice $L \subset V_m$. Abbreviate

$$G_m = \mathrm{SO}(V_m)(\mathbb{Q}_q), \;\; G'_n = \mathrm{Mp}_{2n}(\mathbb{Q}_q). \tag{5.2}$$

For any parabolic subgroup $P \subset G_m$ (resp. $P' \subset G'_n$), we write R_P (resp. $R_{P'}$) for the normalized Jacquet module functor with respect to P (resp. P').

5.1.3. Fix an integer $n \geq 1$, and consider the Weil representation on $\mathcal{S}(V^n, \mathbb{C})$ with respect to the localization, also written ψ, of our fixed global additive character of $\mathbb{Q}$ (Notation 3.2.4). If $k = \overline{\mathbb{F}}_p$, we have fixed an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, and it follows from the discussion in (3.3.2) that $\mathcal{S}(V^n, \breve{\mathbb{Z}}_p) \subset \mathcal{S}(V^n, \overline{\mathbb{Q}}_p)$ is stable under $G_m \times G'_n$; reducing modulo p, we obtain the Weil representation on $\mathcal{S}(V^n, \overline{\mathbb{F}}_p)$. Whether $k = \mathbb{C}$ or $\overline{\mathbb{F}}_p$, we abbreviate $\Omega_{m,n} = \mathcal{S}(V^n, k)$.

5.1.4. If π is an irreducible, admissible k-linear representation of G_m, define $\Theta_{m,n}(\pi)$ to be the $k[G'_n]$-module such that

$$\Omega_{m,n} \twoheadrightarrow \pi \boxtimes \Theta_{m,n}(\pi)$$

is the maximal π-isotypic quotient of $\Omega_{m,n}$. Similarly, if π' is an irreducible, admissible, genuine k-linear representation of G'_n, define $\Theta_{n,m}(\pi')$ to be the $k[G_m]$-module such that

$$\Omega_{m,n} \twoheadrightarrow \Theta_{n,m}(\pi') \boxtimes \pi'$$

is the maximal π'-isotypic quotient of $\Omega_{m,n}$.

5.1.5. Let

$$\widetilde{\mathrm{GL}}_1(\mathbb{Q}_q) \xrightarrow{g \mapsto \overline{g}} \mathbb{Q}_q^\times \tag{5.3}$$

be the double cover described in [34, p. 1661], with canonical genuine character $\chi_\psi : \widetilde{\mathrm{GL}}_1(\mathbb{Q}_q) \to \mu_8 \subset k^\times$.

5.2. **The structure of the Weil representation.** Consider the Schwartz spaces $\mathcal{S}(\mathbb{Q}_q, k)$ and $\mathcal{S}(\mathbb{Q}_q^\times, k)$, viewed as representations of $\mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$ via

$$(g_1, g_2) \cdot \varphi(x) = \varphi(g_1^{-1} x g_2).$$

Lemma 5.2.1. *Fix a character* $\chi : \mathbb{Q}_q^\times \to k^\times$.

(1) *The maximal quotient of* $\mathcal{S}(\mathbb{Q}_q^\times, k)$ *on which the first factor of* $\mathbb{Q}_q^\times$ *acts by* χ *is realized by the map*

$$\begin{aligned} \mathcal{S}(\mathbb{Q}_q^\times, k) &\to \chi \boxtimes \chi^{-1} \\ \varphi &\mapsto \int_{\mathbb{Q}_q^\times} \varphi(t)\chi(t) \mathrm{d}^\times t. \end{aligned}$$

(2) *Assume* χ *is nontrivial. Then the map in (1) extends uniquely to a map* $f_\chi : \mathcal{S}(\mathbb{Q}_q, k) \to \chi \boxtimes \chi^{-1}$ *via*

$$\varphi \mapsto \frac{1}{1-\chi(g_0)} \int_{\mathbb{Q}_q^\times} \left(\varphi(t) - \varphi(g_0^{-1}t)\right) \chi(t) \mathrm{d}^\times t,$$

where $g_0 \in \mathbb{Q}_q^\times$ *is any element such that* $\chi(g_0) \neq 1$.

(3) *For integers* $a \leq b$, *let* $S_{a,b} \subset \mathcal{S}(q^a\mathbb{Z}_q, k)$ *be the subspace of Schwartz functions that are invariant under multiplication by* $\mathbb{Z}_q^\times$ *and translation by* $q^b\mathbb{Z}_q$. *Then for distinct, nontrivial, and unramified characters* $\chi_1, \dots, \chi_m : \mathbb{Q}_q^\times \to k^\times$ *with* $m \leq b - a + 1$, $f_{\chi_1}, \dots, f_{\chi_m}$ *are linearly independent as functions on* $S_{a,b}$.

Proof. Part (1) is elementary. For (2), we have the exact sequence

$$0 \to \mathcal{S}(\mathbb{Q}_q^\times, k) \to \mathcal{S}(\mathbb{Q}_q, k) \xrightarrow{\varphi \mapsto \varphi(0)} k \to 0,$$

which is equivariant for the trivial $\mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$-action on k. Since

$$\mathrm{Hom}_{\mathbb{Q}_q^\times \times \mathbb{Q}_q^\times}(k, \chi \boxtimes \chi^{-1}) = 0,$$

there is at most one extension of the map in (1) to $\mathcal{S}(\mathbb{Q}_q, k)$, and the formula given in (2) exhibits it.

For (3), let $x_i = \chi_i(q)$. A direct calculation shows that, for $\varphi \in S_{a,b}$,

$$\begin{aligned} f_{\chi_i}(\varphi) = \frac{1}{1-x_i} \mathrm{vol}(\mathbb{Z}_q^\times) \Big(& \varphi(q^a)(x_i^a - x_i^{a+1}) + \varphi(q^{a+1})(x_i^{a+1} - x_i^{a+2}) \\ & + \dots + \varphi(q^{b-1})(x_i^{b-1} - x_i^b) + \varphi(q^b)x_i^b \Big). \end{aligned}$$

So it suffices to show that the matrix

$$\begin{pmatrix} x_1^a - x_1^{a+1} & \cdots & x_1^{b-1} - x_1^b & x_1^b \\ x_2^a - x_2^{a+1} & \cdots & x_2^{b-1} - x_2^b & x_2^b \\ \vdots & \ddots & \vdots & \vdots \\ x_m^a - x_m^{a+1} & \cdots & x_m^{b-1} - x_m^b & x_m^b \end{pmatrix}$$

is nondegenerate, and this follows from the Vandermonde determinant formula. □

5.2.2. Assuming that $m \geq 1$, let

$$P = MN \subset G_m$$

be the parabolic subgroup stabilizing the isotropic line $\langle v_1 \rangle$. Then M is isomorphic to $\mathbb{Q}_q^\times \times G_{m-1}$; we normalize the isomorphism such that

$$(\alpha, g) \cdot v_1 = \alpha v_1, \text{ for } (\alpha, g) \in M. \tag{5.4}$$

Similarly, let $P' = M'N' \subset G'_n$ be the preimage of the stabilizer of $e_1 \in W_{2n}$ (Notation 3.2.1(1)); then M' is isomorphic to $\widetilde{\mathrm{GL}}_1(\mathbb{Q}_q) \times_{\mu_2} G'_{n-1}$. We normalize the isomorphism so that

$$(\alpha, g) \cdot e_1 = \overline{\alpha} e_1, \text{ for } (\alpha, g) \in M'. \tag{5.5}$$

Lemma 5.2.3.

(1) The normalized Jacquet module $R_P(\Omega_{m,n})$ fits into an exact sequence

$$0 \to \mathrm{Ind}_{M\times P'}^{M\times G'_n} \chi_\psi \cdot \mathcal{S}(\mathbb{Q}_q^\times, k) \boxtimes \Omega_{m-1,n-1} \to R_P(\Omega_{m,n}) \to |\cdot|^{\frac{2n-2m+1}{2}} \Omega_{m-1,n} \to 0,$$

where $|\cdot|$ is the canonical character of M, and $\chi_\psi \cdot \mathcal{S}(\mathbb{Q}_q^\times, k)$ is a $\mathrm{GL}_1(\mathbb{Q}_q) \times \widetilde{\mathrm{GL}}_1(\mathbb{Q}_q)$-module with action defined by $(g,h)\cdot\varphi(t) = \chi_\psi(h)\varphi(g^{-1}t\overline{h})$.

(2) Similarly, the normalized Jacquet module $R_{P'}(\Omega_{m,n})$ fits into a canonical exact sequence

$$0 \to \mathrm{Ind}_{P\times M'}^{G_m\times M'} \chi_\psi \cdot \mathcal{S}(\mathbb{Q}_q^\times, k) \boxtimes \Omega_{m-1,n-1} \to R_{P'}(\Omega_{m,n}) \to \chi_\psi|\cdot|^{\frac{2m-2n+1}{2}} \boxtimes \Omega_{m,n-1} \to 0.$$

Proof. When $k = \mathbb{C}$, this is [53, Theorem 2.8]; see also [34, Proposition 7.3] for our more convenient normalizations. When $k = \overline{\mathbb{F}}_p$, the proof in [53] applies without change because $p \neq q$. □

Corollary 5.2.4.

(1) Let π_{m-1} be an irreducible admissible representation of G_{m-1}, and let $\chi_0 : \mathbb{Q}_q^\times \to k^\times$ be a character with $\chi_0 \neq |\cdot|^{\frac{2n-2m+1}{2}}$. Then for all admissible $k[G'_n]$-modules $\mathcal{M}$,

$$\mathrm{Hom}_{G_m\times G'_n}\left(\Omega_{m,n}, \left(\mathrm{Ind}_P^{G_m} \chi_0 \boxtimes \pi_{m-1}\right) \boxtimes \mathcal{M}\right) = \mathrm{Hom}_{G'_n}\left(\mathrm{Ind}_{P'}^{G'_n} \chi_\psi \cdot \chi_0^{-1} \boxtimes \Theta_{m-1,n-1}(\pi_{m-1}), \mathcal{M}\right).$$

In particular, if

$$\pi_m := \mathrm{Ind}_P^{G_m} \chi_0 \boxtimes \pi_{m-1}$$

is irreducible, then

$$\Theta_{m,n}(\pi_m) = \mathrm{Ind}_{P'}^{G'_n} \chi_\psi \cdot \chi_0^{-1} \boxtimes \Theta_{m-1,n-1}(\pi_{m-1}).$$

(2) Similarly, let π'_{n-1} be an irreducible admissible genuine representation of G'_{n-1} and $\chi_0 \neq |\cdot|^{\frac{2m-2n+1}{2}}$ a character of $\mathbb{Q}_q^\times$. Then for all $k[G_m]$-modules $\mathcal{M}$,

$$\mathrm{Hom}_{G_m\times G'_n}\left(\Omega_{m,n}, \mathcal{M} \boxtimes \left(\mathrm{Ind}_{P'}^{G'_n} \chi_\psi \cdot \chi_0 \boxtimes \pi'_{n-1}\right)\right) = \mathrm{Hom}_{G_m}\left(\mathrm{Ind}_P^{G_m} \chi_0^{-1} \boxtimes \Theta_{m-1,n-1}(\pi'_{n-1}), \mathcal{M}\right).$$

In particular, if

$$\pi'_n := \mathrm{Ind}_{P'}^{G'_n} \chi_\psi\chi_0 \boxtimes \pi'_{n-1}$$

is irreducible, then

$$\Theta_{n,m}(\pi'_n) = \mathrm{Ind}_P^{G_m} \chi_0^{-1} \boxtimes \Theta_{n-1,m-1}(\pi'_{n-1}).$$

Proof. This is immediate from Lemma 5.2.3 and Lemma 5.2.1(1). □

5.3. **Principal series over k for orthogonal and metaplectic groups.** Some of the arguments in this subsection were inspired by the work of Zorn [130]. The results are new only if $k = \overline{\mathbb{F}}_p$.

5.3.1. Continuing the notation of (5.1.2), we now assume $m = n$.

Notation 5.3.2.

(1) Let $B \subset G_n$ be the stabilizer of the maximal isotropic flag

$$0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \dots, v_n \rangle \tag{5.6}$$

and let $B' \subset G'_n$ be the preimage of the stabilizer of the maximal isotropic flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_n \rangle.$$

(2) The Levi factor T of B is identified with $(\mathbb{Q}_q^\times)^n$ via its action on the associated graded of (5.6), and similarly the Levi factor T' of B' is identified with

$$\underbrace{\widetilde{\mathrm{GL}}_1(\mathbb{Q}_q) \times_{\mu_2} \cdots \times_{\mu_2} \widetilde{\mathrm{GL}}_1(\mathbb{Q}_q)}_{n \text{ times}}.$$

(3) For any character

$$\chi : (\mathbb{Q}_q^\times)^n \to k^\times,$$

we define (normalized) principal series representations

$$I(\chi) = \mathrm{Ind}_B^{G_n} \chi, \quad I'(\chi) = \mathrm{Ind}_{B'}^{G'_n} \chi_\psi \cdot \chi.$$

(4) We write $W = S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ for the Weyl group of T in G_n, which is also the Weyl group of T' in G'_n.

Lemma 5.3.3. *The semi-simplified normalized Jacquet modules are*

$$R_B(I(\chi))^{ss} = \oplus_{w \in W} \chi^w, \quad R_{B'}(I'(\chi))^{ss} = \oplus_{w \in W} \chi^w \chi_\psi.$$

Proof. Over $\overline{\mathbb{Q}}_p$, this follows from the well-known result for $\mathbb{C}$; see [130, Lemma 4.8]. Since $p \neq q$, the p-modular case follows by the proof of [113, Lemme 34]. □

Lemma 5.3.4.

(1) Any nontrivial quotient $I(\chi) \twoheadrightarrow \pi$ extends to a nontrivial intertwining operator

$$I(\chi) \twoheadrightarrow \pi \to I(\chi^w)$$

for some $w \in W$.

(2) Similarly, any nontrivial quotient $I'(\chi) \twoheadrightarrow \pi'$ extends to a nontrivial intertwining operator

$$I'(\chi) \twoheadrightarrow \pi' \to I'(\chi^w)$$

for some $w \in W$.

Proof. Let $\overline{B} \subset G_n$ and $\overline{B}' \subset G'_n$ be the opposite Borel subgroups to B, B'. Recall the "second Frobenius reciprocity"

$$\mathrm{Hom}(I(\chi), \pi) = \mathrm{Hom}(\chi, R_{\overline{B}}(\pi)), \quad \mathrm{Hom}(I'(\chi), \pi') = \mathrm{Hom}(\chi_\psi \chi, R_{\overline{B}'}(\pi')). \tag{5.7}$$

In the orthogonal case, this is [114, II.3.8(2)]. Although the result there is only stated for reductive groups, the proof can be adapted verbatim to the metaplectic case. (The key technical points are the existence of arbitrarily small compact open subgroups admitting an Iwahori factorization, and the conditions in Lemma I.8.13 of *op. cit.* All of the fundamental results on Hecke algebras in Chapter I of *op. cit.* apply to general locally profinite groups.)

Returning to the proof of the lemma, note that

$$R_{\overline{B}}(\pi) \neq 0 \iff R_B(\pi) \neq 0$$

and likewise in the metaplectic case. Hence it follows from (5.7) that any quotient $I(\chi) \twoheadrightarrow \pi$ extends to a nontrivial map

$$I(\chi) \to \pi \to I(\rho)$$

for some character ρ of $(\mathbb{Q}_q^\times)^n$; by Lemma 5.3.3, ρ is a Weyl conjugate of χ, so (1) holds. The argument for (2) is identical. ☐

Lemma 5.3.5. *Suppose $n = 1$ and $\chi^2 \neq |\cdot|^{\pm 1}$. Then $I(\chi)$ and $I'(\chi)$ are both irreducible and $\Theta_{1,1}(I(\chi)) = I'(\chi)$.*

Proof. First of all, $I(\chi)$ is irreducible by [113, Théorème 3]. Then

$$\Theta_{1,1}(I(\chi)) = I'(\chi^{-1})$$

by Corollary 5.2.4(1). Since the intertwining map $I(\chi) \to I(\chi^{-1})$ is an isomorphism by irreducibility, we have

$$I'(\chi^{-1}) \cong \Theta_{1,1}(I(\chi)) \cong \Theta_{1,1}(I(\chi^{-1})) \cong I'(\chi).$$

If $\chi^2 \neq \mathbb{1}$, this shows $I'(\chi)$ is irreducible by Lemma 5.3.3 and Lemma 5.3.4. If $\chi^2 = \mathbb{1}$, we instead use Corollary 5.2.4(1,2) to obtain

$$\begin{aligned}\dim \operatorname{Hom}_{G_1'}(I'(\chi), I'(\chi)) &= \dim \operatorname{Hom}_{G_1 \times G_1'}(\Omega_{1,1}, I(\chi) \boxtimes I'(\chi)) \\ &= \dim \operatorname{Hom}_{G_1}(I(\chi), I(\chi)) \\ &= 1\end{aligned}$$

since $I(\chi) = I(\chi^{-1})$ is irreducible. This shows that the intertwining operator $I'(\chi) \to I'(\chi)$ is unique, so the lemma follows from Lemma 5.3.4. ☐

Lemma 5.3.6. *Let $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n$ be a character such that $\chi_i^2 = \mathbb{1}$ for some $1 \leq i \leq n$. Then for any submodule $\pi \subset I(\chi)$, we have*

$$\chi^{\oplus 2} \subset R_B(\pi)^{ss}.$$

Similarly, for any submodule $\pi' \subset I'(\chi)$, we have

$$\chi_\psi \chi^{\oplus 2} \subset R_{B'}(\pi')^{ss}.$$

Proof. The orthogonal and metaplectic cases are identical, so we just prove the result for $\pi' \subset I'(\chi)$. Let $Q_i = L_i U_i \subset G_n'$ be the rank one standard parabolic subgroup corresponding to the ith long root of T'. Then

$$L_i \cong \underbrace{\widetilde{\mathrm{GL}}_1(\mathbb{Q}_q) \times_{\mu_2} \cdots \times_{\mu_2} \widetilde{\mathrm{GL}}_1(\mathbb{Q}_q)}_{i-1 \text{ times}} \times_{\mu_2} G_1' \times_{\mu_2} \underbrace{\widetilde{\mathrm{GL}}_1(\mathbb{Q}_q) \times_{\mu_2} \cdots \times_{\mu_2} \widetilde{\mathrm{GL}}_1(\mathbb{Q}_q)}_{n-i \text{ times}}.$$

Let

$$\rho = \chi_\psi \cdot (\chi_1 \boxtimes \cdots \boxtimes \chi_{i-1}), \quad \sigma = \chi_\psi \cdot (\chi_{i+1} \boxtimes \cdots \boxtimes \chi_n).$$

Since π' admits a nonzero map to $I'(\chi)$, χ is a quotient of $R_{B'}(\pi') = R_{B' \cap L_i} R_{Q_i}(\pi')$, so we have a nontrivial intertwining operator

$$R_{Q_i}(\pi') \to \operatorname{Ind}_{B' \cap L_i}^{L_i} \chi_\psi \chi = \rho \boxtimes I'(\chi_i) \boxtimes \sigma.$$

This is surjective since $I'(\chi_i)$ is irreducible by Lemma 5.3.5 above. (By (5.1), we have $\chi_i^2 = \mathbb{1} \neq |\cdot|^{\pm 1}$.) By the exactness of the Jacquet functor, we conclude

$$R_{B'}(\pi')^{ss} \twoheadrightarrow R_{B' \cap L_i}(\rho \boxtimes I'(\chi_i) \boxtimes \sigma)^{ss} = \chi_\psi \chi^{\oplus 2}.$$

☐

Lemma 5.3.7. *Suppose* $\chi = \chi_1 \boxtimes \cdots \chi_n : (\mathbb{Q}_q^\times)^n \to k^\times$ *is a character such that* χ_i *are all distinct and* $\chi_i^2 = \mathbb{1}$ *for at most one* $1 \leq i \leq n$*. Then*

$$\dim \operatorname{Hom}(I(\chi), I(\chi^w)) = \dim \operatorname{Hom}(I'(\chi), I'(\chi^w)) = 1$$

for all $w \in W$*.*

Proof. If the Weyl conjugates of χ are all distinct, this is automatic from Lemma 5.3.3; so assume without loss of generality that $\chi_i^2 = \mathbb{1}$ for exactly one $1 \leq i \leq n$. Then the stabilizer of χ in W has order exactly two. The argument is the same for $I(\chi)$ and $I'(\chi)$, so we consider $I(\chi)$ in order to minimize notation.

Claim. Any nontrivial map $f : I(\chi) \to I(\chi)$ is an isomorphism.

Proof of claim. Indeed, if f has nontrivial kernel, then $R_B(\ker f)^{ss}$ contains $\chi^{\oplus 2}$ by Lemma 5.3.6. But χ appears with multiplicity exactly two in $R_B(I(\chi))^{ss}$ by Lemma 5.3.3, so then $R_B(\operatorname{Im} f)^{ss}$ does not contain χ, which is impossible since we are assuming that f is nontrivial. Hence f is injective, so

$$R_B(f) : R_B(I(\chi)) \to R_B(I(\chi))$$

is injective. Since $R_B(I(\chi))$ has finite length, $R_B(f)$ is also surjective. But then $\operatorname{coker} f$ has trivial Jacquet module, which means $\operatorname{coker} f = 0$ by Lemma 5.3.4. So f is also surjective. ☐

Now by the claim, $\operatorname{End}(I(\chi))$ is a division algebra over k, and also a k-vector space of dimension

$$\dim \operatorname{Hom}(I(\chi), I(\chi)) \leq 2.$$

Since k is algebraically closed, we conclude

$$\dim \operatorname{Hom}(I(\chi), I(\chi)) = 1. \tag{5.8}$$

Next observe that there are no non-split extensions of distinct characters of T over k. In particular, we may decompose

$$R_B(I(\chi)) = \bigoplus_{\chi^w} R_B(I(\chi))_{\chi^w},$$

where χ^w runs over the (distinct) Weyl conjugates of χ, and (5.8) implies that $R_B(I(\chi))_\chi$ is a non-split extension of χ by χ. The same argument applies to show $R_B(I(\chi^w))_{\chi^w}$ is a non-split extension of χ^w by χ^w for all $w \in W$. Now for any nonzero intertwining map

$$f : I(\chi) \to I(\chi^w),$$

$R_B(\operatorname{Im} f)^{ss}$ contains $(\chi^w)^{\oplus 2}$ by Lemma 5.3.6. Hence the induced map

$$R_B(f) : R_B(I(\chi))_{\chi^w} \to R_B(I(\chi^w))_{\chi^w}$$

is surjective, in particular an isomorphism since both sides have dimension two over k. Since $R_B(I(\chi^w))_{\chi^w}$ is non-split, so is $R_B(I(\chi))_{\chi^w}$. Hence

$$\dim \operatorname{Hom}(I(\chi), I(\chi^w)) = \dim \operatorname{Hom}(R_B(I(\chi)), \chi^w) = 1$$

for all $w \in W$. ☐

5.3.8. To state the next lemma, we use the following explicit generators for the Weyl group W:

(i) The inversion s sending a character $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n$ to $\chi^s = \chi_1^{-1} \boxtimes \cdots \boxtimes \chi_n$.
(ii) For $1 \leq i < n$, the transposition w_i sending a character $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n$ to $\chi^{w_i} = \chi_1 \boxtimes \cdots \boxtimes \chi_{i-1} \boxtimes \chi_{i+1} \boxtimes \chi_i \boxtimes \chi_{i+2} \boxtimes \cdots \boxtimes \chi_n$.

Lemma 5.3.9. *Let* $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n : (\mathbb{Q}_q^\times)^n \to k^\times$ *be a character.*

(1) Suppose $\chi_1^2 \neq |\cdot|^{\pm 1}$*. Then*

$$I(\chi) \cong I(\chi^s) \text{ and } I'(\chi) \cong I'(\chi^s).$$

(2) Suppose $\chi_i/\chi_{i+1} \neq |\cdot|^{\pm 1}$ *for some* $1 \leq i < n$. *Then*

$$I(\chi) \cong I(\chi^{w_i}) \text{ and } I'(\chi) \cong I'(\chi^{w_i}).$$

Proof. Once again, the orthogonal and metaplectic cases are identical. We give the proof in the orthogonal case. For (1), let $P = MN \subset G_n$ be the rank one standard parabolic subgroup such that the Weyl group of M is generated by s. Then

$$M \cong G_1 \times (\mathbb{Q}_q)^{n-1}.$$

Also write $\rho = \chi_2 \boxtimes \cdots \boxtimes \chi_n : (\mathbb{Q}_q^\times)^{n-1} \to k^\times$. Then

$$I(\chi) = \operatorname{Ind}_P^{G_n} \operatorname{Ind}_{B\cap M}^M \chi = \operatorname{Ind}_P^{G_n} I(\chi_1) \boxtimes \rho \tag{5.9}$$

and similarly

$$I(\chi^s) = \operatorname{Ind}_P^{G_n} I(\chi_1^{-1}) \boxtimes \rho. \tag{5.10}$$

By Lemma 5.3.5, $I(\chi_1)$ is irreducible, with an intertwining isomorphism to $I(\chi_1^{-1})$. By (5.9) and (5.10), this induces an isomorphism $I(\chi) \cong I(\chi^s)$. The proof of (2) is similar: let $Q_i = L_iU_i \subset G_n$ be the rank one standard parabolic with Weyl group generated by w_i. Then

$$L_i \cong \left(\mathbb{Q}_q^\times\right)^{i-1} \times \mathrm{GL}_2(\mathbb{Q}_q) \times \left(\mathbb{Q}_q^\times\right)^{n-i-1}.$$

By [113, Théorème 3] applied to the $\mathrm{GL}_2(\mathbb{Q}_q)$-factor of L_i, we conclude

$$\operatorname{Ind}_{B\cap L_i}^{L_i} \chi \cong \operatorname{Ind}_{B\cap L_i} \chi^{w_i}.$$

(In the metaplectic case, L_i has a $\widetilde{\mathrm{GL}}_2(\mathbb{Q}_q)$ factor, to which we may still apply the results of *loc. cit.* with a twist by χ_ψ.) Then as above we obtain an isomorphism

$$I(\chi) = \operatorname{Ind}_{Q_i}^{G_n} \operatorname{Ind}_{B\cap L_i}^{L_i} \chi \cong \operatorname{Ind}_{Q_i}^{G_n} \operatorname{Ind}_{B\cap L_i}^{L_i} \chi^{w_i} = I(\chi^{w_i}).$$

☐

Definition 5.3.10. Let $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n : (\mathbb{Q}_q^\times)^n \to k^\times$ be a character.

(1) We say χ is *generic* if $\chi_i\chi_j \notin \left\{|\cdot|^{\pm 1}, \mathbb{1}\right\}$ for all $1 \leq i, j \leq n$ and $\chi_i/\chi_j \notin \left\{|\cdot|^{\pm 1}, \mathbb{1}\right\}$ for all $1 \leq i < j \leq n$.

(2) We say χ is *almost generic* if $\chi_i\chi_j, \chi_i/\chi_j \notin \left\{|\cdot|^{\pm 1}, \mathbb{1}\right\}$ for all $1 \leq i < j \leq n$, $\chi_i^2 \neq |\cdot|^{\pm 1}$ for all $1 \leq i \leq n$, and $\chi_i^2 = \mathbb{1}$ for at most one $1 \leq i \leq n$.

Corollary 5.3.11. *Let* $\chi : (\mathbb{Q}_q^\times)^n \to k^\times$ *be generic or almost generic. Then:*

(1) $I(\chi) \cong I(\chi^w)$ *and* $I'(\chi) \cong I'(\chi^w)$ *for all* $w \in W$.

(2) $I(\chi)$ *and* $I'(\chi)$ *are both irreducible.*

(3) $\Theta(I(\chi)) = I'(\chi)$.

Proof. (1) follows from writing w as a product of generators and repeatedly applying Lemma 5.3.9. (2) follows from (1) combined with Lemmas 5.3.4 and 5.3.7. Once we have (2), it follows from repeated applications of Corollary 5.2.4(1) that

$$\Theta(I(\chi)) = I'(\chi^{-1}).$$

Then (3) follows from (1). ☐

Definition 5.3.12. Let $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n : (\mathbb{Q}_q)^\times \to k^\times$ be a character.

(1) We say χ is *level-raising generic* if:

  (i) For exactly one $1 \leq i_0 \leq n$, $\chi_{i_0} = |\cdot|^{\frac{1}{2}}$.

  (ii) For all $1 \leq i < j \leq n$, $\chi_i\chi_j, \chi_i/\chi_j \notin \left\{|\cdot|^{\pm 1}, \mathbb{1}\right\}$.

  (iii) For all $i \neq i_0$, $\chi_i^2 \notin \left\{|\cdot|^{\pm 1}, \mathbb{1}\right\}$.

(2) We say χ is *almost level-raising generic* if it satisfies (i) and (ii) above, and moreover:
   (iii') For all $i \neq i_0$, $\chi_i^2 \neq |\cdot|^{\pm 1}$; and $\chi_i^2 = \mathbb{1}$ for at most $i \neq i_0$.

Notation 5.3.13. For χ almost level-raising generic, the set of Weyl conjugates $W(\chi)$ is naturally divided into two subsets: those $\chi^w = \chi_1' \boxtimes \cdots \boxtimes \chi_n'$ such that $\chi_i' = |\cdot|^{\frac{1}{2}}$ for some i; and those such that $\chi_i' = |\cdot|^{-\frac{1}{2}}$ for some i. We denote these subsets by $W(\chi)^+$ and $W(\chi)^-$, respectively.

Lemma 5.3.14. *Fix $\delta = +$ or $-$. Then for any $\chi^{w_1}, \chi^{w_2} \in W(\chi)^\delta$, we have $I(\chi^{w_1}) \cong I(\chi^{w_2})$ and $I'(\chi^{w_1}) \cong I'(\chi^{w_2})$.*

Proof. For any $\chi^{w_1}, \chi^{w_2} \in W(\chi)^\delta$, we may write $w_1^{-1}w_2 = s_1 \cdots s_k$ where each s_i is one of the generators in (5.3.8) and $\chi^{w_1 s_1 \cdots s_i} \in W(\chi)^\delta$ for all $1 \leq i \leq k$. The lemma then follows from repeated applications of Lemma 5.3.9. ☐

Construction 5.3.15. For any χ which is almost level-raising generic, there is a quotient $J(\chi)$ of $I(\chi)$ defined as follows. Let $Q_{i_0} = L_{i_0}U_{i_0} \subset G_n$ be the standard rank one parabolic corresponding to the short root indexed by i_0. Then we have

$$L_{i_0} \cong (\mathbb{Q}_q^\times)^{i_0 - 1} \times G_1 \times (\mathbb{Q}_q^\times)^{n - i_0}, \tag{5.11}$$

and by [113, Théorème 3], $\mathrm{Ind}_T^{L_{i_0}} \chi$ has a one-dimensional quotient

$$J_{i_0}(\chi) \coloneqq \chi_1 \boxtimes \cdots \boxtimes \chi_{i_0 - 1} \boxtimes \mathbb{1} \boxtimes \chi_{i_0+1} \boxtimes \cdots \boxtimes \chi_n, \tag{5.12}$$

where $\mathbb{1}$ denotes the trivial representation of G_1. We let

$$J(\chi) = \mathrm{Ind}_{Q_{i_0}}^{G_n} J_{i_0}(\chi). \tag{5.13}$$

To study the theta lift of $J(\chi)$, we first have the following calculation in the rank-one case.

Lemma 5.3.16. *If $\mathbb{1}$ is the trivial representation of $\mathrm{SO}(V_1)(\mathbb{Q}_q) = \mathrm{PGL}_2(\mathbb{Q}_q)$, then*

$$\Theta_{1,1}(\mathbb{1}) = I'(|\cdot|^{1/2}),$$

with the corresponding quotient

$$\mathcal{S}(V_1, k) \to \mathbb{1} \boxtimes \Theta_{1,1}(\mathbb{1})$$

induced by

$$\varphi \mapsto (g \mapsto \omega_\psi(1, g)\varphi(0)), \quad g \in \mathrm{Mp}_2(\mathbb{Q}_q).$$

Proof. Using the injection $\mathbb{1} \hookrightarrow I(|\cdot|^{-1/2})$, we obtain by Corollary 5.2.4(1) an embedding

$$\begin{aligned}\mathrm{Hom}_{G_1'}(\Theta_{1,1}(\mathbb{1}), \mathcal{M}) = \mathrm{Hom}\, G_1 \times G_1'(\Omega_{1,1}, \mathbb{1} \boxtimes \mathcal{M}) &\hookrightarrow \mathrm{Hom}_{G_1 \times G_1'}(\Omega_{1,1}, I(|\cdot|^{-1/2}) \boxtimes \mathcal{M}) \\ &= \mathrm{Hom}_{G_1'}(I'(|\cdot|^{1/2}, \mathcal{M})\end{aligned}$$

functorial in $k[G_1']$-modules $\mathcal{M}$. In particular, $\Theta_{1,1}(\mathbb{1})$ is a quotient of $I'(|\cdot|^{1/2})$. On the other hand, it is clear that the map in the lemma defines a nontrivial homomorphism $\Theta_{1,1}(\mathbb{1}) \to I'(|\cdot|^{1/2})$, so we have a nontrivial composite

$$I'(|\cdot|^{1/2}) \twoheadrightarrow \Theta_{1,1}(\mathbb{1}) \to I'(|\cdot|^{1/2}).$$

This must be an isomorphism by Lemma 5.3.7, and the lemma follows. ☐

Corollary 5.3.17. *If χ is almost level-raising generic, then $J(\chi)$ is irreducible and $\Theta_{n,n}(J(\chi)) = I'(\chi)$.*

Proof. First note that, by [113, p. 44], $J(\chi)$ is the image of a nonzero intertwining operator $I(\chi) \to I(\rho)$, where

$$\rho = \chi_1 \boxtimes \cdots \boxtimes \chi_{i_0 - 1} \boxtimes \chi_{i_0}^{-1} \boxtimes \chi_{i_0+1} \boxtimes \cdots \boxtimes \chi_n \in W(\chi)^-.$$

If $J(\chi) \twoheadrightarrow \pi$ is a nonzero quotient, then we obtain, by Lemma 5.3.4, a nonzero intertwining operator

$$I(\chi) \twoheadrightarrow J(\chi) \twoheadrightarrow \pi \to I(\chi^w)$$

for some $w \in W$. However, Lemmas 5.3.7 and 5.3.14 show that this composite (since it is not an isomorphism) must coincide with the intertwining operator $I(\chi) \to I(\rho)$ whose image defines $J(\chi)$. In particular $J(\chi) = \pi$. So indeed $J(\chi)$ is irreducible. Then $\Theta_{n,n}(J(\chi)) = I'(\rho^{-1})$ by Lemma 5.3.16 and repeated applications of Corollary 5.2.4(1). Since $\rho^{-1} \in W(\chi)^+$, we also have $I'(\rho^{-1}) = I'(\chi)$ by Lemma 5.3.14, and this completes the proof. ☐

5.4. **Explicit theta lifting over k for principal series.**

Notation 5.4.1. Given $\varphi \in \Omega_{n,n}$ and $t_1, \dots, t_n \in \mathbb{Q}_q$, define
(5.14)

$$\overline{\varphi}(t_1, \dots, t_n) = \int_{\mathbb{Q}_q^{\frac{n(n-1)}{2}}} \varphi(t_1 v_1, t_2 v_2 + a_1 v_1, t_3 v_3 + a_2 v_1 + a_3 v_2, \dots, t_n v_n + \cdots + a_{\frac{n(n-1)}{2}} v_{n-1}) \mathrm{d}a_1 \cdots \mathrm{d}a_{\frac{n(n-1)}{2}},$$

with $v_1, \dots, v_n$ as in (5.1.2) above.

A direct calculation shows that:

Lemma 5.4.2. *The map $\varphi \mapsto \overline{\varphi}$ defines a morphism of $T \times T'$-modules*

$$R_{B \times B'}(\Omega_{n,n}) \to \Big((|\cdot|^{-\frac{1}{2}})^{\boxtimes n} \boxtimes \chi_\psi \cdot (|\cdot|^{\frac{1}{2}})^{\boxtimes n} \Big) \otimes \mathcal{S}(\mathbb{Q}_q^n, k),$$

where $T \times T'$ acts on $\mathcal{S}(\mathbb{Q}_q^n, k)$ by

$$(x_1, \dots, x_n) \times (y_1, \dots, y_n)(f)(t_1, \dots, t_n) = f(x_1^{-1} t_1 \overline{y}_1, \dots, x_n^{-1} t_n \overline{y}_n). \tag{5.15}$$

Definition 5.4.3. For any character $\chi = \chi_1 \boxtimes \cdots \boxtimes \chi_n : (\mathbb{Q}_q^\times)^n \to k^\times$ with $\chi_i \neq |\cdot|^{-\frac{1}{2}}$ for all i, we consider the following condition on $\varphi \in \Omega_{n,n}$:

(C_χ) There exists $g \in G'_n$ such that $f_\chi(\overline{\omega_\psi(1, g)\varphi}) \neq 0$, where

$$f_\chi : \Big((|\cdot|^{-\frac{1}{2}})^{\boxtimes n} \boxtimes \chi_\psi \cdot (|\cdot|^{\frac{1}{2}})^{\boxtimes n} \Big) \otimes \mathcal{S}(\mathbb{Q}_q^n, k) \to \chi \boxtimes \chi_\psi \cdot \chi^{-1}$$

is the unique projection deduced from Lemma 5.2.1.

The map f_χ also exists and is unique without the assumption $\chi_i \neq |\cdot|^{-\frac{1}{2}}$, so (C_χ) makes sense for all χ; this is elementary but not needed in our applications.

5.4.4. Let $K \subset G_n$ be the hyperspecial subgroup stabilizing the self-dual lattice $L \subset V_n$ (5.1.2).

Lemma 5.4.5. *Let $\chi : (\mathbb{Q}_q^\times)^n \to k^\times$ be almost generic and unramified, and suppose $\varphi \in \Omega_{n,n}^K$ satisfies condition (C_χ). Also let $\mathcal{M}$ be any admissible $k[G'_n]$-module. Then, for any nonzero map*

$$\theta : \Omega_{n,n} \to I(\chi) \boxtimes \mathcal{M}$$

of $k[G_n \times G'_n]$-modules, we have

$$\theta(\varphi) \neq 0.$$

Proof. Since $I(\chi)$ is irreducible, Corollary 5.3.11 implies that θ factors as

$$\Omega_{n,n} \xrightarrow{\theta_0} I(\chi) \boxtimes \Theta_{n,n}(I(\chi)) \cong I(\chi) \boxtimes I'(\chi) \xrightarrow{f} I(\chi) \boxtimes \mathcal{M}$$

for some map of G'_n-modules

$$f : I'(\chi) \to \mathcal{M}.$$

Then since $I'(\chi)$ is also irreducible, it suffices to show $\theta_0(\varphi) \neq 0$. Now, by Lemma 5.4.2, the map

$$\chi \mapsto \Big((h, g) \mapsto f_\chi(\overline{\omega_\psi(h, g)\varphi})\Big)$$

gives a $(G_n \times G'_n)$-intertwining map

$$F_\chi : \Omega_{n,n} \to I(\chi) \boxtimes I'(\chi^{-1}) \cong I(\chi) \boxtimes I'(\chi).$$

Since F_χ is not identically zero, Corollary 5.3.11 shows that θ_0 coincides with F_χ up to a nonzero scalar; in particular $\theta_0(\varphi) \neq 0$ if and only if $F_\chi(\varphi) \neq 0$. Then because φ is K-spherical, $F_\chi(\varphi) \neq 0$ if and only if there exists $g \in G'_n$ with $f_\chi(\overline{\omega_\psi(1, g)\varphi}) \neq 0$, which is condition (C_χ). ☐

Similarly, we have:

Lemma 5.4.6. *Let $\chi : (\mathbb{Q}_q^\times)^n \to k^\times$ be almost level-raising generic and unramified, and suppose $\varphi \in \Omega_{n,n}^K$ satisfies condition (C_χ). Then, for any nonzero map*

$$\theta : \Omega_{n,n} \to J(\chi) \boxtimes \mathcal{M}$$

of $k[G_n \times G'_n]$-modules, we have

$$\theta(\varphi) \neq 0.$$

Proof. Since $J(\chi)$ is irreducible by Corollary 5.3.17, the map θ factors as

$$\Omega_{n,n} \twoheadrightarrow J(\chi) \boxtimes \Theta_{n,n}(J(\chi)) \cong J(\chi) \boxtimes I'(\chi) \xrightarrow{f} J(\chi) \boxtimes \mathcal{M}$$

for some map of G'_n-modules

$$f : I'(\chi) \to \mathcal{M}.$$

By Lemma 5.3.4, we may assume without loss of generality that $f : I'(\chi) \to I'(\chi^w)$ is an intertwining operator. Then by Lemmas 5.3.7 and 5.3.14, we see that it suffices to show φ has nonzero image under the map

$$\theta_0 : \Omega_{n,n} \to J(\chi) \boxtimes I'(\chi) \to J(\chi) \boxtimes I'(\chi^{-1}).$$

By Lemma 5.4.2, the map

$$\varphi \mapsto \Big((h, g) \mapsto f_\chi(\overline{\omega_\psi(h, g)\chi})\Big)$$

gives a $(G_n \times G'_n)$-intertwining map

$$F_\chi : \Omega_{n,n} \to I(\chi) \boxtimes I'(\chi^{-1}).$$

As in the proof of Lemma 5.4.5, since φ is K-invariant, condition (C_χ) is equivalent to $F_\chi(\varphi) \neq 0$. Now project to obtain a composite

$$F'_\chi : \Omega_{n,n} \xrightarrow{F_\chi} I(\chi) \boxtimes I'(\chi^{-1}) \to J(\chi) \boxtimes I'(\chi^{-1}).$$

Now we observe that any K-spherical vector in $I(\chi)$ has nonzero image in $J(\chi)$; indeed, by the construction of $J(\chi)$ it suffices to show this when $n = 1$, in which case it is clear from the explicit intertwining operator in [113, p. 44] and the assumption $q^2 - 1 \neq 0$ in k. In particular, we have $F'_\chi(\varphi) \neq 0$ if and only if $F_\chi(\varphi) \neq 0$. On the other hand, F'_χ must factor as

$$F'_\chi : \Omega_{n,n} \to J(\chi) \boxtimes \Theta_{n,n}(J(\chi)) \to J(\chi) \boxtimes I'(\chi^{-1});$$

the map $\Theta_{n,n}(J(\chi)) = I'(\chi) \to I'(\chi^{-1})$ is unique by Lemma 5.3.7, so F'_χ coincides with θ_0 up to a nonzero scalar. Hence $\theta_0(\chi) \neq 0$ is equivalent to (C_χ). ☐

5.4.7. We end this subsection with a convenient shortcut that we will use to check condition (C_χ) in the characteristic zero, non-level-raising case.

Lemma 5.4.8. *Suppose $k = \mathbb{C}$ and let $\chi : (\mathbb{Q}_q^\times)^2 \to k^\times$ be generic and unramified. Assume that $\varphi \in \Omega_{2,2}^K$ satisfies:*

1. *For all $x, y \in V$, we have $\varphi(x, y) \in \mathbb{Q}_{\geq 0}$.*
2. *φ is supported on $L \times q^{-1}L$ and invariant under translations by $qL \times q^2L$.*
3. *There exist elements $x, y \in V$ with $y \cdot v_1 = 0$ and $\varphi(x, y) \neq 0$.*

Then there exists $\chi^w \in W(\chi)$ such that φ satisfies condition (C_{χ^w}).

Proof. Let w_0 be the Weyl element in (3.3.1); then

$$\overline{\omega_\psi(1, w_0)\varphi} \in \left((|\cdot|^{-\frac{1}{2}})^{\boxtimes 2} \boxtimes \chi_\psi \cdot (|\cdot|^{\frac{1}{2}})^{\boxtimes 2}\right) \otimes \mathcal{S}(\mathbb{Q}_q^2, k)$$

is a unit multiple of the function

$$\begin{aligned} c(t_1, t_2) &= \int_{V^2} \int_{\mathbb{Q}_q} \varphi(x, y) \psi\left(t_1 x \cdot v_1 + t_2 y \cdot v_2 + a y \cdot v_1\right) \mathrm{d}a \mathrm{d}x \mathrm{d}y \\ &= \mathrm{Vol}\left\{q^{-2}\mathbb{Z}_q\right\} \int_{V^2} \varphi(x, y) \psi\left(t_1 x \cdot v_1 + t_2 y \cdot v_2\right) \cdot \mathbb{1}_{y \cdot v_1 \in q^2 \mathbb{Z}_q} \mathrm{d}a \mathrm{d}x \mathrm{d}y. \end{aligned}$$

By condition (2) of the lemma, c is supported on $q^{-1}\mathbb{Z}_q \times q^{-2}\mathbb{Z}_q$ and invariant under translations by $\mathbb{Z}_q \times q\mathbb{Z}_q$. Note that the conditions of the lemma together imply that $c(\mathbb{Z}_q, q\mathbb{Z}_q) \neq 0$. From this, we will deduce that $f_{\chi^w}(c) \neq 0$ for some Weyl conjugate χ^w of χ, which will show the lemma.

Indeed, write $\chi = \chi_1 \boxtimes \chi_2$ and $\chi^w = \chi_1^w \boxtimes \chi_2^w$ for $w \in W$. Then

$$f_{\chi^w}(c) = f_{|\cdot|^{\frac{1}{2}}\chi_1^w} \left(f_{|\cdot|^{\frac{1}{2}}\chi_2^w} c(t_1, \cdot) \right)$$

(where the functions on the right are defined in Lemma 5.2.1(2)). Now, because $c(\mathbb{Z}_q, q\mathbb{Z}_q) \neq 0$, $c(1, \cdot)$ is a nonzero element of the four-dimensional k-vector space $S_{-2,1}$ from Lemma 5.2.1(3). Since $\chi_2^w |\cdot|^{\frac{1}{2}}$ takes on four distinct nontrivial values as w ranges over W, we may therefore replace χ with a Weyl conjugate such that $d := f_{|\cdot|^{\frac{1}{2}}\chi_2} c(t_1, \cdot)$ is not identically zero. Since d lies in $S_{-1,0}$ as a function of t_1, Lemma 5.2.1(3) again implies that either $f_{|\cdot|^{\frac{1}{2}}\chi_1}(d)$ or $f_{|\cdot|^{\frac{1}{2}}\chi_1^{-1}}(d)$ is nonzero. This concludes the proof because $\chi_1^{-1} \boxtimes \chi_2$ is Weyl-conjugate to χ.

□

5.5. **Applications to formal theta lifts.**

5.5.1. For this subsection, fix the following data:

- A quadratic space V of trivial discriminant and dimension $2n + 1 \geq 3$.
- A neat compact open subgroup $K = \prod K_\ell \subset \mathrm{GSpin}(V)(\mathbb{A})$.
- An odd prime q such that K_q is hyperspecial.
- A subring $R \subset \mathbb{C}$ which is either $\mathbb{C}$, or a finite flat extension of $\breve{\mathbb{Z}}_p$ (embedded into $\mathbb{C}$ by a choice of isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$). In the latter case we assume the pro-order of K_q is prime to p. Let $\varpi_R \in R$ generate the maximal ideal (so $\varpi_R = 0$ if $R = \mathbb{C}$), and write $k := R/\varpi_R$.
- A character $\chi : (\mathbb{Q}_q^\times)^n \to k^\times$ that is either almost generic or almost level-raising generic (Definition 5.3.10 and Definition 5.3.12).

With these data, we make the following notation:

Notation 5.5.2.

1. Write $\mathbb{T}_q := \mathbb{T}_{\mathrm{GSpin}_{2n+1},q,R}$. Let $\mathfrak{m}_\chi \subset \mathbb{T}_q$ be the maximal ideal corresponding to χ; explicitly, $\mathfrak{m}_\chi$ is the annihilator of the unique spherical vector in $I(\chi)$.
2. For any ring A, $\mathbb{1}_A$ denotes the trivial $A[K_q]$-module.

Lemma 5.5.3. *We have*

$$\text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_R \otimes_{\mathbb{T}_{q,\mathfrak{m}_\chi}} k \cong I(\chi)$$

as k-linear representations of $\mathrm{GSpin}(V)(\mathbb{Q}_q)$.

Proof. Write

$$\pi_\chi := \text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_R \otimes_{\mathbb{T}_q,\mathfrak{m}_\chi} k.$$

Then we have a map $f : \pi_\chi \to I(\chi)$, sending the generator of π_χ to the unique spherical vector. The first claim is that f is surjective. Indeed, if χ is almost generic, this is automatic by Corollary 5.3.11. If χ is almost level-raising generic, then $I(\chi)$ has no K_q-spherical submodules: since taking the K_q-invariants is exact, $J(\chi)$ is the unique K_q-spherical constituent, and it cannot be both a quotient and a submodule by Lemma 5.3.7. So indeed f is surjective.

It remains to prove f is injective. Because π_χ is generated by K_q-spherical vectors, every $\mathrm{GSpin}(V)(\mathbb{Q}_q)$-stable subspace $V \subset \pi_\chi$ satisfies $R_B(V) \neq 0$; for instance, this follows from [114, Corollaire II.3.5] combined with (I.3.15) of *op. cit.* Thus it suffices to show $R_B(f)$ is injective, or equivalently that

$$\dim_k R_B(\pi_\chi) = \dim_k R_B(I(\chi)) = |W| = n! \cdot 2^n.$$

To compute the dimension of $R_B(\pi_\chi)$, note that $R_B(\text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_k) = k[X^\bullet(\widehat{T})]$ by the Iwasawa decomposition for $\mathrm{GSpin}(V)(\mathbb{Q}_q)$, and the action of $\mathbb{T}_q \otimes_R k$ is the natural one under the Satake isomorphism

$$\mathbb{T}_q \otimes_R k \xrightarrow{\sim} k[X^\bullet(\widehat{T})]^W.$$

Since $k[X^\bullet(\widehat{T})]$ is a finite flat $k[X^\bullet(\widehat{T})]^W$-algebra of degree $|W|$, we have

$$\dim R_B(\pi_\chi) = \dim \left(k[X^\bullet(\widehat{T})] \otimes_{k[X^\bullet(\widehat{T})]^W,\mathfrak{m}_\chi} k \right) = |W|,$$

as desired. □

Notation 5.5.4. For all $\varphi \in \mathcal{S}(V^n \otimes \mathbb{Q}_q, R)$, let $\overline{\varphi}$ be its image in $\mathcal{S}(V^n \otimes \mathbb{Q}_q, k)$.

(Despite the conflict with Notation 5.4.1, we hope that the meaning will always be clear from context.)

5.5.5. For the next proposition, recall the notation on formal theta lifts from §3.5.

Proposition 5.5.6. *Let $\alpha \in \mathrm{Test}_K(V, R)$ be a test vector and $n_0 \geq 1$ an integer such that:*

(1) $\Theta(\alpha, \varphi) \not\equiv 0 \pmod{\varpi_R^{n_0}}$ for some $\varphi = \varphi^q \otimes \varphi_q \in \mathcal{S}(V^n \otimes \mathbb{A}_f, R)^K$.

(2) For all $h \in \mathfrak{m}_\chi \subset \mathbb{T}_q$ and all $\varphi'_q \in \mathcal{S}(V^n \otimes \mathbb{Q}_q, R)^{K_q}$,

$$\Theta(h \cdot \alpha, \varphi^q \otimes \varphi'_q) \equiv 0 \pmod{\varpi_R^{n_0}}.$$

Then for any $\varphi_q^\circ \in \mathcal{S}(V^n \otimes \mathbb{Q}_q, R)^{K_q}$ such that $\overline{\varphi}_q^\circ$ satisfies condition (C_χ),

$$\Theta(\alpha, \varphi^q \otimes \varphi_q^\circ) \not\equiv 0 \pmod{\varpi_R^{n_0}}.$$

Proof. For all $f \in \text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_R$, we can consider the convolution

$$f * \alpha \in \mathrm{Test}_{K^q}(V, R)$$

(notation as in (3.5.3)). By Proposition 3.5.4, the map

$$\text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_R \otimes \mathcal{S}(V^n \otimes \mathbb{Q}_q, R) \to M^n_{n+\frac{1}{2},R} \tag{5.16}$$

defined by

$$(f, \varphi_q') \mapsto \Theta(f * \alpha, \varphi^q \otimes \varphi_q')$$

is $\mathrm{GSpin}(V)(\mathbb{Q}_q) \times \mathrm{Mp}_{2n}(\mathbb{Q}_q)$-equivariant. By condition (2) of the proposition applied to $h = \varpi_R \in \mathfrak{m}_\chi$, the image of (5.16) is contained in $\varpi_R^{n_0-1} M^n_{n+\frac{1}{2},R}$. Abbreviate

$$\mathcal{M} := \frac{\varpi_R^{n_0-1} M^n_{n+\frac{1}{2},R}}{\varpi_R^{n_0} M^n_{n+\frac{1}{2},R}}.$$

Reducing modulo ϖ_R, (5.16) induces a map

$$\text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_k \otimes \mathcal{S}(V^n \otimes \mathbb{Q}_q, k) \to \mathcal{M} \tag{5.17}$$

which remains a map of $\mathrm{GSpin}(V)(\mathbb{Q}_q) \times \mathrm{Mp}_{2n}(\mathbb{Q}_q)$-modules by Proposition 3.4.6. Now note that condition (2) of the proposition implies (5.17) factors through the quotient

$$\text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_k \twoheadrightarrow \text{c-Ind}_{K_q}^{\mathrm{GSpin}(V)(\mathbb{Q}_q)} \mathbb{1}_k \otimes_{\mathbb{T}_q, \mathfrak{m}_\chi} k \cong I(\chi)$$

(Lemma 5.5.3). By duality [114, p. 96, Propriété (vi)], (5.17) is equivalent to a nonzero map

$$\theta : \mathcal{S}(V^n \otimes \mathbb{Q}_q, k) \to I(\chi^{-1}) \otimes \mathcal{M}$$

and, for $\varphi_q^\circ \in \mathcal{S}(V^n \otimes \mathbb{Q}_q, R)^{K_q}$, we have

$$\Theta(\alpha, \varphi^q \otimes \varphi_q^\circ) \not\equiv 0 \pmod{\varpi_R^{n_0}} \iff \theta(\overline{\varphi}_q^\circ) \neq 0.$$

If χ is almost generic, the proposition therefore follows from Corollary 5.3.11 and Lemma 5.4.5. So assume instead that χ is almost level-raising generic.

Claim. Let $\mathcal{M}$ be any admissible $k[\mathrm{Mp}_{2n}(\mathbb{Q}_q)]$-module. Then every map of $\mathrm{GSpin}(V)(\mathbb{Q}_q) \times \mathrm{Mp}_{2n}(\mathbb{Q}_q)$-modules

$$\mathcal{S}(V^n \otimes \mathbb{Q}_q, k) \to I(\chi^{-1}) \boxtimes \mathcal{M}$$

factors as

$$\mathcal{S}(V^n \otimes \mathbb{Q}_q, k) \to J(\chi) \boxtimes \mathcal{M} \to I(\chi^{-1}) \boxtimes \mathcal{M}.$$

Given the claim, the proposition follows from Lemma 5.4.6, because $J(\chi) \hookrightarrow I(\chi^{-1})$ is injective (cf. the proof of Corollary 5.3.17).

Let us now prove the claim. Since the statement is purely local in nature, we resume our local abbreviation $\Omega_{n,n} = \mathcal{S}(V^n \otimes \mathbb{Q}_q, k)$. The claim is also insensitive to replacing χ with any $\chi^w \in W(\chi)^+$ (by Lemma 5.3.14), so suppose without loss of generality that $\chi = |\cdot|^{\frac{1}{2}} \boxtimes \rho$ for some almost generic character $\rho : (\mathbb{Q}_q^\times)^n \to k^\times$. Apply Corollary 5.2.4(1) with $\chi_0 = |\cdot|^{-\frac{1}{2}}$ and $\pi_{m-1} = I(\rho) \cong I(\rho^{-1})$. Since $\Theta_{n-1,n-1}(I(\rho)) = I'(\rho)$ by Corollary 5.3.11, we obtain an isomorphism

$$\mathrm{Hom}(\Omega_{n,n}, I(\chi^{-1}) \boxtimes \mathcal{M}) = \mathrm{Hom}(I'(\chi), \mathcal{M}) \tag{5.18}$$

that is functorial in $\mathcal{M}$. So it suffices to show the claim with $\mathcal{M} = I'(\chi)$. But in this case, (5.18) combined with Lemma 5.3.7 shows that there is a unique non-zero map

$$\Omega_{n,n} \to I(\chi^{-1}) \boxtimes I'(\chi). \tag{5.19}$$

Since $J(\chi)$ injects into $I(\chi^{-1})$, we also have the map induced by the theta lift

$$\Omega_{n,n} \to J(\chi) \boxtimes \Theta_{n,n}(J(\chi)) \cong J(\chi) \boxtimes I'(\chi) \to I(\chi^{-1}) \boxtimes I'(\chi);$$

this must coincide with (5.19) up to a nonzero scalar, which shows the claim. □

5.6. **Main result on changing test functions.**

5.6.1. Let π, S, and E_0 be as in Notation 4.0.1, and fix a prime $\mathfrak{p}$ of E_0, which we suppress from all the notation in this subsection. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ inducing the prime $\mathfrak{p}$, and let $R \subset \overline{\mathbb{Z}}_p$ be the ring of integers of the maximal unramified extension of $O = O_{\mathfrak{p}}$.

We apply the results of §5.5 to study the behavior of $\lambda_n^D(Q, \varphi; K)$ and $\kappa_n^D(Q, \varphi; K)$ as φ changes locally at a prime $q \nmid Q$.

Proposition 5.6.2. *Suppose q is an admissible prime. Then there exists an unramified character $\chi : (\mathbb{Q}_q^\times)^2 \to \overline{\mathbb{F}}_p^\times$ such that χ is almost level-raising generic, and the corresponding maximal ideal $\mathfrak{m}_\chi \subset \mathbb{T}_{q,R}$ contains the kernel of the Hecke action on the unique spherical vector of π_q.*

Proof. Because π_q is unramified with trivial central character, we have $\pi_q = I(\widetilde{\chi})$ for an unramified character $\widetilde{\chi} : (\mathbb{Q}_q^\times)^2 \to \mathbb{C}^\times$, uniquely determined up to Weyl action. Write $\alpha = \iota^{-1}\widetilde{\chi}(q, 1)$, $\beta = \iota^{-1}\widetilde{\chi}(1, q)$; then by Theorem 2.2.10(1), $\rho_\pi(\mathrm{Frob}_q)$ has eigenvalues $\alpha q^{1/2}, \beta q^{1/2}, \alpha^{-1} q^{1/2}, \beta^{-1} q^{1/2}$, which lie in $\overline{\mathbb{Z}}_p^\times \subset \overline{\mathbb{Q}}_p^\times$. By the admissibility of q, we may assume without loss of generality that

$$\alpha q^{1/2} \equiv q \pmod{\mathfrak{p}}, \ \beta q^{1/2} \not\equiv \pm q, \pm 1, q^2, q^{-1} \pmod{\mathfrak{p}}. \tag{5.20}$$

We define the character χ to be the reduction modulo p of $\iota^{-1}\widetilde{\chi}$, and the conditions (5.20) exactly correspond to χ being almost level-raising generic. □

Corollary 5.6.3. *Suppose Qq is admissible with $\nu(DQ)$ odd, and fix an S-level structure K for $\mathrm{GSpin}(V_{DQ})$. Let $\varphi_q^\circ \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, O)^{K_q}$ be a test function whose image in $\mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, \overline{\mathbb{F}}_p)$ satisfies condition (C_χ), where $\chi : (\mathbb{Q}_q^\times)^2 \to \overline{\mathbb{F}}_p^\times$ is the almost level-raising generic character of Proposition 5.6.2. Then for all $n \geq 1$ and all $\varphi = \varphi^q \otimes \varphi_q \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{A}_f, O)^K$, we have*

$$\lambda_n^D(Q, \varphi^q \otimes \varphi_q^\circ; K) \supset \lambda_n^D(Q, \varphi; K).$$

Remark 5.6.4. The same corollary holds for $\kappa_n^D(Q, -)$ if $\nu(DQ)$ is even, but this version will not be used for the main results.

Proof. Suppose $\lambda_n^D(Q, \varphi; K) = (\varpi^{n_0 - 1})$ for some $1 \leq n_0$; without loss of generality we may assume $n_0 \leq n$ and that

$$\lambda_n^D(Q, \varphi^q \otimes \varphi_q'; K) \equiv 0 \pmod{\varpi^{n_0 - 1}}$$

for all $\varphi_q' \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, O)^{K_q}$.

Now choose a vector $\alpha \in \mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$ such that $\alpha(Z(T, \varphi)_K)$ generates $\lambda_n^D(Q, \varphi; K)$ for some $T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$. Lift α arbitrarily to an O-valued test function $\widetilde{\alpha} \in \mathrm{Test}_K(V_{DQ}, O)$. Recall $R \subset \overline{\mathbb{Z}}_p$ is the ring of integers of the maximal unramified extension of O, and let $f_\pi : \mathbb{T}_{q,R} \to R$ be the character associated with the Hecke eigenvalues of π_q, so that $f_\pi(h) \in (\varpi)$ for all $h \in \mathfrak{m}_\chi \subset \mathbb{T}_{q,R}$. Then for $h \in \mathfrak{m}_\chi$ and $\varphi_q' \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, O)^{K_q}$, we have

$$\Theta(h \cdot \widetilde{\alpha}, \varphi^q \otimes \varphi_q') \equiv f_\pi(h) \Theta(\widetilde{\alpha}, \varphi^q \otimes \varphi_q') \equiv 0 \pmod{\varpi^{n_0}},$$

so we may apply Proposition 5.5.6 to conclude. (Note that because $q^4 \not\equiv 1 \pmod{p}$, p does not divide the pro-order of K_q.)

□

We now give an analogue of Corollary 5.6.3 in characteristic zero, which requires $Q = 1$.

Proposition 5.6.5. *Suppose $q \notin S$ is a prime such that $\rho_\pi(\mathrm{Frob}_q)$ has distinct eigenvalues. Let $\chi : (\mathbb{Q}_q^\times)^2 \to \mathbb{C}^\times$ be the unramified character, well-defined up to W-action, such that π_q is a constituent of $I(\chi)$. Then χ is generic.*

Proof. The proof is similar to Proposition 5.6.2, using that $|\alpha| = |\beta| = 1$ because π_q is tempered by Theorem 2.2.10(1). ☐

Corollary 5.6.6. *Let $D \geq 1$ be squarefree with $\nu(D)$ even, and suppose $q \notin S \cup \operatorname{div}(D)$ is a prime such that $\rho_\pi(\mathrm{Frob}_q)$ has distinct eigenvalues. Fix an S-level structure K for $\mathrm{GSpin}(V_D)$, and let $\varphi_q^\circ \in \mathcal{S}(V_D^2 \otimes \mathbb{Q}_q, \mathbb{Z})^{K_q}$ be a test function whose image in $\mathcal{S}(V_D^2 \otimes \mathbb{Q}_q, \mathbb{C})^{K_q}$ satisfies the hypotheses of Lemma 5.4.8, or more generally condition (C_χ), where $\chi : (\mathbb{Q}_q^\times)^2 \to \mathbb{C}^\times$ is the generic character of Proposition 5.6.5. Then for all $\varphi = \varphi^q \otimes \varphi_q \in \mathcal{S}(V_D^2 \otimes \mathbb{A}_f, O)^K$,*

$$\kappa^D(1, \varphi; K) \neq 0 \implies \kappa^D(1, \varphi^q \otimes \varphi_q^\circ; K) \neq 0.$$

Proof. The argument is similar to Corollary 5.6.3. First fix a vector $\alpha \in \mathrm{Test}_K(V_D, \pi, O)$ with $\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T,\varphi)_K) \neq 0$ for some $T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$. Because $H^1(\mathbb{Q}, T_\pi)$ is torsion-free by Lemma 4.1.6(1), we may choose a linear functional $\beta : H^1(\mathbb{Q}, T_\pi) \to \overline{\mathbb{Q}}_p$ such that $\beta(\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T,\varphi)_K)) \neq 0$. Let $\widetilde{\alpha} \in \mathrm{Test}_K(V_D, \mathbb{C})$ denote the composite map

$$\mathrm{CH}^2(\mathrm{Sh}_K(V_D)) \xrightarrow{\partial_{\mathrm{AJ},\mathfrak{m}}} H^1(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}) \xrightarrow{\alpha_*} H^1(\mathbb{Q}, T_\pi) \xrightarrow{\beta} \overline{\mathbb{Q}}_p \xrightarrow{\iota} \mathbb{C}.$$

Then $\widetilde{\alpha}$ is Hecke-equivariant because α is so. Let $\varphi_q' \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, \mathbb{C})^{K_q}$ be any vector. By the Hecke-equivariance of $\widetilde{\alpha}$, we have $\Theta(h \cdot \widetilde{\alpha}, \varphi^q \otimes \varphi_q') = 0$ for all $h \in \mathfrak{m}_\chi$ (Notation 5.5.2), with $R = \mathbb{C}$, $\varpi_R = 0$. We can now apply Proposition 5.5.6 to conclude. ☐

With essentially the same proof, we have the following in the endoscopic case:

Corollary 5.6.7. *With the setup of Corollary 5.6.6, suppose π is endoscopic associated to a pair (π_1, π_2). Then for all $\varphi = \varphi^q \otimes \varphi_q \in \mathcal{S}(V_D^2 \otimes \mathbb{A}_f, O)^K$ and $j = 1$ or 2, we have*

$$\kappa^D(1, \varphi; K)^{(j)} \neq 0 \implies \kappa^D(1, \varphi^q \otimes \varphi_q^\circ; K)^{(j)} \neq 0.$$

6. The ramified GSpin_5 Rapoport-Zink space

6.1. **The moduli problem.**

6.1.1. Fix a prime $q > 2$, and let $\mathcal{O}_q$ be the unique maximal order in the non-split quaternion algebra B over $\mathbb{Q}_q$. Suppose given a q-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}}_q$ of dimension 4 and height 8, equipped with an action $\iota_{\mathbb{X}} : \mathcal{O}_q \hookrightarrow \mathrm{End}(\mathbb{X})$ and a principal polarization $\lambda_{\mathbb{X}} : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ such that the Rosati involution $*$ of $\mathrm{End}(\mathbb{X})$ induces a nebentype involution on $\mathcal{O}_q$ of *unit type* (Definition 1.2.9).

Definition 6.1.2. Let Nilp be the category of schemes over $\breve{\mathbb{Z}}_q$ on which q is locally nilpotent. Let $\mathcal{N}$: Nilp $\to$ Set be the functor sending $S \in$ Nilp to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$ where:

(i) X is a q-divisible group of dimension 4 and height 8 over S.

(ii) $\iota : \mathcal{O}_q \hookrightarrow \mathrm{End}(X/S)$ is an $\mathcal{O}_q$-action such that

$$\det(T - \iota(\alpha) | \operatorname{Lie}(X)) = (T^2 - \operatorname{tr}(\alpha)T + N(\alpha))^2, \ \ \forall \alpha \in \mathcal{O}_q.$$

(iii) $\lambda : X \xrightarrow{\sim} X^\vee$ is a principal polarization such that the Rosati involution on $\mathrm{End}(X/S)$ extends the involution $*$ on $\mathcal{O}_q$.

(iv) If $\overline{S} = S \times_{\breve{\mathbb{Z}}_q} \overline{\mathbb{F}}_q$ denotes the mod q fiber, then ρ is an $\mathcal{O}_q$-linear quasi-isogeny

$$\rho : X \times_S \overline{S} \to \mathbb{X} \times_{\overline{\mathbb{F}}_q} \overline{S}$$

such that $\rho^\vee \circ \lambda_{\mathbb{X}} \circ \rho = c(\rho)\lambda$ for some $c(\rho) \in \mathbb{Q}_q^\times$.

The functor $\mathcal{N}$ is studied in [118]; in this section, we will recall the key points, and prove additional properties needed for our applications.

6.1.3. The functor $\mathcal{N}$ is represented by a formal scheme over $\operatorname{Spf} \breve{\mathbb{Z}}_q$, locally formally of finite type, which admits a decomposition into open and closed formal subschemes

$$\mathcal{N} = \sqcup_{i \in \mathbb{Z}} \mathcal{N}(i) \tag{6.1}$$

according to the q-adic valuation of $c(\rho)$.

6.1.4. Let $\sigma : \breve{\mathbb{Z}}_q \to \breve{\mathbb{Z}}_q$ denote the *arithmetic* Frobenius, lifting the qth power map on $\overline{\mathbb{F}}_q$. The formal scheme $\mathcal{N}$ is equipped with a canonical Weil descent datum $\varphi : \mathcal{N} \xrightarrow{\sim} \sigma^*\mathcal{N}$ over $\breve{\mathbb{Z}}_q$, which we now recall. On S-points, φ is given by the isomorphism $\varphi(S) : \mathcal{N}(S) \xrightarrow{\sim} \mathcal{N}((\sigma^{-1})^*S)$, which sends $(X, \iota, \lambda, \rho) \in \mathcal{N}(S)$ to $((\sigma^{-1})^*X, (\sigma^{-1})^*\iota, (\sigma^{-1})^*\lambda, \rho \circ F_{X/\overline{S}})$, where

$$F_{X/\overline{S}} : (\sigma^{-1})^* X_{\overline{S}} \to X_{\overline{S}}$$

is the relative Frobenius. Since $c(\rho \circ F_{X/\overline{S}}) = c(\rho) + 1$, φ restricts to an isomorphism

$$\varphi_i : \mathcal{N}(i) \xrightarrow{\sim} \sigma^*\mathcal{N}(i+1)$$

for each $i \in \mathbb{Z}$.

6.2. **The Bruhat-Tits stratification.**

6.2.1. Let $\mathcal{M}$ denote the underlying reduced scheme of $\mathcal{N}(0)$. We now recall the description in [118] of the stratification of $\mathcal{M}$ in terms of lattices in the isocrystal N of our fixed q-divisible group $\mathbb{X}$. By the assumption that the involution $*$ on $\mathcal{O}_q$ is of unit type, we may choose coordinates

$$\mathcal{O}_q = \mathbb{Z}_{q^2} \oplus \Pi\mathbb{Z}_{q^2}, \tag{6.2}$$

where $\alpha^* = \Pi\alpha\Pi^{-1} = \sigma\alpha$ for $\alpha \in \mathbb{Z}_{q^2}$, $\Pi^* = \Pi$, and $\Pi^2 = q$.

6.2.2. Label the two embeddings of $\mathbb{Z}_{q^2}$ into $\breve{\mathbb{Z}}_q$ by $j_\bullet$ and $j_\circ$. Then we have a decomposition

$$N = N_\bullet \oplus N_\circ, \tag{6.3}$$

where $\iota(\alpha) = j_?(\alpha)$ on $N_?$ for $? = \bullet, \circ$. Each of $N_\bullet$ and $N_\circ$ is an isocrystal of dimension 4 and slope $\frac{1}{2}$.

The polarization $\lambda_{\mathbb{X}}$ induces a pairing

$$\langle \cdot, \cdot \rangle : N \otimes N \to \breve{\mathbb{Q}}_q = \breve{\mathbb{Z}}_q \otimes \mathbb{Q}_q,$$

with respect to which $N_\bullet$ and $N_\circ$ are each isotropic (since $*$ is nontrivial on $\mathbb{Z}_{q^2} \subset \mathcal{O}_q$). Define a new pairing

$$\langle \cdot, \cdot \rangle_\bullet : N_\bullet \otimes N_\bullet \to \breve{\mathbb{Q}}_q \tag{6.4}$$

by

$$\langle x, y \rangle_\bullet = \langle x, \Pi y \rangle.$$

Then $\langle \cdot, \cdot \rangle_\bullet$ is symplectic and non-degenerate.

The operators Π and V on N both interchange $N_\bullet$ and $N_\circ$, so the operator

$$\tau := \Pi V^{-1} \tag{6.5}$$

stabilizes $N_\bullet$; moreover

$$\langle \tau x, \tau y \rangle_\bullet = \langle x, y \rangle_\bullet^\sigma, \tag{6.6}$$

where again σ is the arithmetic Frobenius of $\breve{\mathbb{Q}}_q$. Hence $W := N_\bullet^{\tau=1}$ is a 4-dimensional $\mathbb{Q}_q$-vector space equipped with a $\mathbb{Q}_q$-valued symplectic form, such that $N_\bullet = W \otimes_{\mathbb{Q}_q} \breve{\mathbb{Q}}_q$.

Definition 6.2.3. For a lattice $\Lambda \subset W$, denote the $\mathbb{Z}_q$-dual lattice by $\Lambda^\vee$. We define the following families of lattices in W:

$$\begin{aligned}
\mathcal{L}_{\{0\}} &= \left\{\text{lattices } \Lambda \subset W \text{ s.t. } \Lambda = \Lambda^\vee\right\}.\\
\mathcal{L}_{\{2\}} &= \left\{\text{lattices } \Lambda \subset W \text{ s.t. } \Lambda = q\Lambda^\vee\right\}.\\
\mathcal{L}_{\{02\}} &= \left\{\text{pairs of lattices } \Lambda_0, \Lambda_2 \subset W \text{ s.t. } q\Lambda_0 = q\Lambda_0^\vee \subset_2 q\Lambda_2^\vee = \Lambda_2 \subset_2 \Lambda_0\right\}.\\
\mathcal{L}_{\{1\}} &= \left\{\text{lattices } \Lambda_1 \subset W \text{ s.t. } q\Lambda_1^\vee \subset_2 \Lambda_1 \subset_2 \Lambda_1^\vee\right\}.
\end{aligned}$$

Here, the notation $\Lambda \subset_n \Lambda'$ was defined in (1.1.8).

Theorem 6.2.4. *The underlying reduced scheme $\mathcal{M}$ of $\mathcal{N}(0)$ admits a stratification*

$$\mathcal{M} = \mathcal{M}^0_{\{0\}} \sqcup \mathcal{M}^0_{\{2\}} \sqcup \mathcal{M}^0_{\{02\}} \sqcup \mathcal{M}^0_{\{1\}},$$

with a decomposition into open and closed subschemes

$$\mathcal{M}^0_? = \bigsqcup_{y \in \mathcal{L}_?} \mathcal{M}^0_?(y)$$

for each $? = \{0\}, \{2\}, \{02\}, \{1\}$, satisfying the following conditions (where $\mathcal{M}_?$ and $\mathcal{M}_?(y)$ denote the Zariski closures of $\mathcal{M}^0_?$ and $\mathcal{M}^0_?(y)$, respectively):

1. *For $? = \{0\}$ or $\{2\}$ and each $\Lambda \in \mathcal{L}_?$, $\mathcal{M}_?(\Lambda)$ is isomorphic to the smooth projective hypersurface in $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ defined by the equation*

   $$X_3^q X_0 - X_0^q X_3 + X_2^q X_1 - X_1^q X_2.$$

   The scheme $\mathcal{M}$ is the union $\mathcal{M}_{\{0\}} \cup \mathcal{M}_{\{2\}}$.
2. *Given $\Lambda_0 \in \mathcal{L}_{\{0\}}$ and $\Lambda_2 \in \mathcal{L}_{\{2\}}$, $\mathcal{M}_{\{0\}}(\Lambda_0)$ meets $\mathcal{M}_{\{2\}}(\Lambda_2)$ if and only if $(\Lambda_0, \Lambda_2) \in \mathcal{L}_{\{02\}}$, in which case the intersection is transverse and*

   $$\mathcal{M}_{\{0\}}(\Lambda_0) \cap \mathcal{M}_{\{2\}}(\Lambda_2) = \mathcal{M}_{\{02\}}(\Lambda_0, \Lambda_2).$$

   For each $(\Lambda_0, \Lambda_2) \in \mathcal{L}_{\{02\}}$, $\mathcal{M}_{\{02\}}(\Lambda_0, \Lambda_2)$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$, and both of the resulting embeddings $\mathbb{P}^1_{\overline{\mathbb{F}}_q} \hookrightarrow \mathbb{P}^3_{\overline{\mathbb{F}}_q}$ are linear.
3. *For each $\Lambda_1 \in \mathcal{L}_{\{1\}}$, $\mathcal{M}^0_{\{1\}}(\Lambda_1) = \mathcal{M}_{\{1\}}(\Lambda_1)$ is an isolated point. Given $\Lambda_0 \in \mathcal{L}_{\{0\}}$ and $\Lambda_2 \in \mathcal{L}_{\{2\}}$, $\mathcal{M}_{\{1\}}(\Lambda_1)$ lies on $\mathcal{M}_{\{0\}}(\Lambda_0)$ if and only if $\Lambda_1 \subset \Lambda_0$, and on $\mathcal{M}_{\{2\}}(\Lambda_2)$ if and only if $\Lambda_2 \subset \Lambda_1$.*
4. *For a pair of distinct $\Lambda_0, \Lambda_0' \in \mathcal{L}_{\{0\}}$, $\mathcal{M}_{\{0\}}(\Lambda_0)$ meets $\mathcal{M}_{\{0\}}(\Lambda_0')$ if and only if $\Lambda_0 \cap \Lambda_0' \in \mathcal{L}_{\{1\}}$; in this case the intersection is transverse and we have*

   $$\mathcal{M}_{\{0\}}(\Lambda_0) \cap \mathcal{M}_{\{0\}}(\Lambda_0') = \mathcal{M}_{\{1\}}(\Lambda_0 \cap \Lambda_0').$$
5. *For a pair of distinct $\Lambda_2, \Lambda_2' \in \mathcal{L}_{\{2\}}$, $\mathcal{M}_{\{2\}}(\Lambda_2)$ meets $\mathcal{M}_{\{2\}}(\Lambda_2')$ if and only if $\Lambda_2 + \Lambda_2' \in \mathcal{L}_{\{1\}}$; in this case the intersection is transverse and we have*

   $$\mathcal{M}_{\{2\}}(\Lambda_2) \cap \mathcal{M}_{\{2\}}(\Lambda_2') = \mathcal{M}_{\{1\}}(\Lambda_2 + \Lambda_2').$$
6. *The stratum $\mathcal{M}_{\{1\}}$ is precisely the nonsmooth locus of $\mathcal{N}(0)$, and the complete local ring of $\mathcal{N}(0)$ at each point in $\mathcal{M}_{\{1\}}$ is isomorphic to*

   $$\breve{\mathbb{Z}}_q[\![X, Y, Z, W]\!]/q - XY + ZW.$$

Proof. Each point except (6) is contained in [118], and (6) is [85, Corollary 4.2]. ☐

6.2.5. For later use, we recall the meaning of each of the strata in Theorem 6.2.4 on the level of $\overline{\mathbb{F}}_q$-points.

Given $s = (X, \iota, \lambda, \rho) \in \mathcal{N}(\overline{\mathbb{F}}_q)$, the covariant Dieudonné module of X gives rise to an $\mathcal{O}_q$-stable lattice $M \subset N$. Such an M admits a decomposition

$$M = M_\bullet \oplus M_\circ \subset N_\bullet \oplus N_\circ$$

as in (6.2.2); here we are using $q \neq 2$. For a lattice $\Lambda \subset W$, define $\breve{\Lambda} := \Lambda \otimes \breve{\mathbb{Z}}_q \subset M_\bullet$. Then we have, for any $s \in \mathcal{M}(\overline{\mathbb{F}}_q)$ and any lattices $\Lambda_0 \in \mathcal{L}_{\{0\}}$, $\Lambda_2 \in \mathcal{L}_{\{2\}}$, $\Lambda_1 \in \mathcal{L}_{\{1\}}$:

- s lies in $\mathcal{M}_{\{1\}}$ if and only if $M_\bullet = \tau M_\bullet$, and is the point $\mathcal{M}_{\{1\}}(\Lambda_1)$ if and only if $M_\bullet = \breve{\Lambda}_1$.
- s lies in $\mathcal{M}_{\{0\}}(\Lambda_0) - \mathcal{M}_{\{1\}}$ if and only if $M_\bullet + \tau M_\bullet = \breve{\Lambda}_0$.
- s lies in $\mathcal{M}_{\{2\}}(\Lambda_2) - \mathcal{M}_{\{1\}}$ if and only if $M_\bullet \cap \tau M_\bullet = \breve{\Lambda}_2$.

By Theorem 6.2.4, at least one of these three options occurs for any point $s \in \mathcal{M}(\overline{\mathbb{F}}_q)$.

Notation 6.2.6. From now on, to ease the notation we shall abbreviate $\mathcal{M}_{\{0\}}(\Lambda_0)$ as $\mathcal{M}(\Lambda_0)$, etc.

6.3. **Deformation theory and the geometry of $\mathcal{N}$.**

6.3.1. Let $(X, \lambda, \iota, \rho)$ be an S-valued point of $\mathcal{N}$, for some $S \in$ Nilp. To X we associate the (covariant) Dieudonné crystal $\mathbb{D}(X)$ [74]; thus for any thickening $S \hookrightarrow \widehat{S}$ in Nilp admitting locally nilpotent divided powers, we obtain a locally free sheaf $\mathbb{D}(\widehat{S})$ of $\mathcal{O}_{\widehat{S}}$-modules, such that $D(X) := \mathbb{D}(X)(S)$ fits into a canonical exact sequence

$$0 \to \omega_{X^\vee} \to D(X) \to \mathrm{Lie}(X) \to 0 \tag{6.7}$$

of locally free $\mathcal{O}_S$-modules.

6.3.2. As in (6.2.5) above, the action of $\mathbb{Z}_{q^2} \subset \mathcal{O}_q$ on $\mathbb{D}(X)$ induces a decomposition

$$\mathbb{D}(X) = \mathbb{D}(X)_\bullet \oplus \mathbb{D}(X)_\circ,$$

and likewise for $D(X)$, $\omega_{X^\vee}$, and $\mathrm{Lie}\, X$; the action of Π interchanges the two components in each case.

The polarization λ induces a perfect alternating pairing

$$\langle \cdot, \cdot \rangle : \mathbb{D}(X) \otimes \mathbb{D}(X) \to \mathcal{O}_S^{\mathrm{cris}}.$$

Since both $\mathbb{D}(X)_\bullet$ and $\mathbb{D}(X)_\circ$ are isotropic, $\langle \cdot, \cdot \rangle$ identifies $\mathbb{D}(X)_\bullet$ with the dual of $\mathbb{D}(X)_\circ$. Finally, the submodule $\omega_{X^\vee}$ of $D(X)$ is also isotropic, so that λ induces perfect pairings of locally free $\mathcal{O}_S$-modules:

$$\langle \cdot, \cdot \rangle : \omega_{X^\vee,\bullet} \otimes \mathrm{Lie}\, X_\circ \to \mathcal{O}_S$$
$$\langle \cdot, \cdot \rangle : \omega_{X^\vee,\circ} \otimes \mathrm{Lie}\, X_\bullet \to \mathcal{O}_S.$$

If $S = \mathrm{Spec}\, \overline{\mathbb{F}}_q$, then $\mathbb{D}(X)$ is equivalent to the data of the Dieudonné module M of X; the exact sequence (6.7) becomes

$$0 \to VM/pM \to M \to M/VM \to 0.$$

6.3.3. Let $S \hookrightarrow \widehat{S}$ be a thickening in Nilp admitting locally nilpotent divided powers, and fix $x = (X, \lambda, \iota, \rho) \in \mathcal{N}(S)$. Denote by $\mathrm{Lift}(x)$ the set of isomorphisms classes of lifts of x to $\widehat{x} = (\widehat{X}, \widehat{\lambda}, \widehat{\iota}, \widehat{\rho}) \in \mathcal{N}(\widehat{S})$, and denote by $\overline{\mathrm{Lift}}(x)$ the set of locally free, $\mathcal{O}_q$-stable, totally isotropic $\mathcal{O}_{\widehat{S}}$-submodules $\widehat{\omega}_{X^\vee} \subset \mathbb{D}(X)(\widehat{S})$ lifting $\omega_{X^\vee}$. From the well-known deformation theory of q-divisible groups [76], one has:

Proposition 6.3.4. *The canonical map*

$$\widehat{x} = (\widehat{X}, \widehat{\lambda}, \widehat{\iota}, \widehat{\rho}) \mapsto \omega_{\widehat{X}^\vee} \subset D(\widehat{X}) = \mathbb{D}(X)(\widehat{S})$$

defines a bijection

$$\mathrm{Lift}(x) \xrightarrow{\sim} \overline{\mathrm{Lift}}(x).$$

6.3.5. Let $\mathcal{N}^{\mathrm{sm}}$ denote the formally smooth locus of $\mathcal{N}(0)$, which by Theorem 6.2.4(6) is the complement of $\bigsqcup_{\Lambda_1\in\mathcal{L}_{\{1\}}}\mathcal{M}(\Lambda_1)$. Before we can calculate the tangent bundle to the mod q fiber of $\mathcal{N}^{\mathrm{sm}}$ in Theorem 6.3.7 below, we need the following lemma.

Lemma 6.3.6. *Suppose given $(X,\lambda,\iota,\rho)\in\mathcal{N}^{\mathrm{sm}}(S)$, for some $\overline{\mathbb{F}}_q$-scheme S. Then Π induces isomorphisms of line bundles on S:*

$$\Pi : \frac{\operatorname{Lie} X_\bullet}{\Pi \operatorname{Lie} X_\circ} \xrightarrow{\sim} \Pi \operatorname{Lie} X_\bullet \subset \operatorname{Lie} X_\circ$$
$$\Pi : \frac{\omega_{X^\vee,\bullet}}{\Pi\omega_{X^\vee,\circ}} \xrightarrow{\sim} \Pi\omega_{X^\vee,\bullet} \subset \omega_{X^\vee,\circ},$$

and likewise with $\bullet$ and $\circ$ reversed.

Proof. We consider the first map; the second, and the versions with $\bullet$ and $\circ$ reversed, are all similar to this case. Without loss of generality, we may assume that $S = \operatorname{Spf} R$ is affine and formally of finite type, and that R is a local ring with maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m} = \overline{\mathbb{F}}_q$. Then $\operatorname{Lie} X_\bullet$ and $\operatorname{Lie} X_\circ$ are each free of rank two over R by the Kottwitz condition, and the map $\Pi_\bullet : \operatorname{Lie} X_\bullet \to \operatorname{Lie} X_\circ$ is nonzero modulo $\mathfrak{m}$; indeed, this amounts to the assertion that $\Pi M_\bullet \neq VM_\bullet$ for the Dieudonné module $M = M_\bullet \oplus M_\circ$ corresponding to the special fiber of X, and this holds because we are away from the nonsmooth locus of $\mathcal{N}(0)$, cf. (6.2.5).

In particular, we can choose bases of $\operatorname{Lie} X_\bullet$ and $\operatorname{Lie} X_\circ$ such that

$$\Pi_\bullet = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : \operatorname{Lie} X_\bullet \to \operatorname{Lie} X_\circ$$

for some $d \in R$. Now, we also know that $\Pi^2 = q = 0$ on $\operatorname{Lie}(X)$, so the matrix g for $\Pi_\circ : \operatorname{Lie} X_\circ \to \operatorname{Lie} X_\bullet$ must satisfy

$$g\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} g = 0.$$

A direct calculation shows that $g = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}$ for some $w \in R$, where $wd = 0$; but the same reasoning as for $\Pi_\bullet$ shows that $\Pi_\circ$ is nonzero modulo $\mathfrak{m}$, so w is a unit and we conclude $d = 0$. From these coordinates for $\Pi_\bullet$ and $\Pi_\circ$, it is clear that $\Pi \operatorname{Lie} X_\bullet$ and $\operatorname{Lie} X_\bullet / \Pi \operatorname{Lie} X_\circ$ are both free of rank one over R and that the map in the lemma is indeed an isomorphism. ☐

Theorem 6.3.7. *Let $\mathcal{T}$ denote the tangent bundle on the mod q fiber $\mathcal{N}^{\mathrm{sm}}_{\overline{\mathbb{F}}_q}$. Then we have a canonical exact sequence:*

$$0 \to \mathcal{T} \to \mathcal{H}om(\omega_{\mathcal{X}^\vee,\bullet}, \operatorname{Lie}\mathcal{X}_\bullet) \to \mathcal{H}om(\Pi\omega_{\mathcal{X}^\vee,\circ}, \operatorname{Lie}\mathcal{X}_\bullet/\Pi\operatorname{Lie}\mathcal{X}_\circ) \to 0,$$

where $\mathcal{X}$ is the universal q-divisible group.

Proof. It suffices to consider deformations of a point $(X,\iota,\lambda,\rho)\in\mathcal{N}^{\mathrm{sm}}(R)$ to points of $\mathcal{N}^{\mathrm{sm}}(R[\epsilon]/\epsilon^2)$, for R an $\overline{\mathbb{F}}_q$-algebra. Let $S = \operatorname{Spec} R$ and $\widehat{S} = \operatorname{Spec} R[\epsilon]/\epsilon^2$. By Proposition 6.3.4, we need to consider lifts of $\omega_{X^\vee}$ to locally free submodules $\widehat{\omega}_{X^\vee} \subset \mathbb{D}(X)(\widehat{S})$ which are $\mathcal{O}_q$-stable and isotropic; this is equivalent to lifting $\omega_{X^\vee,\bullet} \subset D(X)_\bullet$ and $\omega_{X^\vee,\circ} \subset D(X)_\circ$ to locally free submodules $\widehat{\omega}_{X^\vee,\bullet} \subset \mathbb{D}(X)_\bullet(\widehat{S})$ and $\widehat{\omega}_{X^\vee,\circ} \subset \mathbb{D}(X)_\circ(\widehat{S})$, subject to the following conditions.

(i) $\langle \widehat{\omega}_{X^\vee,\bullet}, \widehat{\omega}_{X^\vee,\circ}\rangle = 0$.
(ii) $\Pi\widehat{\omega}_{X^\vee,\bullet} \subset \widehat{\omega}_{X^\vee,\circ}$.
(iii) $\Pi\widehat{\omega}_{X^\vee,\circ} \subset \widehat{\omega}_{X^\vee,\bullet}$.

Now, lifts of $\omega_{X^\vee,\bullet}$ correspond to maps of R-modules $f_\bullet : \omega_{X^\vee,\bullet} \to \operatorname{Lie} X_\bullet$ via

$$f_\bullet \mapsto \operatorname{span}\left\{x + \epsilon f_\bullet(x) + \epsilon\omega_{X^\vee,\bullet} \;:\; x \in \omega_{X^\vee,\bullet}\right\},$$

and likewise for $\omega_{X^\vee,\circ}$. The conditions (i)-(iii) translate to:

(i)' $\langle x, f_\bullet(y)\rangle = \langle y, f_\circ(x)\rangle$, for all $x \in \omega_{X^\vee,\circ}$, $y \in \omega_{X^\vee,\bullet}$.
(ii)' $\Pi f_\bullet(x) = f_\circ(\Pi x)$ for all $x \in \omega_{X^\vee,\bullet}$.
(iii)' $\Pi f_\circ(x) = f_\bullet(\Pi x)$ for all $x \in \omega_{X^\vee,\circ}$.

If we specify $f_\bullet$, then (i)' can be taken as a *definition* of $f_\circ$. In terms of $f_\bullet$ only, conditions (ii)' and (iii)' correspond to:

(ii)" $\langle y, \Pi f_\bullet(x)\rangle = \langle x, \Pi f_\bullet(y)\rangle$, for all $x, y \in \omega_{X^\vee,\bullet}$.
(iii)" $\langle y, f_\bullet(\Pi x)\rangle = \langle x, f_\bullet(\Pi y)\rangle$, for all $x, y \in \omega_{X^\vee,\circ}$.

Now, since we are outside the singular locus, Lemma 6.3.6 implies that $\Pi\omega_{X^\vee,\bullet}$ and $\Pi\omega_{X^\vee,\circ}$ are locally rank-one direct summands of the rank-two projective R-modules $\omega_{X^\vee,\circ}$ and $\omega_{X^\vee,\bullet}$, respectively. In particular, using that (ii)" and (iii)" are clearly satisfied when x and y are linearly dependent, it suffices to check (ii)" and (iii)" for $x \in \Pi\omega_{X^\vee,\circ}$ and $x \in \Pi\omega_{X^\vee,\bullet}$, respectively. Using $\langle \Pi z, \Pi w\rangle = q\langle z, w\rangle = 0$, we find that both conditions (ii)" and (iii)" are equivalent to $\Pi f_\bullet(\Pi\omega_{X^\vee,\circ}) = 0$. Again by Lemma 6.3.6, the kernel of Π on $\operatorname{Lie} X_\bullet$ is $\Pi \operatorname{Lie} X_\circ$, so we are just requiring

$$f_\bullet(\Pi\omega_{X^\vee,\circ}) \subset \Pi \operatorname{Lie} X_\circ,$$

as desired. ☐

6.3.8. *Scheme-theoretic description of the strata $\mathcal{M}(\Lambda_0)$, $\mathcal{M}(\Lambda_2)$.* Fix a lattice $\Lambda \in \mathcal{L}_{\{0\}} \sqcup \mathcal{L}_{\{2\}}$; we recall the construction of $\mathcal{M}(\Lambda)$ as a subscheme of $\mathcal{N}$ given in [118, §4]. Let $\breve{\Lambda} = \Lambda \otimes \breve{\mathbb{Z}}_q \subset N_\bullet$, and let Y denote the q-divisible group over $\overline{\mathbb{F}}_q$ associated to the lattice

$$\breve{\Lambda} \oplus q^\delta \Pi^{-1}\breve{\Lambda} \subset N_\bullet \oplus N_\circ,$$

where $\delta = 0$ if $\Lambda \in \mathcal{L}_{\{0\}}$ and $\delta = 1$ if $\Lambda \in \mathcal{L}_{\{2\}}$. The group Y comes with a natural quasi-isogeny

$$t : Y \to \mathbb{X}.$$

For any $(X, \lambda, \iota, \rho) \in \mathcal{N}(S)$ with S an $\overline{\mathbb{F}}_q$-scheme, consider the two quasi-isogenies:

$$\rho_+ : X \xrightarrow{\rho} \mathbb{X}_S \xrightarrow{q^\delta t^{-1}} Y_S,$$
$$\rho_- : Y_S \xrightarrow{q^{1-\delta} t} \mathbb{X}_S \xrightarrow{\rho^{-1}} X,$$

where $\delta = 0$ if $\Lambda \in \mathcal{L}_{\{0\}}$ and $\delta = 1$ if $\Lambda \in \mathcal{L}_{\{2\}}$.

Then the scheme $\mathcal{M}(\Lambda)$ is constructed as the locus where ρ_+ and ρ_- are both isogenies. In fact, since the dual lattice to $\breve{\Lambda} \oplus \Pi^{-1}\breve{\Lambda}$ is $q^{1-2\delta}(\breve{\Lambda} \oplus \Pi^{-1}\breve{\Lambda})$, we may identify $Y \xrightarrow{\sim} Y^\vee$ by $q^{1-2\delta} t^\vee \circ \lambda_{\mathbb{X}} \circ t$; with respect to this polarization, ρ_+ and ρ_- are duals, so ρ_- is an isogeny if and only if ρ_+ is.

Proposition 6.3.9. *Let $\mathcal{O}(1)$ be the line bundle on $\mathcal{M}(\Lambda)$ corresponding to the embedding into $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ of Theorem 6.2.4 (2). Then, if $(X, \lambda, \iota, \rho)$ is the universal q-divisible group over $\mathcal{M}(\Lambda)$, we have isomorphisms of line bundles on $\mathcal{M}(\Lambda)^{\mathrm{sm}} := \mathcal{M}(\Lambda) \cap \mathcal{N}^{\mathrm{sm}}$:*

$$\mathcal{O}(-q) \cong \Pi \operatorname{Lie} X_\circ \cong \operatorname{Lie} X_\bullet / \Pi \operatorname{Lie} X_\circ.$$

Proof. We give the proof in the case $\Lambda \in \mathcal{L}_{\{0\}}$; for $\Lambda \in \mathcal{L}_{\{2\}}$, the roles of $\bullet$ and $\circ$ are interchanged. Let $S := \mathcal{M}(\Lambda)^{\mathrm{sm}}$, and let Y_S be the constant q-divisible group Y on S.

We need to recall the construction of the projective embedding from [118, §4]. First of all, one has the Dieudonné crystal

$$\mathbb{D}(Y_S) = \mathbb{D}(Y_S)_\bullet \oplus \mathbb{D}(Y_S)_\circ,$$

with notation as in (6.3.2), and

$$D(Y_S)_\circ = D(Y)_\circ \otimes \mathcal{O}_S$$

is a free $\mathcal{O}_S$-module of rank 4. The submodule $\rho_{+,*}D(X)_\circ \subset D(Y_S)_\circ$ is locally a free summand of rank 1, and this is the tautological bundle $\mathcal{O}(-1)$ under the map $\mathcal{M}(\Lambda) \hookrightarrow \mathbb{P}^3_{\overline{\mathbb{F}}_q}$. The canonical Verschiebung map

$$\mathtt{V} : D(Y_S)_\circ \to D(Y_S^{(q)})_\circ \simeq D(Y_S)_\bullet \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{(q)}}$$

is given by the qth power map $\mathcal{O}_S \to \mathcal{O}_{S^{(q)}}$ tensored with the isomorphism

$$V : D(Y)_\circ = \Pi^{-1}\breve{\Lambda}/\Pi\breve{\Lambda} \xrightarrow{\sim} \breve{\Lambda}/q\breve{\Lambda} = D(Y)_\bullet,$$

so we have an isomorphism

$$\mathcal{O}(-q) \simeq \mathtt{V}(\rho_{+,*}(D(X)_\circ).$$

Now we note that the map

$$\mathtt{V} \circ \rho_{+,*} : D(X)_\circ \to D(Y_S)_\bullet \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{(q)}} \tag{6.8}$$

annihilates both $\Pi D(X)_\bullet$ and $\omega_{X^\vee,\circ}$, because $\mathtt{V}(\omega_{Y_S^\vee,\circ}) = \mathtt{V} \circ \Pi(D(Y_S)_\circ) = 0$ by the definition of Y. In particular, (6.8) induces a surjection

$$\mathtt{V} \circ \rho_{+,*} : \frac{\operatorname{Lie} X_\circ}{\Pi \operatorname{Lie} X_\bullet} \xrightarrow{\sim} \mathcal{O}(-q), \tag{6.9}$$

which is a map of line bundles by Lemma 6.3.6 and therefore an isomorphism. Combined with Lemma 6.3.6 for the other isomorphism, this completes the proof. ☐

Theorem 6.3.10. *For any $\Lambda \in \mathcal{L}_{\{0\}} \sqcup \mathcal{L}_{\{2\}}$, the normal bundle to $\mathcal{M}(\Lambda)^{\mathrm{sm}}$ inside $\mathcal{N}^{\mathrm{sm}}_{\overline{\mathbb{F}}_q}$ is isomorphic to $\mathcal{O}(-2q)$ for the embedding into $\mathbb{P}^3_{\overline{\mathbb{F}}_q}$ given in Theorem 6.2.4.*

Proof. For simplicity, we continue to assume $\Lambda \in \mathcal{L}_{\{0\}}$; the other case is similar.

The first step is to compute the tangent bundle to $\mathcal{M}(\Lambda)^{\mathrm{sm}}$. We wish to consider the lifts of $(X, \lambda, \iota, \rho) \in \mathcal{M}(\Lambda)^{\mathrm{sm}}(R)$ to points of $\mathcal{M}(\Lambda)^{\mathrm{sm}}(R[\epsilon]/\epsilon^2)$ for R an $\overline{\mathbb{F}}_q$-algebra, which we may take to be reduced since $\mathcal{M}(\Lambda)$ is reduced. Continuing the notation of (6.3.8), such lifts correspond to the pairs

$$\begin{cases} f_\bullet : \omega_{X^\vee,\bullet} \to \operatorname{Lie} X_\bullet \\ f_\circ : \omega_{X^\vee,\circ} \to \operatorname{Lie} X_\circ \end{cases}$$

satisfying (i)'-(iii)' from the proof of Theorem 6.3.7, subject to the additional condition that

$$f_\bullet(\rho_{-,*}(\omega_{Y_R^\vee,\bullet})) = f_\circ(\rho_{-,*}(\omega_{Y_R^\vee,\circ})) = 0.$$

By the definition of Y, we have $\omega_{Y_R^\vee,\circ} = 0$, so the second condition is automatic.

Claim. On $\mathcal{M}(\Lambda)^{\mathrm{sm}}$ we have

$$\rho_{-,*}(\omega_{Y_R^\vee,\bullet}) = \Pi\omega_{X^\vee,\circ}. \tag{6.10}$$

Given the claim, we conclude by comparing with the proof of Theorem 6.3.7 that the tangent bundle $\mathcal{T}_\Lambda$ to $\mathcal{M}(\Lambda)^{\mathrm{sm}}$ is canonically isomorphic to

$$\mathcal{H}om(\omega_{X^\vee,\bullet}/\Pi\omega_{X^\vee,\circ}, \operatorname{Lie} X_\bullet).$$

The normal bundle is the quotient $\mathcal{T}/\mathcal{T}_\Lambda$, which by Theorem 6.3.7 is

$$\mathcal{H}om(\Pi\omega_{X^\vee,\circ}, \Pi \operatorname{Lie} X_\circ).$$

Since $\Pi\omega_{X^\vee,\circ}$ is dual to $\operatorname{Lie} X_\bullet/\Pi \operatorname{Lie} X_\circ$ on $\mathcal{M}(\Lambda)^{\mathrm{sm}}$, the theorem then follows from Proposition 6.3.9.

Now we turn to the proof of the claim. We have

$$\rho_{-,*}(\omega_{Y_R^\vee,\bullet}) = \rho_{-,*}(D(Y_R)_\bullet) = (\ker(\rho_{+,*} : D(X)_\circ \to D(Y_R)_\circ))^\perp$$

because ρ_+ and ρ_- are duals; the orthogonal complement is with respect to the perfect pairing on $D(X)$. Because R is reduced, the Verschiebung $\mathtt{V} : D(Y_R)_\circ \to D(Y_R)_\bullet$ is injective, so (6.9) implies that

$$\ker\left(\rho_{+,*} : D(X)_\circ \to D(Y_R)_\circ\right) = \omega_{X^\vee,\circ} + \Pi D(X)_\circ.$$

Arguments similar to Lemma 6.3.6 show that $(\Pi D(X)_\circ)^\perp = \Pi D(X)_\bullet$, and so we conclude

$$\rho_{-,*}(\omega_{Y_R^\vee,\bullet}) = \left(\omega_{X^\vee,\circ} + \Pi D(X)_\bullet\right)^\perp = \omega_{X^\vee,\bullet} \cap \Pi D(X)_\circ.$$

Now it follows from Lemma 6.3.6 and the snake lemma for the diagram

$$\begin{tikzcd}
0 \arrow[r] & \omega_{X^\vee,\circ} \arrow[r] \arrow[d, "\Pi"] & D(X)_\circ \arrow[r] \arrow[d, "\Pi"] & \mathrm{Lie}\, X_\circ \arrow[r] \arrow[d, "\Pi"] & 0 \\
0 \arrow[r] & \omega_{X^\vee,\bullet} \arrow[r] & D(X)_\bullet \arrow[r] & \mathrm{Lie}\, X_\bullet \arrow[r] & 0
\end{tikzcd}$$

that $\omega_{X^\vee,\bullet} \cap \Pi D(X)_\bullet = \Pi\omega_{X^\vee,\circ}$, so the proof of the claim is complete. □

6.4. Regularization of $\mathcal{M}$ and intersection theory.

Notation 6.4.1. Let $\widetilde{\mathcal{N}}(0)$ be the blowup of $\mathcal{N}(0)$ along $\mathcal{M}_{\{1\}}$, and let $\widetilde{\mathcal{M}}$ be the strict transform of the reduced locus $\mathcal{M}$; then $\widetilde{\mathcal{M}}$ is smooth. We denote by $C(\Lambda_1)$ the exceptional divisor of $\widetilde{\mathcal{M}}$ above $\mathcal{M}(\Lambda_1)$ for each $\Lambda_1 \in \mathcal{L}_{\{1\}}$. For any $\Lambda_0 \in \mathcal{L}_{\{0\}}$ and $\Lambda_2 \in \mathcal{L}_{\{2\}}$, let $\widetilde{\mathcal{M}}(\Lambda_0)$ and $\widetilde{\mathcal{M}}(\Lambda_2)$ be the strict transforms of $\mathcal{M}(\Lambda_0)$ and $\mathcal{M}(\Lambda_2)$, respectively.

Lemma 6.4.2. *For any $\Lambda_1 \in \mathcal{L}_{\{1\}}$, there exists an isomorphism $C(\Lambda_1) \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q} \times \mathbb{P}^1_{\overline{\mathbb{F}}_q}$ such that:*

1. *For any $\Lambda_0 \in \mathcal{L}_{\{0\}}$ with $\Lambda_1 \subset_1 \Lambda_0$, $\widetilde{M}(\Lambda_0)$ meets $C(\Lambda_1)$ transversely along a divisor with class* $(1, 0)$.
2. *For any $\Lambda_2 \in \mathcal{L}_{\{2\}}$ with $\Lambda_2 \subset_1 \Lambda_1$, $\widetilde{M}(\Lambda_2)$ meets $C(\Lambda_1)$ transversely along a divisor with class* $(0, 1)$.

Proof. By Theorem 6.2.4(6), we may fix one isomorphism $C(\Lambda_1) \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q} \times \mathbb{P}^1_{\overline{\mathbb{F}}_q}$. Let $\mathcal{L}_{\{0\}}(\Lambda_1)$ be the set of $\Lambda_0 \in \mathcal{L}_{\{0\}}$ with $\Lambda_1 \subset_1 \Lambda_0$, and likewise $\mathcal{L}_{\{2\}}(\Lambda_1)$. The actions of $\mathrm{Stab}(\Lambda_1) \subset \mathrm{Sp}(W)(\mathbb{Q}_q)$ on $\mathcal{L}_{\{0\}}(\Lambda_1)$ and $\mathcal{L}_{\{2\}}(\Lambda_1)$ are transitive, and compatible with the natural $\mathrm{Sp}(W)(\mathbb{Q}_q)$-action on $\mathcal{N}(0)$ (see (6.5.1)). For distinct $\Lambda_0, \Lambda_0' \in \mathcal{L}_{\{0\}}(\Lambda_1)$, it follows that the divisor classes

$$D_{\Lambda_0} := \widetilde{\mathcal{M}}(\Lambda_0) \cap C(\Lambda_1)$$

and

$$D_{\Lambda_0'} := \widetilde{\mathcal{M}}(\Lambda_0') \cap C(\Lambda_1)$$

differ by an automorphism of $C(\Lambda_1)$. In particular, if $D_{\Lambda_0} = (\alpha, \beta)$, then $D_{\Lambda_0'} = (\alpha, \beta)$ or (β, α). On the other hand, since $\mathcal{M}(\Lambda_0)$ meets $\mathcal{M}(\Lambda_0')$ transversely at $\mathcal{M}(\Lambda_1)$, we have

$$D_{\Lambda_0} \cdot D_{\Lambda_0} = 0$$

for the intersection product on $C(\Lambda_1)$. Since $(\alpha, \beta) \cdot (\beta, \alpha) = \alpha^2 + \beta^2$, which can only vanish if $\alpha = \beta = 0$, it follows that $D_{\Lambda_0} = D_{\Lambda_0'} = (\alpha, \beta)$ with $\alpha\beta = 0$; without loss of generality, assume $\beta = 0$. By the same reasoning, for any $\Lambda_2 \in \mathcal{L}_{\{2\}}(\Lambda_1)$,

$$D_{\Lambda_2} := \widetilde{\mathcal{M}}(\Lambda_2) \cap C(\Lambda_1)$$

has divisor class (γ, δ) with $\gamma\delta = 0$. However, $\mathcal{M}(\Lambda_0)$ meets $\mathcal{M}(\Lambda_2)$ transversely along $\mathcal{M}(\Lambda_0, \Lambda_2)$, so we have

$$D_{\Lambda_0} \cdot D_{\Lambda_2} = 1.$$

This implies $\alpha\delta = 1$, so we have $D_{\Lambda_0} = (1, 0)$ and $D_{\Lambda_2} = (0, 1)$, as desired. □

Notation 6.4.3. Let $\Lambda \in \mathcal{L}_{\{0\}} \sqcup \mathcal{L}_{\{2\}}$. We denote by $\mathcal{O}(1)$ the line bundle on $\widetilde{\mathcal{M}}(\Lambda)$ arising from the pullback of $\mathcal{O}(1)$ along the composite

$$\widetilde{\mathcal{M}}(\Lambda) \to \mathcal{M}(\Lambda) \hookrightarrow \mathbb{P}^3_{\overline{\mathbb{F}}_q}.$$

Lemma 6.4.4. *For any $\Lambda \in \mathcal{L}_{\{0\}} \sqcup \mathcal{L}_{\{2\}}$, the normal bundle to $\widetilde{\mathcal{M}}(\Lambda)$ inside $\widetilde{\mathcal{N}}(0)$ is $\mathcal{O}(-2q)$.*

Proof. Let $\left\{\Lambda_1^{(0)}, \ldots, \Lambda_1^{(n)}\right\}$ be the set of lattices Λ_1 in $\mathcal{L}_{\{1\}}$ such that $\mathcal{M}(\Lambda_1)$ lies on $\mathcal{M}(\Lambda)$. Then the projection $\widetilde{\mathcal{M}}(\Lambda) \to \mathcal{M}(\Lambda)$ is the blowup along the points $\mathcal{M}(\Lambda_1^{(i)})$, with exceptional divisors

$$E_i := C(\Lambda_1^{(i)}) \cap \widetilde{\mathcal{M}}(\Lambda).$$

Since $\mathcal{M}(\Lambda)$ is a smooth surface, we have $E_i \cdot E_j = -\delta_{ij}$ for the intersection pairing on $\widetilde{\mathcal{M}}(\Lambda)$.

Now, the normal bundle to $\widetilde{\mathcal{M}}(\Lambda)$ inside $\widetilde{\mathcal{N}}(0)$ is locally free of rank one, and Lemma 6.3.6 implies it is isomorphic to

$$\mathcal{O}(-2q + \alpha_0 E_0 + \ldots + \alpha_n E_n)$$

for some $\alpha_i \in \mathbb{Z}$. On the other hand, as $\widetilde{\mathcal{N}}(0)$ is formally smooth, we can compute the triple intersection number

$$m_i = \widetilde{\mathcal{M}}(\Lambda) \cdot \widetilde{\mathcal{M}}(\Lambda) \cdot C(\Lambda_1^{(i)})$$

in two ways, for each $0 \leq i \leq n$:

$$\begin{aligned} m_i &= \left(\widetilde{\mathcal{M}}(\Lambda) \cdot C(\Lambda_1^{(i)})\right) \cdot_{C(\Lambda_1^{(i)})} \left(\widetilde{\mathcal{M}}(\Lambda) \cdot C(\Lambda_1^{(i)})\right) = 0 \ \ \text{(Lemma 6.4.2)} \\ &= \left(\widetilde{\mathcal{M}}(\Lambda) \cdot \widetilde{\mathcal{M}}(\Lambda)\right) \cdot_{\widetilde{\mathcal{M}}(\Lambda)} \left(\widetilde{\mathcal{M}}(\Lambda) \cdot C(\Lambda_1^{(i)})\right) = -\alpha_i. \end{aligned}$$

So we find $\alpha_i = 0$ for all i, as desired. □

6.5. The GSpin action on $\mathcal{N}$.

6.5.1. The endomorphism algebra $\mathrm{End}(W)$ is equipped with an involution † given by the adjoint with respect to $\langle \cdot, \cdot \rangle_\bullet$, and

$$V := \mathrm{End}(W)^{\dagger=1,\mathrm{tr}=0} = \mathrm{End}(\mathbb{X}, \iota_{\mathbb{X}})^{*=1,\mathrm{tr}=0}$$

is a split orthogonal space of dimension 5, where $*$ denotes the Rosati involution. There is a natural projection

$$\pi : \mathrm{GSp}(W) \to \mathrm{SO}(V) \tag{6.11}$$

inducing an isomorphism $\mathrm{GSpin}(V)(\mathbb{Q}_q) \cong \mathrm{GSp}(W)(\mathbb{Q}_q)$. There is also a canonical action of $\mathrm{GSpin}(V)(\mathbb{Q}_q)$ on $\mathcal{N}$ (by modifying ρ); when restricted to $\mathrm{Spin}(V)(\mathbb{Q}_q)$, the resulting action of $\mathrm{Sp}(W)(\mathbb{Q}_q)$ on $\mathcal{M}$ is compatible with the natural actions on $\mathcal{L}_{\{0\}}$, $\mathcal{L}_{\{2\}}$, $\mathcal{L}_{\{02\}}$, and $\mathcal{L}_{\{1\}}$.

Definition 6.5.2. Define the sets of lattices

$$\begin{aligned} \mathscr{L} &= \left\{\Lambda \subset W \ : \ \Lambda = q^n \Lambda^\vee \text{ for some } n \in \mathbb{Z}\right\} \\ \mathscr{L}_{\mathrm{Pa}} &= \left\{\Lambda_{\mathrm{Pa}} \subset W \ : \ q^{n+1}\Lambda_{\mathrm{Pa}}^\vee \subset_2 \Lambda_{\mathrm{Pa}} \subset_2 p^n \Lambda_{\mathrm{Pa}}^\vee \text{ for some } n \in \mathbb{Z}\right\}. \end{aligned}$$

Remark 6.5.3. Both $\mathscr{L}$ and $\mathscr{L}_{\mathrm{Pa}}$ are homogeneous spaces for $\mathrm{GSp}(W)(\mathbb{Q}_q)$; the stabilizer of a point in $\mathscr{L}$ is a hyperspecial subgroup, and the stabilizer of a point in $\mathscr{L}_{\mathrm{Pa}}$ is a paramodular subgroup.

Using $\mathscr{L}$ and $\mathscr{L}_{\mathrm{Pa}}$ rather than $\mathcal{L}_{\{0\}}$, $\mathcal{L}_{\{2\}}$, and $\mathcal{L}_{\{1\}}$, we can extend the combinatorial description of $\mathcal{M}$ to all of $\mathcal{N}_{\mathrm{red}}$.

Definition 6.5.4. For any $\Lambda \in \mathscr{L}$, choose an arbitrary $g \in \mathrm{GSp}(W)(\mathbb{Q}_q)$ such that $g\Lambda \in \mathcal{L}_{\{0\}} \subset \mathscr{L}$, and define

$$\mathcal{M}_+(\Lambda) := g^{-1}\mathcal{M}_{\{0\}}(g\Lambda).$$

Similarly, choose $g' \in \mathrm{GSp}(W)(\mathbb{Q}_q)$ such that $g'\Lambda \in \mathcal{L}_{\{2\}} \subset \mathscr{L}$, and define

$$\mathcal{M}_-(\Lambda) := (g')^{-1}\mathcal{M}_{\{2\}}(g'\Lambda).$$

For any $\Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}$, choose $g \in \mathrm{GSp}(W)(\mathbb{Q}_q)$ such that $g\Lambda_{\mathrm{Pa}} \in \mathcal{L}_{\{1\}} \subset \mathscr{L}_{\mathrm{Pa}}$, and define $\mathcal{M}(\Lambda_{\mathrm{Pa}}) = g^{-1}\mathcal{M}_{\{1\}}(g\Lambda_{\mathrm{Pa}})$.

Proposition 6.5.5. *(1) Definition 6.5.4 yields bijections*

$$\mathscr{L} \times \{\pm\} \xrightarrow{\sim} \{\textit{irreducible components of } \mathcal{N}_{\mathrm{red}}\}$$

and

$$\mathscr{L}_{\mathrm{Pa}} \xrightarrow{\sim} \{\textit{singular points of } \mathcal{N}_{\mathrm{red}}\}\,.$$

(2) Choose any $\Lambda \in \mathscr{L}$. For the Weil descent datum in (6.1.4), we have

$$\varphi\left(\mathcal{M}_+(\Lambda)\right) = \sigma^*\mathcal{M}_-(q\Lambda)$$

and

$$\varphi\left(\mathcal{M}_-(\Lambda)\right) = \sigma^*\mathcal{M}_+(\Lambda).$$

Proof. For (1), it suffices to show there are two $\mathrm{GSpin}(V)(\mathbb{Q}_q)$-orbits of irreducible components of $\mathcal{N}_{\mathrm{red}}$, and only one orbit of singular points. However, since

$$g \cdot \mathcal{N}(i) = \mathcal{N}(i + \mathrm{ord}_q\nu(g))$$

for $g \in \mathrm{GSpin}(V)(\mathbb{Q}_q)$, it suffices to show that there are two $\mathrm{Spin}(V)(\mathbb{Q}_q)$-orbits of irreducible components of $\mathcal{M}$, and one $\mathrm{Spin}(V)(\mathbb{Q}_q)$-orbit of singular points on $\mathcal{M}$. This follows from the transitivity of the $\mathrm{Sp}(W)(\mathbb{Q}_q)$-actions on $\mathcal{L}_{\{0\}}$, $\mathcal{L}_{\{2\}}$, and $\mathcal{L}_{\{1\}}$.

For (2), note that $\varphi^2(\mathcal{M}_+(\Lambda)) = (\sigma^2)^*\mathcal{M}_+(q\Lambda)$, so it suffices to show the first relation. Without loss of generality, assume $\Lambda \in \mathcal{L}_{\{0\}}$. By definition, $\varphi(\mathcal{M}_+(\Lambda))(\overline{\mathbb{F}}_q)$, viewed as a subset of $\mathcal{N}(\overline{\mathbb{F}}_q)$, is the Zariski closure of the set of points corresponding to lattices $M \subset N$ such that

$$(V^{-1}(M + \tau M))_\bullet = \breve{\Lambda}$$

and

$$\Pi^{-1}M_\circ = qM_\bullet^\vee.$$

Since $V\breve{\Lambda} = \Pi\breve{\Lambda}$, the first condition is equivalent to $\Pi^{-1}M_\circ + \Pi^{-1}\tau M_\circ = \breve{\Lambda}$, or dually

$$M_\bullet \cap \tau M_\bullet = q\breve{\Lambda}.$$

Now choose any $g \in \mathrm{GSpin}(V)(\mathbb{Q}_q)$ with $\nu(g) = q^{-1}$; we have $qg\Lambda \in \mathcal{L}_{\{2\}}$. The locus

$$g\varphi(\mathcal{M}_+(\Lambda))(\overline{\mathbb{F}}_q) \subset \mathcal{M}(\overline{\mathbb{F}}_q)$$

is the Zariski closure of the set of points corresponding to lattices with $M_\bullet \cap \tau M_\bullet = qg\breve{\Lambda}$ and $\Pi^{-1}M_\circ = M_\bullet^\vee$. But this is exactly the stratum $\mathcal{M}_{\{2\}}(qg\Lambda)(\overline{\mathbb{F}}_q) = \mathcal{M}_-(qg\Lambda)(\overline{\mathbb{F}}_q)$, so we conclude $\varphi(\mathcal{M}_+(\Lambda)) = \sigma^*g^{-1}\mathcal{M}_-(qg\Lambda) = \sigma^*\mathcal{M}_-(q\Lambda)$, as desired. ☐

From Theorem 6.2.4, we immediately deduce the following relations among the components $\mathcal{M}_\pm(\Lambda)$ and the points $\mathcal{M}(\Lambda_{\mathrm{Pa}})$.

Corollary 6.5.6. *For any $\Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}$ and $\Lambda, \Lambda' \in \mathscr{L}$, we have:*

1. *$\mathcal{M}(\Lambda_{\mathrm{Pa}})$ lies on $\mathcal{M}_+(\Lambda)$ if and only if $\Lambda_{\mathrm{Pa}} \subset_1 \Lambda$.*
2. *$\mathcal{M}(\Lambda_{\mathrm{Pa}})$ lies on $\mathcal{M}_-(\Lambda)$ if and only if $\Lambda \subset_1 \Lambda_{\mathrm{Pa}}$.*
3. *$\mathcal{M}_+(\Lambda)$ meets $\mathcal{M}_-(\Lambda')$ if and only if $q\Lambda \subset_2 \Lambda' \subset_2 \Lambda$.*

(4) If $\Lambda \neq \Lambda'$, then for $\delta = +$ or $-$, $\mathcal{M}_\delta(\Lambda)$ and $\mathcal{M}_\delta(\Lambda')$ can meet only in singular points of $\mathcal{N}_{\mathrm{red}}$.

7. THE FIRST EXPLICIT RECIPROCITY LAW: GEOMETRIC INPUTS

7.1. **Abel-Jacobi maps for schemes with ordinary quadratic singularities.**

7.1.1. Let R_0 be a Henselian discrete valuation ring with uniformizer π, algebraically closed residue field k, and fraction field K_0. The inertia subgroup I_{K_0} of $\mathrm{Gal}(\overline{K}_0/K_0)$ has the canonical tame character $t_p : I_{K_0} \to \mathbb{Z}_p(1)$ for any $p \neq \mathrm{char}(k)$. Let R be the quadratic extension $R_0[\pi^{1/2}]$, and K its field of fractions, with inertia subgroup $I_K \subset I_{K_0}$. We write s_0, s, η_0, η for the closed points and the generic points of $\mathrm{Spec}\, R_0$ and $\mathrm{Spec}\, R$, with corresponding geometric points $\overline{\eta}_0 = \overline{\eta}$. Let X be an irreducible scheme of finite type and pure relative dimension $2r-1$ over $\mathrm{Spec}\, R_0$, for some integer $r \geq 1$. We assume X has ordinary quadratic singularities: this means that X is smooth outside a finite set of closed points $\{x_i\}_{i\in I}$ in X_{s_0}, and, étale locally near each x_i, X is isomorphic to $\mathrm{Spec}\, R_0[y_0, \ldots, y_{2r-1}]/(Q - \pi)$, with Q the equation of a smooth quadric in $\mathbb{P}^{2r-1}_{R_0}$.

7.1.2. The blowup Y of X_R at the points $\{x_i\}_{i\in I}$ is strictly semistable in the sense of [97], with a particularly simple form [47]. The irreducible components of the special fiber Y_s of Y are $\widetilde{X}_s$, the strict transform of X_s, and the exceptional divisors D_i. Each D_i is isomorphic to the smooth projective quadric in $\mathbb{P}^{2r}_k = \mathrm{Proj}(k[y_0, \ldots, y_{2r-1}, t])$ cut out by $Q - t^2$, and the intersection $C_i = D_i \cap \widetilde{X}_s$ is the hyperplane section $t = 0$, so that C_i is a smooth quadric in $\mathbb{P}^{2r-1}_k$. Since Y is semistable, we have

$$\mathcal{N}_{C_i/\widetilde{X}_s} = -\mathcal{N}_{C_i/D_i} = \mathcal{O}(-1) \tag{7.1}$$

in the Picard group of C_i.

7.1.3. Let O be a finite flat extension of $\mathbb{Z}_p$ with p odd and $p \neq \mathrm{char}(k)$, and let $\varpi \in O$ be a uniformizer. We fix a coefficient ring $\Lambda = O$ or O/ϖ^m for some $m \geq 1$. We recall the definition of the nearby cycles complex: let $\overline{j} : Y_{\overline{\eta}} \to Y$ and $\overline{i} : Y_s \to Y$ denote the inclusions of the geometric generic and special fibers, respectively. Then $R\Psi\Lambda = \overline{i}^* \circ \overline{j}_* \circ \overline{j}^* \Lambda$, an element of the bounded derived category of sheaves on Y_s; it has a canonical action of $I_K \subset I_{K_0}$ factoring through the tame character t_p, and the (increasing) monodromy filtration $M_\bullet R\Psi\Lambda$. Fix $T \in I_K$ such that $t_p(T)$ generates $\mathbb{Z}_p(1)$; then the monodromy operator $T-1$ on $R\Psi\Lambda$ induces compatible maps

$$T - 1 : M_i R\Psi\Lambda \to M_{i-2} R\Psi\Lambda$$

for all $i \in \mathbb{Z}$.

Proposition 7.1.4 (Saito). *Let*

$$i_0 : \widetilde{X}_s \sqcup \bigsqcup_{i\in I} D_i \to Y_s$$

and

$$i_1 : \bigsqcup_{i\in I} C_i \to Y_s$$

be the natural maps. Then the graded pieces of $M_\bullet R\Psi\Lambda$ are given by:

$$\begin{aligned}
\mathrm{gr}^M_i R\Psi\Lambda &= 0, & |i| > 1,\\
\mathrm{gr}^M_1 R\Psi\Lambda &= i_{1*}\Lambda(-1)[-1], &\\
\mathrm{gr}^M_0 R\Psi\Lambda &= i_{0*}\Lambda, &\\
\mathrm{gr}^M_{-1} R\Psi\Lambda &= i_{1*}\Lambda[-1]. &
\end{aligned}$$

Proof. This is [97, Proposition 2.2.3]. ☐

Lemma 7.1.5. *The following composite map is an isomorphism:*

$$H^{2r}(Y_s, M_0 R\Psi\Lambda) \to H^{2r}(Y_s, \operatorname{gr}_0^M R\Psi\Lambda) = H^{2r}(\widetilde{X}_s, \Lambda) \oplus \bigoplus_{i \in I} H^{2r}(D_i, \Lambda) \twoheadrightarrow H^{2r}(\widetilde{X}_s, \Lambda).$$

Proof. First, consider the tautological distinguished triangle

$$M_{-1}R\Psi\Lambda \to M_0 R\Psi\Lambda \to \operatorname{gr}_0^M R\Psi\Lambda \to M_{-1}R\Psi\Lambda[+1]. \tag{7.2}$$

This yields an exact sequence

$$H^{2r}(Y_k, M_{-1}R\Psi\Lambda) \to H^{2r}(Y_k, M_0 R\Psi\Lambda) \to H^{2r}(Y_k, \operatorname{gr}_0^M R\Psi\Lambda) \to H^{2r+1}(Y_k, M_{-1}R\Psi\Lambda).$$

Applying Proposition 7.1.4 and using $H^{2r-1}(C_i, \Lambda) = 0$ for all $i \in I$, we obtain

$$0 \to H^{2r}(Y_k, M_0 R\Psi\Lambda) \to H^{2r}(\widetilde{X}_k, \Lambda) \oplus \bigoplus_{i \in I} H^{2r}(D_i, \Lambda) \to H^{2r}(C_i, \Lambda).$$

However, $H^{2r}(D_i, \Lambda) \to H^{2r}(C_i, \Lambda)$ is an isomorphism for each $i \in I$ by the Lefschetz hyperplane theorem, and the lemma follows. ☐

7.1.6. Let $R\Psi_X\Lambda$ be the nearby cycles complex for X, defined as in (7.1.3); it also coincides with the nearby cycles complex for X_R. From now on, we will assume:

(BC$_X$) the base change map $H^i(X_{\overline{\eta}_0}, \Lambda) \to H^i(X_{s_0}, R\Psi_X\Lambda)$ is an isomorphism for all i.

Since the blowup map $f : Y \to X_R$ is proper and is an isomorphism on generic fibers, we have a canonical isomorphism

$$f_* R\Psi\Lambda = R\Psi_X\Lambda$$

by [28, §2.1.7]. In particular, (BC$_X$) implies:

(BC$_Y$) the base change map $H^i(X_{\overline{\eta}}, \Lambda) = H^i(Y_{\overline{\eta}}, \Lambda) \to H^i(Y_s, R\Psi\Lambda)$ is an isomorphism for all i.

Lemma 7.1.7. *Let $j : \bigsqcup_{i \in I} C_i \hookrightarrow \widetilde{X}_s$ be the natural embedding. Then the monodromy operator $T - 1$ on $H^{2r-1}(Y_s, R\Psi\Lambda)$ fits into a commutative diagram with exact rows:*

$$\begin{tikzcd}
H^{2r-1}(X_{\overline{\eta}}, \Lambda) \arrow[r, "\alpha"] \arrow[d, "T-1"] & \oplus_{i \in I} H^{2r-2}(C_i, \Lambda(-1)) \arrow[r, "j_*"] \arrow[d, "t"] & H^{2r}(\widetilde{X}_s, \Lambda) \arrow[r, "\gamma"] & H^{2r}(X_{\overline{\eta}}, \Lambda) \\
H^{2r-1}(X_{\overline{\eta}}, \Lambda) & \oplus_{i \in I} H^{2r-2}(C_i, \Lambda) \arrow[l, "\beta"'] & H^{2r-2}(\widetilde{X}_s, \Lambda) \arrow[l, "j^*"'] & H^{2r-2}(X_{\overline{\eta}}, \Lambda). \arrow[l]
\end{tikzcd}$$

Here, t is the isomorphism $- \otimes t_p(T)$.

Proof. From the vanishing of $\operatorname{gr}_i^M R\Psi\Lambda$ for $|i| > 1$ (Proposition 7.1.4), we have a canonical factorization

$$T - 1 : R\Psi\Lambda \to \operatorname{gr}_1^M R\Psi\Lambda \to M_{-1}R\Psi\Lambda \to R\Psi\Lambda; \tag{7.3}$$

taking cohomology and applying Proposition 7.1.4, (7.3) induces a commutative diagram:

$$\begin{tikzcd}
H^{2r-1}(Y_s, R\Psi\Lambda) \arrow[r, "\alpha"] \arrow[d, "T-1"] & \oplus_{i \in I} H^{2r-2}(C_i, \Lambda(-1)) \arrow[d, "t"] \\
H^{2r-1}(Y_s, R\Psi\Lambda) & \oplus_{i \in I} H^{2r-2}(C_i, \Lambda). \arrow[l, "\beta"']
\end{tikzcd}$$

The description of t is [97, Corollary 2.2.4.2].

We now explain the exactness of the top row

$$H^{2r-1}(X_{\overline{\eta}}, \Lambda) \xrightarrow{\alpha} \oplus_{i \in I} H^{2r-2}(C_i, \Lambda(-1)) \xrightarrow{j_*} H^{2r}(\widetilde{X}_s, \Lambda) \to H^{2r}(X_{\overline{\eta}}, \Lambda) \tag{7.4}$$

of the diagram in the lemma. From the tautological distinguished triangle

$$M_0 R\Psi\Lambda \to R\Psi\Lambda \to \mathrm{gr}_1^M R\Psi\Lambda \to M_0 R\Psi\Lambda[+1], \tag{7.5}$$

we deduce the exact sequence

$$H^{2r-1}(Y_s, R\Psi\Lambda) \xrightarrow{\alpha} \oplus_{i\in I} H^{2r-2}(C_i, \Lambda(-1)) \xrightarrow{\delta} H^{2r}(Y_s, M_0 R\Psi\Lambda) \to H^{2r}(Y_s, R\Psi\Lambda). \tag{7.6}$$

Combined with Lemma 7.1.5 and (BC$_Y$), it suffices to show that the composite

$$\oplus_{i\in I} H^{2r-2}(C_i, \Lambda(-1)) \xrightarrow{\delta} H^{2r}(Y_s, M_0 R\Psi\Lambda) \xrightarrow{\sim} H^{2r}(\widetilde{X}_s, \Lambda) \tag{7.7}$$

is the pushforward map, and this follows from [97, Proposition 2.2.6]. The argument for the exactness of the bottom row

$$H^{2r-2}(X_{\overline{\eta}}, \Lambda) \to H^{2r-2}(\widetilde{X}_s, \Lambda) \xrightarrow{j^*} \bigoplus_{i\in I} H^{2r-2}(C_i, \Lambda) \xrightarrow{\beta} H^{2r-1}(X_{\overline{\eta}}, \Lambda)$$

is essentially identical. ☐

7.1.8. By (BC_Y), the monodromy filtration $M_\bullet$ of $R\Psi\Lambda$ induces filtrations $M_\bullet$ on

$$H^{2r-1}(X_{\overline{\eta}}, \Lambda)$$

and on

$$H^1(I_{K_0}, H^{2r-1}(X_{\overline{\eta}}, \Lambda)) = H^1(I_K, H^{2r-1}(X_{\overline{\eta}}, \Lambda))$$

(using $p \neq 2$).

Proposition 7.1.9. *The diagram in Lemma 7.1.7 induces an exact sequence*

$$0 \to M_{-1} H^1(I_{K_0}, H^{2r-1}(X_{\overline{\eta}}, \Lambda)) \xrightarrow{\zeta} \frac{H^{2r}(\widetilde{X}_s, \Lambda)}{\mathrm{Im}(j_* \circ t^{-1} \circ j^*)} \xrightarrow{\gamma} \frac{H^{2r}(X_{\overline{\eta}}, \Lambda)}{\mathrm{Im}(\gamma \circ j_* \circ t^{-1} \circ j^*)},$$

such that $\zeta(c) = j_* \circ t^{-1}(y)$ for any $y \in \oplus_{i\in I} H^{2r-2}(C_i, \Lambda)$ such that $c(T) = \beta(y)$.

Proof. By definition, $M_{-1} H^{2r-1}(X_{\overline{\eta}}, \Lambda)$ is the image of

$$\beta : \bigoplus_{i\in I} H^{2r-2}(C_i, \Lambda) \to H^{2r-1}(Y_s, R\Psi\Lambda) \cong H^{2r-1}(X_{\overline{\eta}}, \Lambda).$$

Then, by the left half of the commutative diagram in Lemma 7.1.7, the map $c \mapsto c(T)$ identifies

$$M_{-1} H^1(I_{K_0}, H^{2r-1}(X_{\overline{\eta}}, \Lambda)) \simeq \frac{\mathrm{Im}\,\beta}{\mathrm{Im}(\beta \circ t \circ \alpha)}.$$

Using the exactness of the rows in Lemma 7.1.7, we also have the exact sequence

$$0 \to \frac{\mathrm{Im}\,\beta}{\mathrm{Im}(\beta \circ t \circ \alpha)} \xrightarrow{j_* \circ t^{-1}} \frac{H^{2r}(\widetilde{X}_s, \Lambda)}{\mathrm{Im}(j_* \circ t^{-1} \circ j^*)} \xrightarrow{\gamma} \frac{H^{2r}(X_{\overline{\eta}}, \Lambda)}{\mathrm{Im}(\gamma \circ j_* \circ t^{-1} \circ j^*)},$$

and the proposition follows. ☐

7.1.10. Let $\mathrm{CH}^r(X_{\eta_0})^0$ denote the Chow group of cohomologically trivial algebraic cycles of pure codimension r. We have the Abel-Jacobi map

$$\partial_{\mathrm{AJ}} : \mathrm{CH}^r(X_{\eta_0})^0 \to H^1(I_{K_0}, H^{2r-1}(X_{\overline{\eta}}, \Lambda(r))),$$

which we are now ready to compute in terms of the geometry of the special fiber of X.

If Z_{η_0} is a closed irreducible subvariety of X_{η_0}, then write Z for its Zariski closure in X, Z_Y for the strict transform of Z_R under the blowup $Y \to X_R$, and Z_{Y_s} for $Z_Y \times_Y Y_s$. The intersection $Z_{Y_s} \times_Y \widetilde{X}_s$ is $\widetilde{Z}_s$, the strict transform of Z_{s_0} under the blowup $\widetilde{X}_s \to X_{s_0}$. Extending this construction linearly, for any algebraic cycle $z_\eta = \Sigma n_j Z_\eta^{(j)}$ of pure codimension r, we obtain a codimension-r algebraic cycle

$$\widetilde{z}_s = \Sigma n_j \widetilde{Z}_s^{(j)}$$

on $\widetilde{X}_s$.

Theorem 7.1.11. *Let z_{η_0} be an algebraic cycle of codimension r on X_{η_0}, whose class in $\mathrm{CH}^r(X_{\eta_0})$ is cohomologically trivial, and assume (BC$_X$). Then $\partial_{\mathrm{AJ}}(z)$ lies in $M_{-1}H^1(I_{K_0}, H^{2r-1}(X_{\overline{\eta}}, \Lambda(r)))$. If, for each irreducible component Z_η of the support of z_η, Z_{Y_s} is generically smooth, then*

$$\zeta\left(\partial_{\mathrm{AJ}}(z)\right) \in \frac{H^{2r}(\widetilde{X}_s, \Lambda(r))}{j_* \circ t^{-1} \circ j^* \left(H^{2r-2}(\widetilde{X}_s, \Lambda(r-1))\right)}$$

coincides with the algebraic cycle class of $\widetilde{z}_s$, where ζ is the map from Proposition 7.1.9.

Proof. In the proof of [67, Theorem 2.18] there is constructed:

- An element F of the bounded derived category of abelian sheaves on Y_s fitting into a commutative diagram:

(7.8)
$$\begin{tikzcd} F \arrow[r, "T-1"] \arrow[d] & R\Psi\Lambda \arrow[d] \\ R\Psi\Lambda/M_0R\Psi\Lambda \arrow[r, "T-1"] & R\Psi\Lambda/M_{-2}R\Psi\Lambda. \end{tikzcd}$$

- A class $[z^\sharp]_0' \in H^{2r-1}(Y_s, F(r))$ such that $\partial_{\mathrm{AJ}}(z)|_{I_K}$ is represented by the cocycle that factors through $t_p : I_K \twoheadrightarrow \mathbb{Z}_p(1)$ and satisfies

$$\partial_{\mathrm{AJ}}(z)(T) = (T-1)[z^\sharp]_0' \in H^{2r-1}(Y_s, R\Psi\Lambda(r)) = H^{2r-1}(X_{\overline{\eta}}, \Lambda(r)).$$

In our context, since $\mathrm{gr}_i^M R\Psi\Lambda = 0$ for $|i| > 1$, the diagram (7.8) becomes

$$\begin{tikzcd} F \arrow[r, "T-1"] \arrow[d] & R\Psi\Lambda \\ R\Psi\Lambda/M_0R\Psi\Lambda \arrow[r, "T-1"] & M_{-1}R\Psi\Lambda. \arrow[u] \end{tikzcd}$$

In particular, $(T-1)H^{2r-1}(Y_s, F(r))$ lies inside $M_{-1}H^{2r-1}(X_{\overline{\eta}}, \Lambda(r))$, and by construction $\zeta\left(\partial_{\mathrm{AJ}}(z)\right)$ is represented by the image of $[z^\sharp]_0'$ under the composite map

(7.9)
$$\begin{aligned} H^{2r-1}(Y_s, F(r)) &\to H^{2r-1}(Y_s, \mathrm{gr}_1^M R\Psi\Lambda(r)) \to H^{2r}(Y_s, \mathrm{gr}_0^M R\Psi\Lambda(r)) \\ &= H^{2r}(\widetilde{X}_s, \Lambda(r)) \oplus \bigoplus_{i \in I} H^{2r}(D_i, \Lambda(r) \twoheadrightarrow H^{2r}(\widetilde{X}_s, \Lambda(r)). \end{aligned}$$

By [67, Proposition 2.19], under the generic smoothness assumption of the theorem, this image is exactly the cycle class of $\widetilde{z}_s$. ☐

7.1.12. We conclude this section with a related lemma.

Lemma 7.1.13. *In addition to (BC$_X$), assume*

(BC$_{X,c}$) *the base change map* $H^i_c(X_{\overline{\eta}_0}, \Lambda) \to H^i_c(X_{s_0}, R\Psi_X\Lambda)$ *is an isomorphism for all* i.

Suppose $\mathcal{H}$ *is a commutative* O*-algebra (not necessarily finitely generated) of correspondences on* X*, such that the singular locus of* X *is stable under* $\mathcal{H}$*. Let* $\mathfrak{m} \subset \mathcal{H}$ *be a maximal ideal such that the natural map induces an isomorphism*

$$H^i_{\text{ét},c}(X_{\overline{\eta}}, \Lambda)_{\mathfrak{m}} \to H^i_{\text{ét}}(X_{\overline{\eta}}, \Lambda)_{\mathfrak{m}}$$

for all i*. Then for all* i*, the natural map*

$$H^i_{\text{ét},c}(\widetilde{X}_s, \Lambda)_{\mathfrak{m}} \to H^i_{\text{ét}}(\widetilde{X}_s, \Lambda)_{\mathfrak{m}}$$

is an isomorphism as well.

Proof. The tautological distinguished triangle (7.5) gives a commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_c(Y_s, M_0R\Psi\Lambda)_{\mathfrak{m}} \arrow[r] \arrow[d] & H^i_c(Y_s, R\Psi\Lambda)_{\mathfrak{m}} \arrow[r] \arrow[d, "\wr"'] & H^i_c(Y_s, \mathrm{gr}^M_1 R\Psi\Lambda)_{\mathfrak{m}} \arrow[r] \arrow[d, "\wr"'] & \cdots \\
\cdots \arrow[r] & H^i(Y_s, M_0R\Psi\Lambda)_{\mathfrak{m}} \arrow[r] & H^i(Y_s, R\Psi\Lambda)_{\mathfrak{m}} \arrow[r] & H^i(Y_s, \mathrm{gr}^M_1 R\Psi\Lambda)_{\mathfrak{m}} \arrow[r] & \cdots
\end{tikzcd}$$

The first marked isomorphism is by (BC$_X$) and (BC$_{X,c}$), and the second is by Proposition 7.1.4 and the compactness of each C_i. By the five lemma, we have an isomorphism

$$H^i_c(Y_s, M_0R\Psi\Lambda)_{\mathfrak{m}} \xrightarrow{\sim} H^i(Y_s, M_0R\Psi\Lambda)_{\mathfrak{m}}$$

for all i. Arguing similarly with the distinguished triangle (7.2), we find a natural isomorphism

$$H^i_c(Y_s, \mathrm{gr}^M_0 R\Psi\Lambda)_{\mathfrak{m}} \xrightarrow{\sim} H^i(Y_s, \mathrm{gr}^M_0 R\Psi\Lambda)_{\mathfrak{m}}$$

for all i, which implies the lemma by Proposition 7.1.4 once again. ☐

7.2. **Semistable reduction of** GSpin_5 **Shimura varieties.**

7.2.1. Let $D \neq 1$ be a squarefree product of an even number of primes, and fix an odd prime $q|D$. With V_D as in (1.1.6), we suppose fixed a q-adic uniformization datum $(*, A_0, \iota_0, \lambda_0, i_D, i_{D/q})$ for V_D (Definition 1.4.4(2)); we will choose this uniformization datum more precisely in Construction 7.6.4 below.

Let $\mathcal{D}$ and $\mathscr{D}$ be the associated PEL data and self-dual q-integral refinement from Definition 1.4.2.

7.2.2. For the entirety of this section, we fix a neat level subgroup

$$K^q = \prod_{\ell \neq q} K_\ell \subset \mathrm{GSpin}(V_D)(\mathbb{A}^q_f) \cong \prod_{\ell \neq q} \mathrm{GSpin}(V_{D/q})(\mathbb{A}^q_f),$$

with the isomorphism arising from Remark 1.4.5(2); then we obtain a flat, quasi-projective scheme $X = \mathcal{M}_{K^q}$ over $\mathbb{Z}_{(q)}$ representing the PEL-type moduli problem defined by $\mathscr{D}$ at level K^q. The generic fiber $X_{\mathbb{Q}}$ of X is isomorphic to $\mathrm{Sh}_{K^qK^{\mathrm{ram}}_q}(V_D)$, where $K^{\mathrm{ram}}_q \subset \mathrm{GSpin}(V_D)(\mathbb{Q}_q)$ is a paramodular subgroup in the sense of Notation 2.6.1.

Lemma 7.2.3. *Let* $R\Psi_X O$ *denote the nearby cycles complex on* $X_{\overline{\mathbb{F}}_q}$*. Then the natural maps*

$$H^i_{c,\text{ét}}(X_{\overline{\mathbb{Q}}}, O) \to H^i_{c,\text{ét}}(X_{\overline{\mathbb{F}}_q}, R\Psi_X O)$$

and

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, O) \to H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, R\Psi_X O)$$

are isomorphisms for all i.

Proof. This is a special case of [57, Theorem 6.8]. ☐

Let $O_D \subset B_D$ be the unique maximal $\mathbb{Z}_{(q)}$-order. We now take the q-divisible group $\mathbb{X} := \overline{A}_0[q^\infty]$, with its induced polarization and $(O_D \otimes \mathbb{Z}_q)$-action, to be the base point for the Rapoport-Zink space $\mathcal{N}$ (Definition 6.1.2).

Theorem 7.2.4.

(1) *Let $\mathcal{X}$ be the completion of X along the supersingular locus $X^{ss}_{\overline{\mathbb{F}}_q}$. Then we have a canonical isomorphism*

$$\mathcal{X} \cong \mathrm{GSpin}(V_{D/q})(\mathbb{Q})\backslash \mathrm{GSpin}(V_{D/q})(\mathbb{A}_f^q) \times \mathcal{N}/K^q,$$

where $\mathrm{GSpin}(V_{D/q})(\mathbb{Q}) \hookrightarrow \mathrm{GSpin}(V_{D/q})(\mathbb{Q}_q)$ *acts on* $\mathcal{N}$ *as described in §6.5.*

(2) *The singular locus of $X_{\overline{\mathbb{F}}_q}$ is the discrete set of points*

$$X^{\mathrm{sing}}_{\overline{\mathbb{F}}_q} = \mathrm{GSpin}(V_{D/q})(\mathbb{Q})\backslash \mathrm{GSpin}(V_{D/q})(\mathbb{A}_f^q) \times \mathcal{N}^{\mathrm{sing}}_{\mathrm{red}}/K^q.$$

The complete local ring of X at any point of $X^{\mathrm{sing}}_{\overline{\mathbb{F}}_q}$ is isomorphic to

$$\breve{\mathbb{Z}}_q[\![x, y, z, w]\!]/(xy - zw - q).$$

Proof. Part (1) is the Rapoport-Zink uniformization theorem for X; part (2) follows from (1) and Theorem 6.2.4(6), after noting that all singularities of $X_{\overline{\mathbb{F}}_q}$ lie in the supersingular locus by [85, Theorem 7.5]. ☐

7.2.5. In particular, Theorem 7.2.4(2) asserts that X has ordinary quadratic singularities, so that the results of §7.1 apply. Following the notation therein, let $\widetilde{X}_{\overline{\mathbb{F}}_q}$ be the blowup of $X_{\overline{\mathbb{F}}_q}$ along the singular locus; note that $\widetilde{X}_{\overline{\mathbb{F}}_q}$ inherits an action of the full prime-to-q Hecke algebra.

7.3. **Tate classes.** The goal of this subsection is to show that the full cohomology group $H^2_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O)$ is generated by Tate classes from the supersingular locus, after a Hecke localization.

Notation 7.3.1. Recall the sets $\mathscr{L}$ and $\mathscr{L}_{\mathrm{Pa}}$ from Definition 6.5.2.

(1) For

$$g = (g^q, \Lambda) \in \mathrm{GSpin}(V_{D/q})(\mathbb{Q})\backslash \mathrm{GSpin}(V_{D/q})(\mathbb{A}_f^q) \times \mathscr{L}/K^q,$$

let $B_\pm(g)$ be the image of $(g^q, \mathcal{M}_\pm(\Lambda))$ under the uniformization in Theorem 7.2.4, and let $\widetilde{B}_\pm(g)$ be its strict transform under the blowup $\widetilde{X}_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$.

(2) For

$$g = (g^q, \Lambda_{\mathrm{Pa}}) \in \mathrm{GSpin}(V_{D/q})(\mathbb{Q})\backslash \mathrm{GSpin}(V_{D/q})(\mathbb{A}_f^q) \times \mathscr{L}_{\mathrm{Pa}}/K^q,$$

let $y(g) \in X^{\mathrm{sing}}_{\overline{\mathbb{F}}_q}$ be the image of $(g^q, \mathcal{M}(\Lambda_{\mathrm{Pa}}))$, and let $C(g)$ be the exceptional divisor of $\widetilde{X}_{\overline{\mathbb{F}}_q}$ over the point $y(g)$.

(3) Recall that $\mathscr{L}$ and $\mathscr{L}_{\mathrm{Pa}}$ are homogeneous spaces for $\mathrm{GSpin}(V_{D/q})(\mathbb{Q}_q) \cong \mathrm{GSp}_4(\mathbb{Q}_q)$, with the stabilizer of any point a hyperspecial or paramodular subgroup, respectively. We will therefore abbreviate the two sets in (1) and (2) by $\mathrm{Sh}_{K^qK_q}(V_{D/q})$ and $\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})$, respectively, even though the identifications actually depend on a non-canonical choice of base point which we do not need to make.

Remark 7.3.2. Notation 7.3.1 identifies $\mathrm{Sh}_{K^qK_q}(V_{D/q}) \times \{\pm\}$ and $\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})$ with the set of irreducible components of $X^{ss}_{\overline{\mathbb{F}}_q}$ and the set of points of $X^{\mathrm{sing}}_{\overline{\mathbb{F}}_q}$, respectively.

To study the intersections of the divisors $\widetilde{B}_\pm(g)$ and $C(g)$, we need to define some additional Hecke operators.

Definition 7.3.3. Let

$$\delta_\pm : O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})] \to O[\mathrm{Sh}_{K^q K_q}(V_{D/q})]$$

be the maps defined by

$$\delta_+ : (g^q, \Lambda_{\mathrm{Pa}}) \mapsto \sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda \\ \Lambda \in \mathscr{L}}} (g^q, \Lambda),$$

$$\delta_- : (g^q, \Lambda_{\mathrm{Pa}}) \mapsto \sum_{\substack{\Lambda \subset_1 \Lambda_{\mathrm{Pa}} \\ \Lambda \in \mathscr{L}}} (g^q, \Lambda).$$

Similarly, let

$$\theta_\pm : O[\mathrm{Sh}_{K^q K_q}(V_{D/q})] \to O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})]$$

be the maps defined by

$$\theta_+ : (g^q, \Lambda) \mapsto \sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda \\ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}}} (g^q, \Lambda_{\mathrm{Pa}}), \quad \theta_- : (g^q, \Lambda) \mapsto \sum_{\substack{\Lambda \subset_1 \Lambda_{\mathrm{Pa}} \\ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}}} (g^q, \Lambda_{\mathrm{Pa}}).$$

These are incarnations of the level-lowering and level-raising operators in [95, §3].

Definition 7.3.4. We define the natural composite maps

$$\begin{aligned}(7.10)\quad \mathrm{inc}^* : H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2)) \longrightarrow \\ \bigoplus_{g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})} H^4_{\text{ét}}(C(g), O(2)) \oplus \bigoplus_{g \in \mathrm{Sh}_{K^q K_q}(V_{D/q})} \Big(H^4_{\text{ét}}(\widetilde{B}_+(g), O(2)) \oplus H^4_{\text{ét}}(\widetilde{B}_-(g), O(2)) \Big) \\ \xrightarrow{\sim} O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2}\end{aligned}$$

and

$$\begin{aligned}(7.11)\quad \mathrm{inc}_{c,*} : O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2} \xrightarrow{\sim} \\ \bigoplus_{g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})} H^0_{\text{ét}}(C(g), O) \oplus \bigoplus_{g \in \mathrm{Sh}_{K^q K_q}(V_{D/q})} \Big(H^0_{\text{ét}}(\widetilde{B}_+(g), O) \oplus H^0_{\text{ét}}(\widetilde{B}_-(g), O) \Big) \\ \longrightarrow H^2_{c,\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(1)).\end{aligned}$$

We also denote by inc_c^* the composite

$$\mathrm{inc}_c^* : H^4_{c,\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2)) \to H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2)) \xrightarrow{\mathrm{inc}^*} O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2}$$

and likewise by inc_* the composite

$$\mathrm{inc}_* : O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2} \xrightarrow{\mathrm{inc}_{c,*}} H^2_{c,\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(1)) \longrightarrow H^2_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(1)).$$

Notation 7.3.5. For $\alpha \in O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]$ and $g \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$, let $m(g;\alpha) \in O$ denote the coefficient of g in α, and similarly for $\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})$.

Lemma 7.3.6. *Fix* $g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})$.

(1) There exists an isomorphism $C(g) \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q} \times \mathbb{P}^1_{\overline{\mathbb{F}}_q}$ *with the following property: for* $h \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$*, the intersection* $\widetilde{B}_+(h) \cdot C(g)$ *has the cycle class* $(m(h;\delta_+(g)), 0)$ *on* $C(g)$*; and the intersection* $\widetilde{B}_-(h) \cdot C(g)$ *has the cycle class* $(0, m(h;\delta_-(g)))$*.*

(2) Let $\iota_g : C(g) \hookrightarrow \widetilde{X}_{\overline{\mathbb{F}}_q}$ *be the natural inclusion. If we fix an isomorphism* $C(g) \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q} \times \mathbb{P}^1_{\overline{\mathbb{F}}_q}$ *as in (1), then we have*

$$\mathrm{inc}^*(\iota_g)_*[(1,0)] = (-g, 0, \delta_-(g))$$

and

$$\mathrm{inc}^*(\iota_g)_*[(0,1)] = (-g, \delta_+(g), 0).$$

Proof. (1) is immediate from Lemma 6.4.2 and Corollary 6.5.6; (2) follows from (1), using that the Chern class of the normal bundle on $C(g)$ is $(-1,-1)$ (by (7.1)) to compute the first coordinate. ☐

Notation 7.3.7. For any $g \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$, let

$$[\mathcal{O}_{\widetilde{B}_\pm(g)}(1)] \in H^4_{\text{ét},c}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))$$

denote the pushforward of the class of the line bundle $\mathcal{O}(1)$ on $\widetilde{B}_\pm(g)$ (which is the one induced by Notation 6.4.3).

Recall the explicit Hecke algebra generators from (2.1.10).

Lemma 7.3.8. *For all* $g \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$*, we have*

$$\mathrm{inc}_c^*[\mathcal{O}_{\widetilde{B}_+(g)}(1)] = (0, -2q(q+1)\cdot g, T_{q,2}\cdot g)$$

and

$$\mathrm{inc}_c^*[\mathcal{O}_{\widetilde{B}_-(g)}(1)] = (0, \langle q\rangle^{-1} T_{q,2}\cdot g, -2q(q+1)\cdot g).$$

Proof. The two calculations are similar, so we consider $\left[\mathcal{O}_{\widetilde{B}_+(g)}(1)\right]$. We must calculate the intersection pairings with divisor classes $[C(h)]$ and $\left[\widetilde{B}_\pm(g')\right]$, for $h \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})$ and $g' \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$.

Since $C(h)\cdot \widetilde{B}_+(g)$ always lies in the exceptional divisor of the blowup $\widetilde{B}_+(g) \to B_+(g)$, we have

$$\left[\mathcal{O}_{\widetilde{B}_+(g)}(1)\right]\cdot [C(h)] = [\mathcal{O}(1)] \cdot_{\widetilde{B}_+(g)} \left[C(h)\cdot \widetilde{B}_+(g)\right] = 0. \tag{7.12}$$

Now, $\widetilde{B}_+(g)$ and $\widetilde{B}_+(g')$ meet only if $g = g'$, in which case we find

$$\left[\mathcal{O}_{\widetilde{B}_+(g)}(1)\right]\cdot \left[\widetilde{B}_+(g)\right] = [\mathcal{O}(1)] \cdot_{\widetilde{B}_+(g)} \left[\mathcal{N}_{\widetilde{B}_+(g)/\widetilde{X}_{\overline{\mathbb{F}}_q}}\right] = -2q(q+1), \tag{7.13}$$

since the normal bundle to $\widetilde{B}_+(g)$ is $\mathcal{O}(-2q)$ (Lemma 6.4.4) and $B_+(g)$ has degree $q+1$ (Theorem 6.2.4(1)). Finally, we compute

$$\left[\mathcal{O}_{\widetilde{B}_+(g)}(1)\right]\cdot \left[\widetilde{B}_-(g')\right] = [\mathcal{O}(1)] \cdot_{\widetilde{B}_+(g)} \left[\widetilde{B}_+(g)\cdot \widetilde{B}_-(g')\right] = m(g'; T_{q,2}\cdot g), \tag{7.14}$$

since $B_+(g) \cap B_-(g')$ consists of $m(g'; T_{q,2}\cdot g)$ linearly embedded copies of $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$ inside $B_+(g) \subset \mathbb{P}^3_{\overline{\mathbb{F}}_q}$ (by Theorem 6.2.4(2) and Corollary 6.5.6). Combining these calculations gives

$$\mathrm{inc}_c^*\left[\mathcal{O}_{\widetilde{B}_+(g)}(1)\right] = (0, -2q(q+1)\cdot g, T_{q,2}\cdot g),$$

as desired. ☐

Definition 7.3.9. Fix a finite set S of places of $\mathbb{Q}$ containing q, and all primes $\ell \neq q$ such that K_ℓ is not hyperspecial. A maximal ideal $\mathfrak{m} \subset \mathbb{T}_O^S$ will be called *weakly q-generic* if the map

$$\langle q\rangle^{-1}T_{q,2}^2 - 4q^2(q+1)^2 : O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}} \to O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}$$

is an isomorphism.

Remark 7.3.10. Using (2.3), one calculates the following: if π is a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ unramified at q with trivial central character and $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ is any isomorphism with $p \neq q$, then $\langle q\rangle^{-1}T_{q,2}^2 - 4q^2(q+1)^2$ acts on the spherical vector of π_q with eigenvalue

$$q^2\left(\iota\,\mathrm{tr}(\mathrm{Frob}_q\,|V_{\pi,\iota})^2 - 4(q+1)^2\right).$$

Lemma 7.3.11. *Let $\mathfrak{m}$ be a weakly q-generic maximal ideal of $\mathbb{T}_O^S$. Then the map*

$$\mathrm{inc}_{c,\mathfrak{m}}^* : H^4_{c,\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}} \to O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}} \oplus O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$$

is surjective.

We note that Lemma 7.3.11 is slightly stronger than the corresponding statement for $\mathrm{inc}_{\mathfrak{m}}^*$.

Proof. By Lemma 7.3.6, it suffices to show that the image of $\mathrm{inc}_{c,\mathfrak{m}}^*$ contains $O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$. Define a map

$$\mu = (\mu_+, \mu_-) : O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2} \to H^4_{\text{ét},c}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}} \tag{7.15}$$

by linearly extending

$$\mu_\pm(g) = \left[\mathcal{O}_{\widetilde{B}_\pm(g)}(1)\right]. \tag{7.16}$$

Then by Lemma 7.3.8, the composite $\mathrm{inc}_{c,\mathfrak{m}}^* \circ \mu$ is given by the matrix map

$$\begin{pmatrix} 0 & 0 \\ -2q(q+1) & \langle q\rangle^{-1}T_{q,2} \\ T_{q,2} & -2q(q+1). \end{pmatrix}$$

Since

$$\det\begin{pmatrix} -2q(q+1) & \langle q\rangle^{-1}T_{q,2} \\ T_{q,2} & -2q(q+1) \end{pmatrix} = 4q^2(q+1)^2 - \langle q\rangle^{-1}T_{q,2}^2$$

and $\mathfrak{m}$ is weakly q-generic, we have

$$\mathrm{Im}(\mathrm{inc}_{c,\mathfrak{m}}^*) \supset \mathrm{Im}(\mathrm{inc}_{c,\mathfrak{m}}^* \circ \mu) = O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2},$$

as desired. ☐

In Theorem 7.3.14, we will see that $\mathrm{inc}_{c,\mathfrak{m}}^*$ also has torsion kernel.

Lemma 7.3.12. *Let $\mathfrak{m}$ be a generic and non-Eisenstein maximal ideal of $\mathbb{T}_O^S$. There is a canonical injection induced by pullback*

$$H^2(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(1))_{\mathfrak{m}} \hookrightarrow \bigoplus_{g \in \mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})} H^2_{\text{ét}}(C(g), O(1)) \simeq O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$$

and a canonical surjection induced by pushforward

$$O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2} \simeq \bigoplus_{g \in \mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})} H^2_{\text{ét}}(C(g), O(1))_{\mathfrak{m}} \twoheadrightarrow H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}}.$$

Proof. This follows from Lemma 7.1.7 and Theorem 2.7.5(2). ☐

Corollary 7.3.13 (Ihara's Lemma)**.** *Let* $\mathfrak{m}$ *be a generic, non-Eisenstein, and weakly* q*-generic maximal ideal of* $\mathbb{T}^S_O$*. Then the degeneracy map*

$$(\delta_+, \delta_-) : O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}} \longrightarrow O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$$

is surjective.

Proof. Combining Lemmas 7.3.12 and 7.3.11, we see that the composite map

$$O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2} \simeq \oplus H^2_{\text{ét}}(C(g), O(1))_{\mathfrak{m}} \longrightarrow H^4_{\text{ét},c}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}} \xrightarrow{\mathrm{inc}^*_{c,\mathfrak{m}}} \\ O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}} \oplus O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$$

is surjective. On the other hand, by Lemma 7.3.6, this composite is given as a matrix by

$$\begin{pmatrix} -1 & 0 & \delta_- \\ -1 & \delta_+ & 0 \end{pmatrix},$$

and the corollary follows by restricting to the preimage of $O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$. ☐

Theorem 7.3.14. *Let* $\mathfrak{m}$ *be a generic, non-Eisenstein, and weakly* q*-generic maximal ideal of* $\mathbb{T}^S_O$*. Then*

$$\mathrm{inc}^*_{\mathfrak{m}} : H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}} \to O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}} \oplus O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2}$$

and

$$\mathrm{inc}_{*,\mathfrak{m}} : O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}} \oplus O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}^{\oplus 2} \to H^2_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(1))_{\mathfrak{m}}$$

are both surjective. Moreover $\mathrm{inc}_{*,\mathfrak{m}}$ *is injective, and* $\mathrm{inc}^*_{\mathfrak{m}}$ *is injective modulo* O*-torsion.*

In fact, only the surjectivity of $\mathrm{inc}_{*,\mathfrak{m}}$ is needed for the main result.

Proof. We claim that it suffices to show

$$\dim H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_p)_{\mathfrak{m}} \leq \dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}}\right) + 2\dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}\right); \tag{7.17}$$

indeed, this combined with Lemma 7.3.11 implies that $\mathrm{inc}^*_{\mathfrak{m}}$ is injective modulo torsion as well as surjective, and the other assertions follow by duality along with Lemma 7.1.13 and Theorem 2.7.5(2). (We also use that $H^2(\widetilde{X}_{\overline{\mathbb{F}}_q}, O)_{\mathfrak{m}}$ is O-torsion-free by Lemma 7.3.12.) Inspecting the diagram in Lemma 7.1.7 and using Theorem 2.7.5(2), we see that

$$\dim H^4(\widetilde{X}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_p)_{\mathfrak{m}} = 2\dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}}\right) - \mathrm{rank}\left(T - 1 | H^3_{\text{ét}}\left(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p\right)_{\mathfrak{m}}\right),$$

where $T \in I_{\mathbb{Q}_q}$ is a generator of tame inertia, so we wish to show

$$\dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}}\right) \leq 2\dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}\right) + \mathrm{rank}\left(T - 1 | H^3_{\text{ét}}\left(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p\right)_{\mathfrak{m}}\right). \tag{7.18}$$

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Applying Lemma 2.7.6, it suffices to show

$$\dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]\left[\iota^{-1}\pi_f^q\right]\right) \leq 2\dim\left(\overline{\mathbb{Q}}_p\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]\left[\iota^{-1}\pi_f^q\right]\right) + \\ \mathrm{rank}\left(T - 1 | H^3_{\text{ét}}\left(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p\right)\left[\iota^{-1}\pi_f^q\right]\right) \tag{7.19}$$

for all relevant automorphic representations π of $\mathrm{GSpin}(V_{D/q})(\mathbb{A})$ such that $\pi_f^{K^qK_q^{\mathrm{Pa}}} \neq 0$ and the Hecke action on $\iota^{-1}\pi_f^{K^qK_q^{\mathrm{Pa}}}$ factors through $\mathbb{T}^S_{O,\mathfrak{m}}$.

By Lemma 2.6.2(1,2), π_q is uniquely determined by π_f^q, and is either spherical or of type IIa. Assume first that π_q is spherical. Then π_f^q cannot be completed to a relevant automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$ (by Corollary 2.5.3), so by Corollary 2.7.7 the final term in (7.19) vanishes. Since $\dim \pi_q^{K_q} = 1$ and $\dim \pi_q^{K_q^{\mathrm{Pa}}} = 2$ by [95, Table A.13], both sides of (7.19) are 2.

On the other hand, suppose that π_q is of type IIa.

Case 1: π is non-endoscopic. Then π_f^q can be completed to a relevant automorphic representation π' of $\mathrm{GSpin}(V_D)(\mathbb{A})$ by Theorem 2.4.6. By Lemma 2.6.2(3), π'_q has a unique fixed vector for $K_q^{\mathrm{ram}} \subset \mathrm{GSpin}(V_D)(\mathbb{Q}_q)$, so by Corollary 2.7.7,

$$H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)\left[\iota^{-1}\pi_f^q\right] = \rho_{\pi,\iota}(-2) \otimes \iota^{-1}(\pi_f^q)^{K^q}.$$

By Lemma 2.6.2(3), $\rho_{\pi,\iota}$ is tamely ramified at q with monodromy of rank one. On the other hand, π_q has a unique paramodular fixed vector, and π has automorphic multiplicity one for $\mathrm{GSpin}(V_{D/q})(\mathbb{A})$ by Theorem 2.4.6(3). So in this case we see that both sides of (7.19) are 1.

Case 2: π is endoscopic, associated to a pair of cuspidal automorphic representations (π_1, π_2) of GL_2 with discrete series archimedean components of weights 2 and 4, respectively. Theorem 2.5.2 implies that there exist (uniquely determined) quaternion algebras B_1 and B_2 such that π is the theta lift $\Theta(\pi_1^{B_1} \boxtimes \pi_2^{B_2})$, with $\pi_i^{B_i}$ the Jacquet-Langlands transfers. Moreover, $B_1 \otimes \mathbb{R}$ is ramified and $B_2 \otimes \mathbb{R}$ is split. Since π_q is of type IIa with a paramodular fixed vector, we can conclude from Lemma 2.6.2(3) and Theorem 2.2.1(1) that exactly one of $\pi_{i,q}$ is a twist of a Steinberg representation, and the other is unramified. Let B'_i be the quaternion algebras obtained from B_i by changing invariants at q and ∞. Then π_f^q has a unique completion to an automorphic representation of $\mathrm{GSpin}(V_D)$, which is

$$\begin{cases} \Theta(\pi_1^{B'_1} \boxtimes \pi_2^{B_2}), & \pi_{1,q} \text{ ramified}, \\ \Theta(\pi_1^{B_1} \boxtimes \pi_2^{B'_2}), & \pi_{2,q} \text{ ramified}. \end{cases} \tag{7.20}$$

We therefore have (applying Corollary 2.7.7)

$$H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)\left[\pi_f^q\right] \cong (\pi_f^q)^{K^q} \otimes \rho,$$

with

$$\rho = \begin{cases} \rho_{\pi_1,\iota}(-2), & \pi_{1,q} \text{ ramified}, \\ \rho_{\pi_2,\iota}(-2), & \pi_{2,q} \text{ ramified}. \end{cases}$$

In particular, the monodromy at q has rank one in either case, so again both sides of (7.19) are 1. □

7.4. **Level-raising and potential map.**

Lemma 7.4.1. *The Hecke operators $\theta_\pm$, $\delta_\pm$ satisfy:*

$$\begin{aligned} \delta_+ \circ \theta_+ &= \delta_- \circ \theta_- = T_{q,1} + (q+1)(q^2+1) \\ \delta_- \circ \theta_+ &= (q+1)T_{q,2} \\ \delta_+ \circ \theta_- &= \langle q \rangle^{-1}(q+1)T_{q,2}. \end{aligned}$$

Proof. By definition, $\delta_+ \circ \theta_+$ is induced by

$$\begin{aligned}(g^q, \Lambda) &\mapsto \delta_+ \left(\sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda \\ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}}} (g^q, \Lambda_{\mathrm{Pa}}) \right) \\ &= \sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda' \\ \Lambda' \in \mathscr{L}}} \sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda \\ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}}} (g^q, \Lambda') \\ &= \sum_{\Lambda' \in \mathscr{L}} e(\Lambda, \Lambda')(g^q, \Lambda'),\end{aligned}$$

where

$$e(\Lambda, \Lambda') = \# \left\{ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}} \, : \, \Lambda_{\mathrm{Pa}} \subset_1 \Lambda, \, \Lambda_{\mathrm{Pa}} \subset_1 \Lambda' \right\}.$$

If $e(\Lambda, \Lambda') \neq 0$, then either $\Lambda = \Lambda'$ or $\Lambda' \in T_{q,1} \cdot \Lambda$. In the latter case, $\Lambda_{\mathrm{Pa}} = \Lambda \cap \Lambda'$ is uniquely determined, so $e(\Lambda, \Lambda') = 1$. On the other hand, $e(\Lambda, \Lambda)$ is the number of lattices $\Lambda_{\mathrm{Pa}} \subset_1 \Lambda$, or equivalently the number of rational 3-planes in the symplectic space $\Lambda / q\Lambda$. Thus

$$e(\Lambda, \Lambda) = \#\mathbb{P}^3(\mathbb{F}_q) = (q+1)(q^2+1).$$

This shows

$$\delta_+ \circ \theta_+ = T_{q,1} + (q+1)(q^2+1),$$

and the calculation for $\delta_- \circ \theta_-$ is similar. We now compute $\delta_- \circ \theta_+$, which is induced by

$$\begin{aligned}(g^q, \Lambda) &\mapsto \delta_- \left(\sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda \\ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}}} (g^q, \Lambda_{\mathrm{Pa}}) \right) \\ &= \sum_{\substack{\Lambda_{\mathrm{Pa}} \subset_1 \Lambda \\ \lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}}} \sum_{\substack{\Lambda' \subset_1 \Lambda_{\mathrm{Pa}} \\ \Lambda' \in \mathscr{L}}} (g^q, \Lambda') \\ &= \sum_{\Lambda' \in \mathscr{L}} e'(\Lambda, \Lambda')(g^q, \Lambda'),\end{aligned}$$

where

$$e'(\Lambda, \Lambda') = \# \left\{ \Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}} \, : \, \Lambda' \subset_1 \Lambda_{\mathrm{Pa}} \subset_1 \Lambda \right\}.$$

If $e'(\Lambda, \Lambda') \neq 0$, then $\Lambda' \in T_{q,2} \cdot \Lambda$. On the other hand, given $\Lambda' \in T_{q,2} \cdot \Lambda$, then the choices of Λ_{Pa} with $\Lambda' \subset_1 \Lambda_{\mathrm{Pa}} \subset_1 \Lambda$ are in bijection with rational lines in the 2-dimensional $\mathbb{F}_q$-vector space Λ/Λ'; hence

$$e'(\Lambda, \Lambda') = \#\mathbb{P}^1(\mathbb{F}_q) = q+1.$$

This shows $\delta_- \circ \theta_+ = (q+1)T_{q,2}$, and the computation of $\delta_+ \circ \theta_-$ is similar. □

7.4.2. Recall from Lemma 7.1.7 the natural embedding

$$j : \bigsqcup C(g) \hookrightarrow \widetilde{X}_{\overline{\mathbb{F}}_q}.$$

Lemma 7.4.3. *The composite map*

$$\begin{aligned}\mathrm{inc}^* \circ j_* \circ j^* \circ \mathrm{inc}_* : O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2} \longrightarrow \\ O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2}\end{aligned}$$

is given by the matrix

$$\begin{pmatrix} 2 & -\theta_+ & -\theta_- \\ -\delta_+ & 0 & (q+1)\langle q\rangle^{-1}T_{q,2} \\ -\delta_- & (q+1)T_{q,2} & 0. \end{pmatrix}$$

Proof. We begin by calculating $\mathrm{inc}^* \circ j_* \circ j^*\,[C(g)]$, for $g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})$. Let $\iota_g : C(g) \hookrightarrow \widetilde{X}_{\overline{\mathbb{F}}_q}$ be the natural embedding. Since the $C(g)$ are all disjoint,

$$j_* \circ j^*\,[C(g)] \in H^4_{\text{ét},c}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))$$

is the pushforward of the class of the normal bundle, i.e. $\iota_{g*}[(-1,-1)]$ in the notation of Lemma 7.3.6. Then

$$\mathrm{inc}^*\,\iota_{g*}[(-1,-1)] = -\,\mathrm{inc}^*\,\iota_{g*}[(1,0)] - \mathrm{inc}^*\,\iota_{g*}[(0,1)] = (2\cdot g, -\delta_+(g), -\delta_-(g))$$

by Lemma 7.3.6 (2), which gives the first column of the matrix.

For the second column, we must calculate $\mathrm{inc}^* \circ j_* \circ j^* \left[\widetilde{B}_+(g)\right]$, for $g \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$. By Lemma 7.3.6 (1), the class

$$j_* \circ j^* \left[\widetilde{B}_+(g)\right] \in H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))$$

is

$$\sum_{h \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})} m(h;\theta_+(g))\iota_{h*}\,[(1,0)]\,.$$

Then by Lemma 7.3.6 (2),

$$\mathrm{inc}^* \circ j_* \circ j^* \left[\widetilde{B}_+(g)\right] = (-\theta_+(g), 0, \delta_- \circ \theta_+(g))\,.$$

By Lemma 7.4.1, $\delta_- \circ \theta_+ = (q+1)T_{q,2}$, so this gives the second column of the matrix; the third column is similar. ☐

Definition 7.4.4. Define the potential map

$$\nabla : H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2)) \to O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]$$

as the composite

$$H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2)) \xrightarrow{\mathrm{inc}^*} O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})\right] \oplus O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]^{\oplus 2} \xrightarrow{M} O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right],$$

with M the matrix map

$$\begin{pmatrix} \delta_+ + \delta_- & 2 & 2 \end{pmatrix}.$$

Definition 7.4.5. The level-raising Hecke operator $\mathtt{T}_q^{\mathrm{lr}}$ is defined by

$$\mathtt{T}_q^{\mathrm{lr}} := T_{q,1} + (q+1)(q^2+1) - T_{q,2}(q+1).$$

Remark 7.4.6. Using (2.3), one calculates the following: If π is a relevant automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ unramified at q with trivial central character and $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ is any isomorphism with $p \neq q$, then $\mathtt{T}_q^{\mathrm{lr}}$ acts on the spherical vector of π_q with eigenvalue

$$q^{-1}\iota\det(\mathrm{Frob}_q - q | V_{\pi,\iota}).$$

Theorem 7.4.7. *Let* $\mathfrak{m}$ *be a generic, non-Eisenstein, and weakly* q*-generic maximal ideal of* $\mathbb{T}^S_O$*. The composite map*

$$\nabla \circ j_* \circ t^{-1} \circ j^* : H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}} \to O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]_{\mathfrak{m}}$$

has image contained in

$$\left(\langle q\rangle^{-1} - 1, \mathtt{T}_q^{\mathrm{lr}}\right) \cdot O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right]_{\mathfrak{m}}.$$

In particular, $\nabla \circ \zeta$ gives a well-defined surjection

$$\nabla \circ \zeta : M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \twoheadrightarrow \frac{O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}}{\left(\langle q\rangle^{-1}-1, \mathtt{T}_q^{\mathrm{lr}}\right)},$$

where ζ is the map from Proposition 7.1.9.

Proof. By Theorem 7.3.14, the image of $\nabla \circ j_* \circ t^{-1} \circ j^*$ coincides with the image of

$$\nabla \circ j_* \circ j^* \circ \mathrm{inc}_{*,\mathfrak{m}} : O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right]_{\mathfrak{m}} \oplus O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right)^{\oplus 2}_{\mathfrak{m}} \to O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}.$$

By Lemma 7.4.3, this is the composite of matrix maps

$$\begin{aligned}\begin{pmatrix}\delta_+ + \delta_- & 2 & 2\end{pmatrix} \circ \begin{pmatrix} 2 & -\theta_+ & -\theta_- \\ -\delta_+ & 0 & (q+1)\langle q\rangle^{-1}T_{q,2} \\ -\delta_- & (q+1)T_{q,2} & 0\end{pmatrix} \\ = \begin{pmatrix}0 & -\mathtt{T}_q^{\mathrm{lr}} & -\mathtt{T}_q^{\mathrm{lr}} + \left(\langle q\rangle^{-1}-1\right)(q+1)T_{q,2}\end{pmatrix}\end{aligned}$$

(using Lemma 7.4.1 to compute $\delta_\pm \circ \theta_\pm$). ☐

7.5. **Siegel cycles on the special fiber.**

Definition 7.5.1. Recall the set $\mathscr{L}$ from Definition 6.5.2. We define

$$\mathscr{L}_{\mathrm{Sie}} := \left\{\text{pairs } (\Lambda_+, \Lambda_-) \in \mathscr{L}^2 \ : \ q\Lambda_+ \subset_2 \Lambda_- \subset_2 \Lambda_+\right\}.$$

Notation 7.5.2. Note that $\mathscr{L}_{\mathrm{Sie}}$ is a homogeneous space for $\mathrm{GSpin}(V_{D/q})(\mathbb{Q}_q)$, where the stabilizer of any point is a Siegel parahoric subgroup. As in Notation 7.3.1, we abbreviate

$$\mathrm{Sh}_{K^qK_q^{\mathrm{Sie}}}(V_{D/q}) = \mathrm{GSpin}(V_{D/q})(\mathbb{Q})\backslash \mathrm{GSpin}(V_{D/q})(\mathbb{A}_f^q) \times \mathscr{L}_{\mathrm{Sie}}/K^qK_q^{\mathrm{Sie}}, \tag{7.21}$$

although the identification depends on a choice of base point of $\mathscr{L}_{\mathrm{Sie}}$ which is not necessary for our discussion.

Definition 7.5.3.

(1) We define degeneracy maps

$$\delta_\pm^{\mathrm{Sie}} : \mathscr{L}_{\mathrm{Sie}} \to \mathscr{L}$$

by

$$\delta_\pm^{\mathrm{Sie}}(\Lambda_+, \Lambda_-) = \Lambda_\pm.$$

(2) We define the operator

$$\theta_{\mathrm{Sie}}^{\mathrm{Pa}} : \mathscr{L}_{\mathrm{Sie}} \to \mathbb{Z}\left[\mathscr{L}_{\mathrm{Pa}}\right]$$

by

$$\theta_{\mathrm{Sie}}^{\mathrm{Pa}}(\Lambda_+, \Lambda_-) = \sum_{\substack{\Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}} \\ \Lambda_- \subset \Lambda_{\mathrm{Pa}} \subset \Lambda_+}} \left[\Lambda_{\mathrm{Pa}}\right].$$

(3) We extend these maps linearly to

$$\delta_\pm^{\mathrm{Sie}} : O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Sie}}}(V_{D/q})\right] \to O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]$$

and

$$\theta_{\mathrm{Sie}}^{\mathrm{Pa}} : O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Sie}}}(V_{D/q})\right] \to O\left[\mathrm{Sh}_{K^qK_q^{\mathrm{Pa}}}(V_{D/q})\right].$$

Notation 7.5.4. For each $g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Sie}}}(V_{D/q})$, let

$$D(g) = B_+(\delta_1^{\mathrm{Sie}}(g)) \cap B_-(\delta_2^{\mathrm{Sie}}(g)),$$

which is a closed subscheme of $X_{\overline{\mathbb{F}}_q}$ isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$. We write $\widetilde{D}(g) \hookrightarrow \widetilde{X}_{\overline{\mathbb{F}}_q}$ for the strict transform of $D(g)$ under the blowup $\widetilde{X}_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$, and $[\widetilde{D}(g)] \in H^4_{\text{ét}}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2))$ for its algebraic cycle class.

Lemma 7.5.5. *For $g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Sie}}}(V_{D/q})$, we have*

$$\nabla \left[\widetilde{D}(g)\right] = (\delta_+ + \delta_-) \circ \theta^{\mathrm{Pa}}_{\mathrm{Sie}}(g) - 4q\delta_+^{\mathrm{Sie}}(g) - 4q\delta_-^{\mathrm{Sie}}(g).$$

Proof. We first calculate $\mathrm{inc}^* \left[\widetilde{D}(g)\right]$. For $g' \in \mathrm{Sh}_{K^q K_q}(V_{D/q})$, we have

$$\left[\widetilde{D}(g)\right] \cdot \left[\widetilde{B}_+(g')\right] = \left[\widetilde{B}_+(\delta_1^{\mathrm{Sie}}(g))\right] \cdot \left[\widetilde{B}_-(\delta_2^{\mathrm{Sie}}(g))\right] \cdot \left[\widetilde{B}_+(g')\right] = 0$$

unless $g' = \delta_1^{\mathrm{Sie}}(g)$, in which case the intersection number is $-2q$ (cf. the proof of Lemma 7.3.8). Similarly,

$$\left[\widetilde{D}(g)\right] \cdot \left[\widetilde{B}_-(g')\right] = \begin{cases} -2q, & g' = \delta_2^{\mathrm{Sie}}(g), \\ 0, & \text{else.} \end{cases}$$

Now consider the intersections with $C(h)$, for $h \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{D/q})$. We see from Lemma 7.3.6 that $\widetilde{D}(g)$ meets $C(h)$ transversely with multiplicity $m(h; \theta^{\mathrm{Pa}}_{\mathrm{Sie}}(g))$. Hence

$$\mathrm{inc}^* \left[\widetilde{D}(g)\right] = \left(\theta^{\mathrm{Pa}}_{\mathrm{Sie}}(g), -2q\delta_1^{\mathrm{Sie}}(g), -2q\delta_2^{\mathrm{Sie}}(g)\right).$$

The claimed formula then follows from the formula for ∇ in Definition 7.4.4. □

7.6. **Special cycles on ramified** GSpin_5 **Shimura varieties.** The goal of this section is to compute the local ramification of Abel-Jacobi images of special cycles $Z(T, \varphi)$ on the generic fiber of X, by applying the results of §7.1. However, our first task is to make a good choice of the uniformization datum from (7.2.1).

Notation 7.6.1. Fix a matrix $T \in \mathrm{Sym}_2(\mathbb{Z}_{(q)})_{\geq 0}$ such that

$$T \equiv \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \pmod{q},$$

for some $\alpha \in \mathbb{F}_q^\times$. Then we make the following notations.

1. Let $V_\star$ be the two-dimensional quadratic space over $\mathbb{Q}_q$ with basis $\{e_1^\star, e_2^\star\}$ and pairing matrix given by T.
2. Let $B_\star$ be the quaternion algebra $C(V_\star)$, with its natural positive nebentype involution $*$.
3. Let d be the discriminant of the unique quaternion algebra B_d such that $B_D \otimes B_\star \simeq M_2(B_d)$, and let $O_d \subset B_d$ be the unique maximal $\mathbb{Z}_{(q)}$-order.

Remark 7.6.2. Because $B_\star$ is split at q and ∞, B_d is ramified at q and split at ∞. In particular, O_d is well-defined.

Proposition 7.6.3. *Fix a nebentype involution $*$ on O_D, of unit type. Then there exists an isomorphism*

$$\beta : B_D \otimes B_\star \xrightarrow{\sim} M_2(B_d)$$

such that:

(1) $\beta(O_D) \subset M_2(O_d)$.
(2) The induced involution $\dagger$ on $M_2(B_d)$ stabilizes $M_2(O_d)$, and is of non-unit type.

(3) *If* $\Pi_D \in O_D$ *and* $\Pi_d \in O_d$ *are uniformizers, then*

$$\beta(\Pi_D) \cdot \begin{pmatrix} \Pi_d^{-1} & 0 \\ 0 & \Pi_d^{-1} \end{pmatrix}$$

lies in $\mathrm{GL}_2(O_d)$.

(4) *The* $\mathbb{Z}_{(q)}$*-lattice* $\Lambda_\star := M_2(O_d)^{\dagger=1,\mathrm{tr}=0} \cap B_\star \subset V_\star$ *has basis* $\{e_1^\star, qe_2^\star\}$.

Proof. Let

$$X = \mathrm{Isom}(B_D \otimes B_\star, M_2(B_d)),$$

viewed as algebraic variety over $\mathbb{Q}$; X is a (split) torsor for the algebraic group $\mathrm{GL}_2(B_d)$. Note that all the conditions of the proposition can be checked after tensoring with $\mathbb{Z}_q$, and define an open subset $U \subset X(\mathbb{Q}_q)$ in the q-adic topology. Since $X(\mathbb{Q})$ is dense in $X(\mathbb{Q}_q)$, it suffices to show $U \neq \emptyset$.

Since the involution $*$ on B_D is nebentype and of unit type, we can fix a unit $j \in O_D^\times$ such that $j^{*_D} = -j^{*_D}$ and $\alpha^* = j(\alpha^{*_D})j^{-1}$ for $\alpha \in B_D$ (recall $*_D$ is the canonical involution on B_D). Also choose a uniformizer $\Pi \in O_D$ satisfying $\mathrm{tr}\,\Pi = 0$ and $\Pi j = -j\Pi$, and let $K := \mathbb{Q}(\Pi) \subset B_D$. We have a decomposition

$$B_D = K \oplus j \cdot K,$$

which defines an embedding

$$\iota : B_D \hookrightarrow M_2(K) \hookrightarrow M_2(B_D),$$

satisfying $\alpha(O_D) \subset M_2(O_D)$. Now let † be the non-unit type involution on O_D defined by

$$j^\dagger = j,\ \Pi^\dagger = \Pi,\ \text{and}\ (\Pi j)^\dagger = -\Pi j.$$

We extend † to an involution of non-unit type on $M_2(B_D)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha^\dagger & -\gamma^\dagger j^2 \\ -\beta^\dagger / j^2 & \delta^\dagger \end{pmatrix}.$$

A simple calculation shows $\iota(\alpha)^\dagger = \iota(\alpha^*)$, for all $\alpha \in B_D$. Moreover, the centralizer Z of $\iota(B_D)$ inside $M_2(B_D)$ satisfies

$$L := Z \cap M_2(O_D)^{\dagger=1,\mathrm{tr}=0} = \mathbb{Z}_{(q)} \cdot \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix} \oplus \mathbb{Z}_{(q)} \cdot \begin{pmatrix} 0 & \Pi j \\ \Pi j^{-1} & 0 \end{pmatrix}.$$

We have a natural quadratic form $x \mapsto x^2$ on L, which is represented by $\begin{pmatrix} q\alpha & 0 \\ 0 & -q\alpha \end{pmatrix}$ in the basis above, for a unit $\alpha \in \mathbb{Z}_{(q)}^\times$. Let $L_\mathbb{Q} = L \otimes_{\mathbb{Z}_{(q)}} \mathbb{Q}$.

We may then choose the following two isomorphisms:

(i) An isomorphism $\beta_{1,q} : Z \otimes \mathbb{Q}_q = C(L_\mathbb{Q}) \otimes \mathbb{Q}_q \xrightarrow{\sim} C(V_\star) \otimes \mathbb{Q}_q = B_\star \otimes \mathbb{Q}_q$ compatible with the involutions, such that $\beta_{1,q}(L \otimes \mathbb{Z}_q) \subset V_\star \otimes \mathbb{Q}_q$ is the lattice spanned by $e_1^\star$ and $qe_2^\star$.

(ii) An isomorphism $\beta_{2,q} : M_2(O_D) \otimes \mathbb{Z}_q \xrightarrow{\sim} M_2(O_d) \otimes \mathbb{Z}_q$.

. One checks readily that the induced isomorphism

$$\beta_q : (B_D \otimes \mathbb{Q}_q) \otimes (B_\star \otimes \mathbb{Q}_q) \xrightarrow[\sim]{\mathrm{id} \otimes \beta_{1,q}^{-1}} B_D \otimes Z \otimes \mathbb{Q}_q \xrightarrow[\sim]{\iota} M_2(B_D) \otimes \mathbb{Q}_q \xrightarrow[\sim]{\beta_{2,q}} M_2(B_d) \otimes \mathbb{Q}_q$$

lies in U, so indeed $U \neq \emptyset$. ☐

Now we use Proposition 7.6.3 to construct some particular integral models of special cycles on $X_\mathbb{Q} = \mathrm{Sh}_K(V_D)$.

Construction 7.6.4. Fix once and for all a positive, unit type involution on B_D, and a choice of β as in Proposition 7.6.3 for this choice. We will now also write $*$ for the induced involution on $M_2(B_d)$.

(1) By Corollary 1.3.5 and Remark 7.6.2, we fix an abelian scheme A_0 over $\breve{\mathbb{Z}}_q$ with supersingular reduction, equipped with an embedding $\iota_0^\diamond : M_2(O_d) \hookrightarrow \mathrm{End}(A_0)\otimes\mathbb{Z}_{(q)}$ and a polarization $\lambda_0 : A_0 \to A_0^\vee$ such that
$$\iota_0^\diamond(\alpha)^\vee \circ \lambda_0 = \lambda_0 \circ \iota_0^\diamond(\alpha^*).$$
We choose our q-uniformization datum for V_D in (7.2.1) to be of the form $(*, A_0, \iota_0^\diamond \circ \beta, \lambda_0, i_D, i_{D/q})$. With notation as in Definition 1.4.4, we also obtain the PEL datum $\mathcal{D}^\diamond = (M_2(B_d), \dagger, H, \psi)$, with self-dual q-integral refinement $\mathscr{D}^\diamond = (M_2(O_d), \dagger, \Lambda, \psi)$.

(2) We observe that, following Remark 1.4.5(2), our choice of q-adic uniformization datum defines an orthogonal decomposition
$$V_D \xrightarrow[\sim]{i_D} \mathrm{End}(H, B_D)^{\dagger=1,\mathrm{tr}=0} = V_\star \oplus V_d^\diamond$$
where
$$V_d^\diamond := \mathrm{End}(H, M_2(B_d))^{\dagger=1,\mathrm{tr}=0}.$$

(3) Now fix an element
$$g^q = \prod_{\ell\neq q} g_\ell \in \mathrm{GSpin}(V_D)(\mathbb{A}_f^q).$$
Let
$$K_\ell^\diamond := g_\ell K_\ell g_\ell^{-1} \cap \mathrm{GSpin}(V_d^\diamond)(\mathbb{Q}_\ell)$$
for each $\ell \neq q$; we define
$$K^{q\diamond} = \prod_{\ell\neq q} K_\ell^\diamond,$$
a neat compact open subgroup by [89, §0.1]. We let $\boldsymbol{Z}(g^q, V_\star, V_D)$ be the $\mathbb{Z}_{(q)}$-scheme representing the moduli functor $\mathcal{M}^\diamond_{K^{q\diamond}}$ associated to $\mathscr{D}^\diamond$ at level $K^{q\diamond}$, with special fiber $Z(g^q, V_\star, V_D)_{\mathbb{F}_q}$.

Remark 7.6.5. Note that (by Corollary 1.3.4 combined with Proposition 1.2.4), $\boldsymbol{Z}(g^q, V_\star, V_D)$ is the usual integral model of the quaternionic Shimura curve associated to B_d at level $K^{q\diamond}$, with maximal compact level structure at q.

7.6.6. We have the obvious forgetful finite morphism
$$j : \boldsymbol{Z}(g^q, V_\star, V_D) \to X \tag{7.22}$$
defined on the moduli problems by $(A, \iota^\diamond, \lambda, \eta^q) \mapsto (A, \iota^\diamond \circ \beta, \lambda, g^q \cdot \eta^q)$.

Lemma 7.6.7. *The generic fiber of (7.22) coincides with the special cycle*
$$Z(g^q, V_\star, V_D)_{K^q K_q^{\mathrm{ram}}} \to \mathrm{Sh}_{K^q K_q^{\mathrm{ram}}}(V_D).$$

Proof. See the proof of [56, Proposition 2.5]. ☐

Notation 7.6.8. We now consider the special fiber $Z(g^q, V_\star, V_D)_{\mathbb{F}_q}$.

(1) Similarly to Construction 7.6.4(2), we obtain a natural orthogonal decomposition
$$V_{D/q} \xrightarrow[\sim]{i_{D/q}} \mathrm{End}^0(\overline{A}_0, B_D)^{\dagger=1,\mathrm{tr}=0} = V_\star \oplus V_{d/q}^\diamond,$$
with
$$V_{d/q}^\diamond := \mathrm{End}^0(\overline{A}_0, M_2(B_d))^{\dagger=1,\mathrm{tr}=0}$$
a three-dimensional quadratic space whose Hasse invariant coincides with that of $B_{d/q}$.

(2) Let $\mathcal{N}^\diamond$ be the Rapoport-Zink space parametrizing framed, polarized deformations of the q-divisible group $A_0[q^\infty]$, with action of $M_2(O_d \otimes \mathbb{Z}_q)$. (The details are analogous to Definition 6.1.2). Let $j^{\mathrm{loc}} : \mathcal{N}^\diamond \hookrightarrow \mathcal{N}$ be the natural closed immersion induced by $\beta \otimes \mathbb{Z}_q : O_D \otimes \mathbb{Z}_q \hookrightarrow M_2(O_d \otimes \mathbb{Z}_q)$, and let
$$\mathcal{N}^\diamond = \sqcup_{i\in\mathbb{Z}} \mathcal{N}^\diamond(i)$$
be the decomposition defined analogously to (6.1), or equivalently defined by $\mathcal{N}^\diamond(i) = \mathcal{N}^\diamond \cap \mathcal{N}(i)$.

Remark 7.6.9. By the local analogue of Corollary 1.3.4, $\mathcal{N}^\diamond$ is isomorphic to the formal scheme considered in [12, §I.3], which is the one encountered in the well-known Čerednik-Drinfeld uniformization of quaternionic Shimura curves.

Lemma 7.6.10. *The image of $\mathcal{M}^\diamond := \mathcal{N}^\diamond_{\mathrm{red}}(0)$ under j^{loc} is contained in $\mathcal{M}_{\{02\}}$ (cf. Theorem 6.2.4).*

Proof. It suffices to consider $\overline{\mathbb{F}}_q$-valued points, so suppose given $x = (X_x, \iota_x, \lambda_x, \rho_x) \in \mathcal{M}^\diamond(\overline{\mathbb{F}}_q)$, with $\iota : M_2(O_d) \otimes \mathbb{Z}_q \hookrightarrow \mathrm{End}(X_x)$ an embedding compatible with involutions. By (6.2.5), it suffices to show the Dieudonné module M of X_x satisfies $M + \tau M = \tau M + \tau^2 M$ for $\tau = (\iota \circ \beta(\Pi_D)) \cdot V^{-1}$. (Since M is self-dual, $M \cap \tau M$ is then τ-stable as well.) If $\tau' = \iota \begin{pmatrix} \Pi_d & 0 \\ 0 & \Pi_d \end{pmatrix} \cdot V^{-1}$, then we have
$$(\tau')^n M = \tau^n M$$
for all $n \in \mathbb{Z}$ by Proposition 7.6.3(3). So it suffices to show $M + \tau' M$ is τ'-stable. The action of
$$M_2(\mathbb{Z}_q) \subset M_2(O_d) \otimes \mathbb{Z}_q$$
on M defines a decomposition $M = M_0 \oplus M_0$, where M_0 inherits an action ι_0 of O_d. We can further decompose $M_0 = M_{0,\bullet} \oplus M_{0,\circ}$ according to the eigenvalues of the action of $\mathbb{Z}_{q^2} \subset O_d \otimes \mathbb{Z}_q$. Then $M_{0,\bullet}$ and $M_{0,\circ}$ are both free of rank two over $\breve{\mathbb{Z}}_q$ and stable under $\tau_0 := \iota_0(\Pi_d)V^{-1}$. Hence $M_0 + \tau_0 M_0$ is τ_0-stable by [93, Proposition 2.17]; so $M + \tau' M$ is τ'-stable, as desired. □

Definition 7.6.11. We define a subset $\mathscr{L}^\diamond \subset \mathscr{L}_{\mathrm{Sie}}$ by
$$j^{\mathrm{loc}}(\mathcal{N}^\diamond_{\mathrm{red}}) = \bigcup_{(\Lambda_+,\Lambda_-)\in\mathscr{L}^\diamond} \left(\mathcal{M}_+(\Lambda_+) \cap \mathcal{M}_-(\Lambda_-)\right),$$
which makes sense by Lemma 7.6.10.

By [12, Proposition III.2] (and Remark 7.6.5), the special fiber $Z(g^q, V_\star, V_D)_{\mathbb{F}_q}$ is purely supersingular. Hence by Rapoport-Zink uniformization, we have:

Proposition 7.6.12. *The special fiber $Z(g^q, V_\star, V_D)_{\overline{\mathbb{F}}_q}$ is isomorphic to*
$$\mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{Q}) \backslash \mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{A}^q_f) \times \mathcal{N}^\diamond_{\mathrm{red}}/K^{q\diamond}_{g^q},$$
in such a way that the special fiber of $j : \boldsymbol{Z}(g^q, V_\star, V_D) \to X$ is given by $(h^q, x) \mapsto (h^q g^q, j^{\mathrm{loc}}(x))$ under the uniformization of Theorem 7.2.4.

Notation 7.6.13. Let
$$\widetilde{j} : \widetilde{Z}(g^q, V_\star, V_D)_{\overline{\mathbb{F}}_q} \to \widetilde{X}_{\overline{\mathbb{F}}_q} \tag{7.23}$$
be the strict transform of
$$Z(g^q, V_\star, V_D)_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q},$$
and
$$\left[\widetilde{Z}(g^q, V_\star, V_D)_{\overline{\mathbb{F}}_q}\right] \in H^4_{\text{ét},c}(\widetilde{X}_{\overline{\mathbb{F}}_q}, O(2)) \tag{7.24}$$

its algebraic cycle class.

Corollary 7.6.14. *We have*

$$\nabla\left[\widetilde{Z}(g^q, V_\star, V_D)_{\overline{\mathbb{F}}_q}\right] = \sum_{\substack{[(h^q,\Lambda)]\in \\ \mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{Q})\backslash\,\mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{A}_f^q)\times\mathscr{L}/K^{q\diamond}_{g^q}}} m(\Lambda)[(h^q g^q, \Lambda)],$$

where

$$m(\Lambda) = \sum_{y\in\mathscr{L}^\diamond} \mathrm{mult}(\Lambda, -4q\delta_+^{\mathrm{Sie}}(y) - 4q\delta_-^{\mathrm{Sie}}(y) + (\delta_+ + \delta_-)\circ\theta^{\mathrm{Pa}}_{\mathrm{Sie}}(y)).$$

Proof. This is immediate from Lemma 7.5.5, Proposition 7.6.12, and Definition 7.6.11. ☐

To compute the multiplicities $m(\Lambda)$ in the Corollary 7.6.14 above, it is better to work with lattices in the split 5-dimensional quadratic space $V := V_{D/q}\otimes\mathbb{Q}_q$ rather than the symplectic space W (see (6.5.1)).

Definition 7.6.15.

(1) A vertex lattice $L\subset V$ is a $\mathbb{Z}_q$-lattice satisfying

$$L^\vee \supset L \supset qL^\vee.$$

For $0\le i\le 2$, set

$$\mathrm{VL}(2i) := \left\{\text{vertex lattices } L\subset V \ :\ \dim_{\mathbb{F}_q} L^\vee/L = 2i\right\}.$$

(The analogous sets $\mathrm{VL}(2i+1)$ are all empty.)

(2) For any $\mathbb{Z}_q$-lattice $\Lambda\subset W$, we define

$$L_\Lambda = \mathrm{End}_{\mathbb{Z}_q}(\Lambda)\cap V,$$

which makes sense because $V = \mathrm{End}(W)^{\dagger=1,\mathrm{tr}=0}$.

Lemma 7.6.16. *The map $\Lambda\mapsto L_\Lambda$ induces an isomorphism*

$$\mathscr{L}/q^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{VL}(0)$$

and a surjection

$$\mathscr{L}_{\mathrm{Pa}}/q^{\mathbb{Z}} \twoheadrightarrow \mathrm{VL}(4).$$

Moreover, the map

$$(\Lambda_+, \Lambda_-)\mapsto L_{\Lambda_+}\cap L_{\Lambda_-}$$

induces a surjection

$$\mathscr{L}_{\mathrm{Sie}}/q^{\mathbb{Z}} \twoheadrightarrow \mathrm{VL}(2).$$

Proof. For each $0\le i\le 2, \mathrm{VL}(2i)$ is a homogeneous space for $\mathrm{GSpin}(V)(\mathbb{Q}_q)$. Since the maps

$$\mathscr{L}/q^{\mathbb{Z}} \to \mathrm{VL}(0)$$
$$\mathscr{L}_{\mathrm{Sie}}/q^{\mathbb{Z}} \to \mathrm{VL}(2)$$
$$\mathscr{L}_{\mathrm{Pa}}/q^{\mathbb{Z}} \to \mathrm{VL}(4)$$

are all $\mathrm{GSpin}(V)(\mathbb{Q}_q)$-equivariant, they are automatically surjective. Finally, for the injectivity of the first map, it suffices to note that the stabilizers coincide, i.e. the hyperspecial subgroup of $\mathrm{SO}(V)(\mathbb{Q}_q)$ is the image of a hyperspecial subgroup of $\mathrm{GSpin}(V)(\mathbb{Q}_q)$. ☐

Lemma 7.6.17. *The projections of Lemma 7.6.16 fit into the following commutative diagrams:*

$$\begin{tikzcd}
\mathscr{L}_{\mathrm{Pa}} \arrow[r, "\delta_+ + \delta_-"] \arrow[d] & \mathbb{Z}[\mathscr{L}] \arrow[d] & \mathscr{L}_{\mathrm{Sie}} \arrow[r, "\delta_+^{\mathrm{Sie}} + \delta_-^{\mathrm{Sie}}"] \arrow[d] & \mathbb{Z}[\mathscr{L}] \arrow[d] & \mathscr{L}_{\mathrm{Sie}} \arrow[r, "\theta_{\mathrm{Sie}}^{\mathrm{Pa}}"] \arrow[d] & \mathbb{Z}[\mathscr{L}_{\mathrm{Pa}}] \arrow[d] \\
\mathrm{VL}(4) \arrow[r, "\overline{\delta}"] & \mathbb{Z}[\mathrm{VL}(0)] & \mathrm{VL}(2) \arrow[r, "\overline{\delta}^{\mathrm{Sie}}"] & \mathbb{Z}[\mathrm{VL}(0)] & \mathrm{VL}(2) \arrow[r, "\overline{\theta}_{\mathrm{Sie}}^{\mathrm{Pa}}"] & \mathbb{Z}[\mathrm{VL}(4)]
\end{tikzcd}$$

Here, the bottom maps are:

$$\overline{\delta}(L_4) = \sum_{\substack{L_0 \in \mathrm{VL}(0) \\ L_0 \supset L_4}} [L_0], \qquad \textit{for } L_4 \in \mathrm{VL}(4),$$

$$\overline{\delta}^{\mathrm{Sie}}(L_2) = \sum_{\substack{L_0 \in \mathrm{VL}(0) \\ L_0 \supset L_2}} [L_0], \qquad \textit{for } L_2 \in \mathrm{VL}(2),$$

$$\overline{\theta}_{\mathrm{Sie}}^{\mathrm{Pa}}(L_2) = \sum_{\substack{L_4 \in \mathrm{VL}(4) \\ L_4 \subset L_2}} [L_4], \qquad \textit{for } L_2 \in \mathrm{VL}(2).$$

Proof. Given $\Lambda_{\mathrm{Pa}} \in \mathscr{L}_{\mathrm{Pa}}$ with

$$q^{n+1}\Lambda_{\mathrm{Pa}}^\vee \subset_2 \Lambda_{\mathrm{Pa}} \subset_2 q^n \Lambda_{\mathrm{Pa}}^\vee$$

and $\Lambda \in \mathscr{L}$ with $\Lambda \in \delta_+(\Lambda_{\mathrm{Pa}})$, we first claim $L_\Lambda \supset L_{\Lambda_{\mathrm{Pa}}}$. Indeed, for any $\ell \in L_{\Lambda_{\mathrm{Pa}}}$, ℓ induces a self-adjoint, trace-zero endomorphism of the two-dimensional symplectic space $q^n\Lambda_{\mathrm{Pa}}^\vee/\Lambda_{\mathrm{Pa}}$; hence $\ell(q^n\Lambda_{\mathrm{Pa}}^\vee) \subset \Lambda_{\mathrm{Pa}}$. Since Λ fits into a chain

$$\Lambda_{\mathrm{Pa}} \subset_1 \Lambda = q^n\Lambda^\vee \subset_1 q^n\Lambda_{\mathrm{Pa}}^\vee,$$

we conclude

$$\ell(\Lambda) \subset \Lambda_{\mathrm{Pa}} \subset \Lambda,$$

so $\ell \in L_\Lambda$. Similarly, we see that, for $\Lambda \in \mathscr{L}$ appearing in $\delta_-(\Lambda_{\mathrm{Pa}})$, we have $L_\Lambda \supset L_{\Lambda_{\mathrm{Pa}}}$. Now to prove the first diagram commutes, it suffices to show

$$\deg \overline{\delta}(L_{\Lambda_{\mathrm{Pa}}}) = \deg(\delta_+(\Lambda_{\mathrm{Pa}})) + \deg(\delta_-(\Lambda_{\mathrm{Pa}})). \tag{7.25}$$

The left-hand side is the number of lattices $L \in \mathrm{VL}(0)$ containing $L_{\Lambda_{\mathrm{Pa}}}$; such lattices are in bijection with isotropic planes in the split 4-dimensional $\mathbb{F}_q$-quadratic space $L_{\Lambda_{\mathrm{Pa}}}^\vee/L_{\Lambda_{\mathrm{Pa}}}$, hence there are $2(q+1)$ of them. Meanwhile $\deg(\delta_\pm(\Lambda_{\mathrm{Pa}})) = q+1$, since lattices $\Lambda \supset_1 \Lambda_{\mathrm{Pa}}$ (resp. $\Lambda \subset_1 \Lambda_{\mathrm{Pa}}$) are in bijection with lines in the 2-dimensional $\mathbb{F}_q$-symplectic space $q^n\Lambda_{\mathrm{Pa}}^\vee/\Lambda_{\mathrm{Pa}}$ (resp. $\Lambda_{\mathrm{Pa}}/q^{n+1}\Lambda_{\mathrm{Pa}}^\vee$). This proves (7.25), hence the commutativity of the first diagram; and the rest are similar. ☐

Notation 7.6.18.

(1) For convenience, we now abbreviate

$$V_{\star,q} = V_\star \otimes \mathbb{Q}_q.$$

Let $L_\star \subset V_{\star,q}$ be the $\mathbb{Z}_q$-lattice spanned by $\{e_1^\star, qe_2^\star\}$, and let $L_\star^{(0)} = \mathrm{Span}_{\mathbb{Z}_q}\left\{q^{-1}e_1^T, qe_2^T\right\}$ and $L_\star^{(1)} = \mathrm{Span}_{\mathbb{Z}_q}\left\{e_1^T, e_2^T\right\}$ be the two self-dual lattices in $V_{\star,q}$ containing $L_\star$.

(2) We write $\mathrm{VL}(2)^\diamond \subset \mathrm{VL}(2)$ for the subset consisting of lattices of the form

$$L_2 = L_\star \oplus L_\diamond,$$

where

$$L_\diamond \subset V_\diamond := V_{d/q}^\diamond \otimes \mathbb{Q}_q$$

is self-dual.

Lemma 7.6.19. *$\mathscr{L}^\diamond$ consists of two $\mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{Q}_q)$-orbits; if $\mathcal{O} \subset \mathscr{L}^\diamond$ is one orbit, then the map of Lemma 7.6.16 induces an isomorphism*

$$\mathcal{O}/q^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{VL}(2)^\diamond.$$

Proof. By [12, Théorème 9.3] and Remark 7.6.5, $\mathscr{L}^\diamond$ consists of two $\mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{Q}_q)$-orbits, and the stabilizer of any point in $\mathscr{L}^\diamond$ is a hyperspecial subgroup. On the other hand, for any point in $\mathscr{L}^\diamond$ with image $L_2 \in \mathrm{VL}(2)$, we know

$$L_2 \cap V_{\star,q} \supset L_\star$$

by Proposition 7.6.3(4) and the definition of the strata $\mathcal{M}_+(\Lambda_+)$ and $\mathcal{M}_-(\Lambda_-)$. Since the stabilizer of L_2 in $\mathrm{SO}(V^\diamond_{d/q})(\mathbb{Q}_q)$ is hyperspecial, this forces

$$L_2 = L_\star \oplus L_\diamond,$$

for some $L_\diamond \subset V_\diamond$ self-dual. Hence the image of either orbit $\mathcal{O} \subset \mathscr{L}^\diamond$ in $\mathrm{VL}(2)$ is contained in $\mathrm{VL}(2)^\diamond$; since $\mathrm{VL}(2)^\diamond$ is a single orbit, we have a surjection $\mathcal{O}/q^{\mathbb{Z}} \twoheadrightarrow \mathrm{VL}(2)^\diamond$. Finally, we see that this map is an isomorphism because the stabilizers coincide. ☐

Lemma 7.6.20. *The multiplicity $m(\Lambda)$ in Corollary 7.6.14 depends only on $L_\Lambda \in \mathrm{VL}(0)$, and is given by:*

$$m(\Lambda) = \begin{cases} 4, & \text{if } L_\Lambda \cap V_{\star,q} = L_\star, \\ 4-4q, & \text{if } L_\Lambda \cap V_{\star,q} = L_\star^{(0)} \text{ or } L_\star^{(1)}, \\ 0, & \text{else.} \end{cases}$$

Proof. Let $\mathcal{O}$ be one of the two $\mathrm{GSpin}(V^\diamond_{d/q})(\mathbb{Q}_q)$-orbits in $\mathscr{L}^\diamond$; we will compute

$$m_{\mathcal{O}}(\Lambda) = \sum_{y\in\mathcal{O}} \mathrm{mult}\left(\Lambda, -4q\delta_+^{\mathrm{Sie}}(y) - 4q\delta_-^{\mathrm{Sie}}(y) + (\delta_+ + \delta_-)\circ\theta^{\mathrm{Pa}}_{\mathrm{Sie}}(y)\right).$$

Let $m(\Lambda, y)$ be the summand above, so that $m_{\mathcal{O}}(\Lambda) = \sum_{y\in\mathcal{O}} m(\Lambda, y)$. Fix some $y \in \mathcal{O}$ corresponding to $L_y \in \mathrm{VL}(2)$. Then

$$m(\Lambda, q^n y) = m(q^{-n}\Lambda, y)$$

is nonzero for at most one n, so

$$\sum_{n\in\mathbb{Z}} m(\Lambda, q^n y) = \mathrm{mult}(L_\Lambda, -4q\overline{\delta}^{\mathrm{Sie}}(L_y) + \overline{\delta}\circ\overline{\theta}^{\mathrm{Pa}}_{\mathrm{Sie}}(L_y))$$

by Lemma 7.6.17. Since $\mathcal{O}/q^{\mathbb{Z}}$ maps isomorphically to $\mathrm{VL}(2)^\diamond$ by Lemma 7.6.19, we find

$$m_{\mathcal{O}}(\Lambda) = \sum_{L_2\in\mathrm{VL}(2)^\diamond} \mathrm{mult}\left(L_\Lambda, -4q\overline{\delta}^{\mathrm{Sie}}(L_2) + \overline{\delta}\circ\overline{\theta}^{\mathrm{Pa}}_{\mathrm{Sie}}(L_2)\right).$$

Next, we calculate, for any $L_2 \in \mathrm{VL}(2)$,

$$\begin{aligned}\overline{\delta}\circ\overline{\theta}^{\mathrm{Pa}}_{\mathrm{Sie}}(L_2) &= \sum_{\substack{L_4\in\mathrm{VL}(4)\\ L_4\subset L_2}} \sum_{\substack{L_0\in\mathrm{VL}(0)\\ L_0\supset L_4}} [L_0] \\ &= \sum_{\substack{L_0\in\mathrm{VL}(0)\\ L_0\supset L_2}} \#\{L_4\in\mathrm{VL}(4) : L_4\subset L_2\} + \sum_{\substack{L_0\in\mathrm{VL}(0)\\ L_0\cap L_2\in\mathrm{VL}(4)}} [L_0] \\ &= (q+1)\overline{\delta}^{\mathrm{Sie}}(L_2) + \sum_{\substack{L_0\in\mathrm{VL}(0)\\ L_0\cap L_2\in\mathrm{VL}(4)}} [L_0].\end{aligned}$$

(The $q+1$ choices of $L_4 \subset L_2$ correspond to flags

$$qL_2^\vee \subset qL_4^\vee \subset L_4 \subset L_2,$$

hence to complete isotropic flags in the split 3-dimensional $\mathbb{F}_q$-quadratic space $L_2/qL_2^\vee$.) Hence

$$m_{\mathcal{O}}(\Lambda) = \sum_{L_2 \in \mathrm{VL}(2)^\diamond} \mathrm{mult}\left(L_\Lambda, (1-3q)\overline{\delta}^{\mathrm{Sie}}(L_2) + \sum_{\substack{L_0 \in \mathrm{VL}(0)\\ L_0 \cap L_2 \in \mathrm{VL}(4)}} [L_0] \right). \tag{7.26}$$

Now observe that, for any $L_2 \in \mathrm{VL}(2)^\diamond$ and $L_4 \in \mathrm{VL}(4)$ with $L_4 \subset L_2$, we have

$$L_4 \supset qL_4^\vee \supset qL_2^\vee \supset L_\star.$$

Hence, if $L_2 \in \mathrm{VL}(2)^\diamond$, all L_0 appearing in

$$(1-3q)\overline{\delta}^{\mathrm{Sie}}(L_2) + \sum_{\substack{L_0 \in \mathrm{VL}(0)\\ L_0 \cap L_2 \in \mathrm{VL}(4)}} [L_0]$$

have $L_0 \cap V_{\star,q} \supset L_\star$. In particular, if $m_{\mathcal{O}}(\Lambda) \neq 0$, then

$$L_\Lambda \cap V_{\star,q} = L_\star^{(0)},\ L_\star^{(1)},\ \text{or}\ L_\star,$$

since these are the only lattices containing $L_\star$ on which the pairing is $\mathbb{Z}_q$-valued.

Suppose first $L_\Lambda \cap V_{\star,q} = L_\star^{(i)}$, for $i = 0$ or 1. Then we may write

$$L_\Lambda = L_\diamond \oplus L_\star^{(i)}$$

with $L_\diamond \subset V_\diamond$ self-dual, and

$$m_{\mathcal{O}}(\Lambda) = (1-3q)\#\left\{L_2 \in \mathrm{VL}(2)^\diamond \, : \, L_\Lambda \in \overline{\delta}^{\mathrm{Sie}}(L_2)\right\} + \#\left\{L_2 \in \mathrm{VL}(2)^\diamond \, : \, L_2 \cap L_\Lambda \in \mathrm{VL}(4)\right\}.$$

Recall that all $L_2 \in \mathrm{VL}(2)^\diamond$ are of the form $L'_\diamond \oplus L_\star$, with $L'_\diamond \subset V_\diamond$ self-dual, and

$$\overline{\delta}^{\mathrm{Sie}}(L'_\diamond \oplus L_\star) = \left[L'_\diamond \oplus L_\star^{(0)}\right] + \left[L'_\diamond \oplus L_\star^{(1)}\right].$$

Hence

$$\begin{aligned} m_{\mathcal{O}}(\Lambda) &= (1-3q) + \#\left\{L'_\diamond \subset V_\diamond \,\text{self-dual} \, : \, (L'_\diamond \cap L_\diamond) \oplus L_\star \in \mathrm{VL}(4)\right\} \\ &= (1-3q) + (q+1) \\ &= 2 - 2q; \end{aligned}$$

the $(q+1)$ choices of $L'_\diamond$ correspond bijectively to the isotropic lines in the 3-dimensional $\mathbb{F}_q$-space $L_\diamond/qL_\diamond$.

Now suppose $L_\Lambda \cap V_{\star,q} = L_\star$. Then L_Λ does not appear in $\overline{\delta}^{\mathrm{Sie}}(L_2)$ for any $L_2 \in \mathrm{VL}(2)^\diamond$, so by (7.26), we have

$$m_{\mathcal{O}}(\Lambda) = \#\left\{L_2 \in \mathrm{VL}(2)^\diamond \, : \, L_\Lambda \cap L_2 \in \mathrm{VL}(4)\right\}.$$

Recall that any L_4 contained in $L_2 \in \mathrm{VL}(2)^\diamond$ satisfies $qL_4^\vee \supset L_\star$; we first show that there is a unique such L_4 contained in L_Λ. Indeed, any such L_4 fits in a chain

$$qL_\Lambda \subset_2 qL_4^\vee \subset_1 L_4 \subset_2 L_\Lambda,$$

and so we must have

$$qL_4^\vee = L_\star + qL_\Lambda,$$

which determines L_4. Now, for this L_4, we claim there are exactly two $L_2 \in \mathrm{VL}(2)^\diamond$ with $L_2 \supset L_4$. Such an L_2 fits into a chain

$$L_4 \subset_1 L_2 \subset_2 L_2^\vee \subset_1 L_4^\vee.$$

Since $L_\star \subset qL_4^\vee$, we have

$$\frac{1}{q}L_\star \subset L_4^\vee.$$

Thus

$$L_4^\vee/L_4 \cong \frac{1}{q}L_\star/L_\star \oplus \mathbb{H},$$

with $\mathbb{H}$ a hyperbolic plane over $\mathbb{F}_q$. For L_2 to lie in $\mathrm{VL}(2)^\diamond$ is equivalent to $\frac{1}{q}L_\star \subset L_2^\vee$, so choices of L_2 correspond to choices of isotropic lines in $L_4^\vee/L_4$ orthogonal to $\frac{1}{q}L_\star/L_\star$; these are isotropic lines in $\mathbb{H}$, so there are exactly two. Hence $m_{\mathcal{O}}(\Lambda) = 2$ if $L_\Lambda \cap V_{\star,q} = L_\star$. We have now shown

$$m_{\mathcal{O}}(\Lambda) = \begin{cases} 2, & \text{if } L_\Lambda \cap V_{\star,q} = L_\star, \\ 2-2q, & \text{if } L_\Lambda \cap V_{\star,q} = L_\star^{(0)} \text{or } L_\star^{(1)}, \\ 0, & \text{else.} \end{cases}$$

Since $m_{\mathcal{O}}(\Lambda)$ is evidently independent of the choice of orbit $\mathcal{O}$, we have

$$m(\Lambda) = 2m_{\mathcal{O}}(\Lambda),$$

which completes the proof. □

Combining Lemma 7.6.20 and Corollary 7.6.14 gives the following simple formula.

Corollary 7.6.21. *We have*

$$\begin{aligned}\nabla\left[\widetilde{Z}(g^q, V_\star, V_D)_{\overline{\mathbb{F}}_q}\right] &= 4Z(g^q g_q^\star, V_\star, V_{D/q})_{K^q K_q} \\ &\quad + 4(1-q)\left(Z(g^q g^{(0)}, V_\star, V_{D/q})_{K^q K_q}) + Z(g^q g^{(1)}, V_\star, V_{D/q})_{K^q K_q}\right) \in O\left[\mathrm{Sh}_{K^q K_q}(V_{D/q})\right],\end{aligned}$$

where $g_q^\star$, $g_q^{(0)}$, $g_q^{(1)} \in \mathrm{GSpin}(V_{D/q})(\mathbb{Q}_q)$ represent the cosets in

$$\mathrm{GSpin}(V_{d/q}^\diamond)(\mathbb{Q}_q)\backslash \mathrm{GSpin}(V_{D/q})(\mathbb{Q}_q)/K_q$$

corresponding to the lattices $L_\star$, $L_\star^{(0)}$, $L_\star^{(1)}$ under Proposition 3.1.8.

7.7. **Interpretation in terms of test functions.**

7.7.1. We now define a specific test function $\varphi_q^{\mathrm{tot}} \in \mathcal{S}(V_{D/q}^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ as follows. First, define a subset $X \subset V_{D/q}^2 \otimes \mathbb{Q}_q$ by

$$X = \left\{(x,y) \in V_{D/q}^2 \otimes \mathbb{Q}_q \,|\, x\cdot x \in q\mathbb{Z}_q,\, y \cdot y \in q\mathbb{Z}_q,\, x \cdot y \in \mathbb{Z}_q^\times\right\}. \tag{7.27}$$

Let $L \subset V_{D/q} \otimes \mathbb{Q}_q$ be a self-dual lattice, and let $\varphi_q^{(0)}$, $\varphi_q^{(1)}$, $\varphi_q^\star \in \mathcal{S}(V_{D/q}^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ be indicator functions of the following compact open subsets of $V_{D/q}^2 \otimes \mathbb{Q}_q$:

$$\begin{aligned} X^{(0)} &= \{(x,y) \in X \,:\, x, y \in L - qL\} \\ X^{(1)} &= \left\{(x,y) \in X \,:\, x \in qL - q^2L,\, y \in q^{-1}L - L\right\} \\ X^\star &= \left\{(x,y) \in X \,:\, x \in L - qL,\, y \in q^{-1}L - L\right\}. \end{aligned}$$

Let

$$\varphi_q^{\mathrm{tot}} = \varphi_q^\star + (1-q)\left(\varphi_q^{(0)} + \varphi_q^{(1)}\right).$$

Corollary 7.6.21 can then be reformulated as follows:

Theorem 7.7.2. *Let $K = \prod K_v \subset \mathrm{GSpin}(V_{D/q})(\mathbb{A}_f)$ be neat with K_q hyperspecial, and let*

$$\varphi^q \in \mathcal{S}(V_{D/q}^2 \otimes \mathbb{A}_f^q, O)$$

be a K^q-invariant Schwartz function. Let $\mathfrak{m} \subset \mathbb{T}_O^S$ be a generic, non-Eisenstein, and weakly q-generic maximal ideal.

Then, for all $T \in \mathrm{Sym}^2(\mathbb{Q})_{\geq 0}$, there exists a choice of uniformization datum for V_D and a test function $\varphi_q^{\mathrm{ram}} \in \mathcal{S}(V_D^2 \otimes \mathbb{Q}_q, \mathbb{Z})^{K_q^{\mathrm{ram}}}$ such that

$$\nabla \circ \zeta \circ \mathrm{res}_{\mathbb{Q}_q} \circ \partial_{\mathrm{AJ},\mathfrak{m}} \left(Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}})_{K^q K_q^{\mathrm{ram}}} \right) = 4 Z(T, \varphi^q \otimes \varphi_q^{\mathrm{tot}})_{K^q K_q} \in \frac{O \left[\mathrm{Sh}_{K^q K_q}(V_{D/q})(V_Q) \right]_{\mathfrak{m}}}{(\langle q \rangle^{-1} - 1, \mathrm{T}_q^{\mathrm{lr}})}.$$

Remark 7.7.3.

(1) The map $\partial_{\mathrm{AJ},\mathfrak{m}}$ is defined as in (4.4.2), and we are using Theorem 7.1.11 to apply ζ on the left-hand side of the identity.

(2) The choice of uniformization datum intervenes in two points in the displayed equation: first in the definition of ∇, and second in the isomorphism $V_{D/q} \otimes \mathbb{A}_f^q \cong V_D \otimes \mathbb{A}_f^q$ from Remark 1.4.5(2), which we are using to view φ^q as a test function in $\mathcal{S}(V_D^2 \otimes \mathbb{A}_f^q, O)$ and $K^q K_q^{\mathrm{ram}}$ as a compact open subgroup of $\mathrm{GSpin}(V_D)(\mathbb{A}_f)$.

(3) Without any great difficulty, φ_q^{ram} can be chosen not to depend on T. Since this is not used in the proofs of the main results, we omit the details.

Proof. Without loss of generality, we may assume T is of the form considered in Notation 7.6.1; otherwise $Z(T, \varphi^q \otimes \varphi_q^{\mathrm{tot}})_K = 0$, so $\varphi_q^{\mathrm{ram}} = 0$ satisfies the conclusion of the theorem. We fix the uniformization datum as in Construction 7.6.4(1); in particular, we are identifying $V_\star$ with a subspace of V_D, so we may choose $\varphi_q^{\mathrm{ram}} \in \mathcal{S}(V_D^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ such that $\varphi_q^{\mathrm{ram}}|_{\Omega_{T,V_D}(\mathbb{Q}_q)}$ is the indicator function of the coset $K_q^{\mathrm{ram}} \cdot (e_1^\star, e_2^\star)$. For any $\varphi^q \in \mathcal{S}(V_{D/q}^2 \otimes \mathbb{A}_f^q, O)$, write

$$\mathrm{supp}(\varphi^q) \cap \Omega_{T,V_{D/q}}(\mathbb{A}_f^q) = \sqcup\, \mathrm{GSpin}(V_{D/q}^\diamond)(\mathbb{A}_f^q) g_i^q K^q.$$

Then we have

$$Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}})_{K^q K_q^{\mathrm{ram}}} = \sum_i Z(g_i^q, V_\star, V_D)_{K^q K_q^{\mathrm{ram}}} \varphi^q((g_i^q)^{-1} e_1^\star, (g_i^q)^{-1} e_2^\star).$$

Now, by Theorem 7.1.11, we conclude that

$$\nabla \circ \zeta \circ \mathrm{res}_{\mathbb{Q}_q} \circ \partial_{\mathrm{AJ},\mathfrak{m}} \left(Z(T, \varphi^q \otimes \varphi_q^{\mathrm{ram}})_{K^q K_q^{\mathrm{ram}}} \right) \in \frac{O \left[\mathrm{Sh}_{K^q K_q}(V_{D/q})(V_Q) \right]_{\mathfrak{m}}}{(\langle q \rangle^{-1} - 1, \mathrm{T}_q^{\mathrm{lr}})}$$

coincides with

$$\sum_i \varphi^q((g_i^q)^{-1} e_1^\star, (g_i^q)^{-1} e_2^\star) \cdot \nabla \left[\widetilde{Z}(g_i^q, V_\star, V_D)_{K^q K_q^{\mathrm{ram}}} \right].$$

Then the theorem follows from Corollary 7.6.21 and the construction of φ_q^{tot}. □

8. First explicit reciprocity law

For this section, let π, S, and E_0 be as in Notation 4.0.1, fix a prime $\mathfrak{p}$ of E_0 satisfying Assumption 4.1.1, and put $\mathfrak{m} := \mathfrak{m}_{\pi,\mathfrak{p}}$ as usual. Our goal for this section is to combine Theorem 7.7.2 with Corollary 5.6.3 to prove Theorems 8.5.1 and 8.5.2 below. First we make some deformation-theoretic preparations in §8.1-§8.3; then we check the criterion from Corollary 5.6.3 in §8.4; and we complete the proofs in §8.5.

8.1. **Typic modules.** The following definition is a generalization of [98, Definition 5.2].

Definition 8.1.1. Let G be a group, R a Noetherian local ring with maximal ideal $\mathfrak{m}_R$, and $\sigma_1, \cdots, \sigma_m$ a finite collection of representations $\sigma_i : G \to \mathrm{GL}_{n_i}(R)$ such that the residual representations $\overline{\sigma}_i := \sigma \otimes_R R/\mathfrak{m}_R$ are distinct and absolutely irreducible for each $i = 1, \cdots, m$.

An $R[G]$-module M is called σ_i-typic if it is isomorphic to $\sigma_i \otimes_R M_0$ for an R-module M_0, and $(\sigma_1, \cdots, \sigma_m)$-typic if it is isomorphic to a direct sum $\oplus M_i$ with each M_i σ_i-typic.

Proposition 8.1.2. *With notation as in Definition 8.1.1, let $N \subset M$ be an inclusion of $R[G]$-modules.*

1. *If M is σ_i-typic for some $i \in \{1, \cdots, m\}$, then N is σ_i-typic.*
2. *If M is $(\sigma_1, \cdots, \sigma_m)$-typic, then N is $(\sigma_1, \cdots, \sigma_m)$-typic.*

Proof. Part (1) is proved in [98, Proposition 5.4]. For (2), as in *loc. cit.* we may assume without loss of generality that M and N are finitely generated. Let $M = \oplus M_i$ be the decomposition of M into σ_i-typic parts, and let $\pi_i : M \to M_i$ be the projection map for each $i = 1, \cdots, m$. Without loss of generality, we may assume $\pi_i(N) = M_i$. We claim that the natural injection

$$\oplus \pi_i : N \hookrightarrow \oplus M_i$$

is an isomorphism, i.e. $\oplus \pi_i$ is surjective. Let $\overline{N} := N \otimes_R R/\mathfrak{m}_R$ and $\overline{M}_i := M_i \otimes_R R/\mathfrak{m}_R$; by Nakayama's lemma, it suffices to show the induced map

$$\oplus \overline{\pi}_i : \overline{N} \to \oplus \overline{M}_i$$

is surjective. Because the residual representations $\overline{\sigma}_i$ are all distinct, any $R[G]$-stable submodule of $\oplus \overline{M}_i$ is a direct sum $\oplus \overline{M}_i'$ for some $\overline{M}_i' \subset \overline{M}_i$. So if $\oplus \overline{\pi}_i$ is not surjective, then $\overline{\pi}_i$ is not surjective for some $i = 1, \cdots, m$; but this contradicts our assumption that $\pi_i(N) = M_i$ for all i, so $\oplus \pi_i$ is surjective, which shows (2). ☐

8.2. **Deformation theory: non-endoscopic case.** We assume for this subsection that π is not endoscopic. We will apply the results and notations of Appendix B to $\rho_\pi = \rho_{\pi,\mathfrak{p}}$, which we view as valued in $\mathrm{GSp}_4(O)$ via Remark 4.1.4. First note:

Lemma 8.2.1. *The representation ρ_π satisfies Assumptions B.1.3 and B.1.5 from Appendix B.*

Proof. By Assumption 4.1.1, $\overline{\rho}_\pi$ is absolutely irreducible, so $H^0(\mathbb{Q}, \mathrm{ad}^0 \overline{\rho}_\pi) = 0$. Also, $\overline{\rho}_\pi \not\cong \overline{\rho}_\pi(1)$ by considering the similitude characters (since $p > 3$). So, again using the absolute irreducibility,

$$H^0(\mathbb{Q}, \mathrm{ad}^0 \overline{\rho}_\pi(1)) \subset \mathrm{Hom}_{G_\mathbb{Q}}(\overline{T}_\pi, \overline{T}_\pi(1)) = 0$$

as well. This shows Assumption B.1.3(1). Assumptions B.1.3(2,3) are clear from Theorem 2.2.10. We now consider Assumption B.1.5. By Lemma B.1.6 it suffices to show $H^0(\mathbb{Q}_v, \mathrm{WD}(\mathrm{ad}^0 \rho_\pi)) = 0$ for all non-archimedean v. But

$$H^0(\mathbb{Q}_v, \mathrm{WD}(\mathrm{ad}^0 \rho_\pi)) \subset \mathrm{Hom}_{\mathrm{WD}_v}(\mathrm{WD}(V_\pi), \mathrm{WD}(V_\pi(1))),$$

with WD_v the local Weil-Deligne group, which vanishes by purity (Theorem 2.2.10(1)). ☐

8.2.2. Suppose q is an admissible prime, and let $\mathcal{D}_q$ and R_q be as in Notation B.1.4. For any $A \in \mathrm{CNL}_O$ and $\rho_A \in \mathcal{D}_q(A)$, let M_A be the free, rank-four A-module with $G_{\mathbb{Q}_q}$ action determined by ρ_A. Then by [65, Lemma 3.21], M_A admits a $G_{\mathbb{Q}_q}$-stable decomposition

$$M_A = M_0 \oplus M_1, \tag{8.1}$$

where:

- Each of M_0 and M_1 is free of rank two over A.
- $M_0 \otimes_A k$ has Frob_q eigenvalues 1 and q.

- $M_1 \otimes_A k$ has generalized Frob_q eigenvalues a and q/a, with $a \neq 1, q$.

(Here $k = O/\varpi$, and $\mathrm{Frob}_q \in G_{\mathbb{Q}_q}$ is any lift of Frobenius.)

Definition 8.2.3. Let $\mathcal{D}_q^{\mathrm{ord}} \subset \mathcal{D}_q$ be the subfunctor of lifts ρ_A such that

$$\det(\rho_A(\mathrm{Frob}_q) - T|M_0) = (1-T)(q-T).$$

Lemma 8.2.4. *The functor $\mathcal{D}_q^{\mathrm{ord}}$ is represented by a formally smooth quotient R_q^{ord} of R_q, of relative dimension 10 over O.*

Proof. This follows the same argument as [65, Proposition 3.35] (cf. also [121, Proposition 3.8]). ☐

8.2.5. In light of Lemma 8.2.4, we take the admissible primes in Notation B.2.4 to be the ones of Definition 4.2.1, and the notion of R_q^{ord} to be the one from Lemma 8.2.4; the definitions of n-admissible primes in Definition 4.2.1 and Definition B.2.5(3) then coincide.

Notation 8.2.6.

1. Let $Q \geq 1$ be admissible, and denote by $\widetilde{R}_{\mathfrak{m}}^{Q}$ the global GSp_4-valued deformation ring of $\overline{\rho}_\pi$ as a representation of $G_{\mathbb{Q},S\cup\mathrm{div}(Qp)}$, with fixed similitude character χ_p^{cyc}. Let R^Q and R_Q be the quotients of $\widetilde{R}_{\mathfrak{m}}^{Q}$ defined in Notation B.4.4 (identifying Q with $\mathrm{div}(Q)$ for notational convenience).
2. Let
$$\rho_Q^{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(R_Q)$$
be a framing of the universal deformation, and let M_Q^{univ} be the free R_Q-module of rank four with $G_{\mathbb{Q}}$-action defined by ρ_Q^{univ}.

8.2.7. Let $\mathrm{pr}_p : I_{\mathbb{Q}_q} \to \mathbb{Z}_p(1)$ be the maximal pro-p quotient.

Lemma 8.2.8. *Suppose Q is admissible and $q|Q$ is a prime. In the decomposition $M_Q^{\mathrm{univ}}|_{G_{\mathbb{Q}_q}} = M_0 \oplus M_1$ of (8.1), M_1 is unramified. Moreover, there exists a basis of M_0 and an element $t_q \in R_Q$ such that*

$$\rho_Q^{\mathrm{univ}}|_{M_0} = \begin{pmatrix} \chi_{p,\mathrm{cyc}} & * \\ 0 & 1 \end{pmatrix},$$

and

$$\rho_Q^{\mathrm{univ}}(g)|_{M_0} = \begin{pmatrix} 1 & t_q\,\mathrm{pr}_p(g) \\ 0 & 1 \end{pmatrix}, \ \forall g \in I_{\mathbb{Q}_q}.$$

Proof. Since M_0 and M_1 are $G_{\mathbb{Q}_q}$-stable, this follows from [102, Propositions 5.3, 5.5]. ☐

Definition 8.2.9. Suppose Q and $q \nmid Q$ are admissible. Then:

1. We set $P_q(T) = \det(\rho_Q^{\mathrm{univ}}(\mathrm{Frob}_q) - T|M_Q^{\mathrm{univ}}) \in R_Q[T]$.
2. We set $R_{Q,q}^{\mathrm{cong}} := R_Q \otimes_{R^{Qq}} R_{Qq}$.

Lemma 8.2.10. *Suppose $Q \geq 1$ is admissible, and $q \nmid Q$ is an admissible prime. Then*

$$R_{Q,q}^{\mathrm{cong}} = R_Q/(P_q(q)) = R_{Qq}/(t_q).$$

Proof. We have $R_{Q,q}^{\mathrm{cong}} = R_Q/(P_q(q))$ because an unramified deformation of $\overline{\rho}_\pi|_{G_{\mathbb{Q}_q}}$ is ordinary if and only if q is an eigenvalue of Frob_q; on the other hand, it is clear from Lemma 8.2.8 that $R_{Q,q}^{\mathrm{cong}} = R_{Qq}/(t_q)$. ☐

Lemma 8.2.11. *Suppose q is n-admissible. Then:*

(1) $H^1_{\mathrm{ord}}(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi,n}) + H^1_{\mathrm{unr}}(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi,n}) = H^1(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi,n})$.

(2) The quotients

$$\frac{H^1(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}, \quad \frac{H^1(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}$$

are both free of rank one over O/ϖ^n.

In particular, q is standard in the sense of Definition B.4.7.

Proof. First note that

$$\frac{H^1(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})} = \mathrm{Hom}_{\mathrm{Frob}_q}(\mathbb{Z}_p(1), \operatorname{ad}^0 \rho_{\pi,n})$$

is free of rank one over O/ϖ^n since q is n-admissible. On the other hand, $H^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})$ clearly surjects onto this quotient by definition, which shows (1). Since $Z^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})$ and $Z^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})$ both contain all coboundaries, we see that

$$\frac{Z^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{Z^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n}) \cap Z^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})} = \frac{H^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n}) \cap H^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})} = \frac{H^1(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}$$

is free of rank 1 over O/ϖ^n. On the other hand, $Z^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})$ and $Z^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})$ are both free of rank 10 over O/ϖ^n because R_q^{ord} and the unramified local deformation ring R_q^{unr} are formally smooth, so we conclude that $Z^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n}) \cap Z^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})$ is free of rank 9 over O/ϖ^n. In particular,

$$\frac{Z^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{Z^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n}) \cap Z^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})} = \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n}) \cap H^1_{\mathrm{unr}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})} = \frac{H^1(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}{H^1_{\mathrm{ord}}(\mathbb{Q}_q, \operatorname{ad}^0 \rho_{\pi,n})}$$

is also free of rank one over O/ϖ^n, as desired. □

8.2.12. To state the next lemma, we establish some temporary notation. Suppose Q is admissible and let K be an S-tidy level structure K for $\mathrm{GSpin}(V_{DQ})$ (Definition 4.3.1). Abbreviate $\mathbb{T} := \mathbb{T}^{S\cup\mathrm{div}(Q)}_{K,V_{DQ},O,\mathfrak{m}}$, which may be the zero ring. Also fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ inducing $\mathfrak{p}$. Then we write $\mathcal{T}$ for the set of relevant automorphic representations Π of $\mathrm{GSpin}(V_{DQ})(\mathbb{A})$ with $\Pi_f^K \neq 0$ such that the Hecke action on $\iota^{-1}\Pi_f^K$ factors through $\mathbb{T}$. Recall from Corollary 2.7.8 that we have an embedding of $\mathbb{T}$-algebras

$$\mathbb{T} \hookrightarrow \bigoplus_{\Pi\in\mathcal{T}} \overline{\mathbb{Q}}_p(\Pi), \tag{8.2}$$

where $\overline{\mathbb{Q}}_p(\Pi)$ is $\overline{\mathbb{Q}}_p$ with Hecke action through the eigenvalues on $\iota^{-1}\Pi_f^K$. By the same argument as [13, Theorem 7.9.4], there exists a Galois representation

$$\rho : G_{\mathbb{Q},S\cup\mathrm{div}(Qp)} \to \mathrm{GSp}_4(\mathbb{T})$$

such that, for each $\Pi \in \mathcal{T}$, the composite

$$G_{\mathbb{Q},S\cup\mathrm{div}(Qp)} \xrightarrow{\rho} \mathrm{GSp}_4(\mathbb{T}) \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p(\Pi))$$

is conjugate to the Galois representation $\rho_{\Pi,\iota}$ from Remark 2.5.4.

Lemma 8.2.13. *With notation as in (8.2.12), we have:*

(1) The composite

$$G_{\mathbb{Q}} \xrightarrow{\rho} \mathrm{GSp}_4(\mathbb{T}) \xrightarrow{\nu} \mathbb{T}^\times$$

is given by $\chi_{p,\mathrm{cyc}}$, and the corresponding O-algebra map $r_\rho : \widetilde{R}^Q_{\mathfrak{m}} \to \mathbb{T}$ factors through R_Q.

(2) Suppose $\sigma(DQ)$ is even. Then $H^3_{\mathrm{\acute{e}t}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}$ is ρ_Q^{univ}-typic when viewed as a $R_Q[G_{\mathbb{Q}}]$-module via (1).

Proof. We have $\nu \circ \rho = \chi_{p,\mathrm{cyc}}$ because each $\Pi \in \mathcal{T}$ has trivial similitude character by Lemma 4.3.2. To complete the proof of (1), it suffices to show that all the composite maps

$$\widetilde{R}_{\mathfrak{m}}^{Q} \xrightarrow{r_\rho} \mathbb{T} \to \overline{\mathbb{Q}}_p(\Pi)$$

factor through R_Q for $\Pi \in \mathcal{T}$. Because K_p is hyperspecial, each $\rho_{\Pi,\iota}|_{G_{\mathbb{Q}_p}}$ is crystalline with Hodge-Tate weights $\{-1, 0, 1, 2\}$ by Theorem 2.2.10(2). So it suffices to check that $\rho_{\Pi,\iota}|_{G_{\mathbb{Q}_q}}$ is ordinary for all $q|Q$. Indeed, $\rho_{\Pi,\iota}|_{G_{\mathbb{Q}_q}}$ is tamely ramified because $\overline{\rho}_\pi$ is unramified, but ramified by Corollary 2.5.3(2) and Theorem 2.2.10(1). In particular, for any lift of $\mathrm{Frob}_q \in G_{\mathbb{Q}_q}$, $\rho_{\Pi,\iota}(\mathrm{Frob}_q)$ has a pair of eigenvalues of ratio q. It follows that $\rho_{\Pi,\iota}|_{G_{\mathbb{Q}_q}}$ is ordinary, and this shows (1).

For (2), by Proposition 8.1.2(1) and Theorem 2.7.5(2) it suffices to show $H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(2))_{\mathfrak{m}}$ is ρ_Q^{univ}-typic. However, this is immediate from Corollary 2.7.7 and the construction of the map in (1); note that each $\Pi \in \mathcal{T}$ is non-endoscopic because $\overline{\rho}_\pi$ is absolutely irreducible. □

Definition 8.2.14. For any admissible Q with $\sigma(DQ)$ even and any S-tidy level structure K for $\mathrm{GSpin}(V_{DQ})$, we define $H_Q(K) = \mathrm{Hom}_{R_Q[G_{\mathbb{Q}}]}\left(M_Q^{\mathrm{univ}}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)$.

Remark 8.2.15. In the context of Definition 8.2.14:

(1) By [98, Proposition 5.3] and Lemma 8.2.13(2), we have

$$H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}} \simeq M_Q^{\mathrm{univ}} \otimes_{R_Q} H_Q(K)$$

as $R_Q[G_{\mathbb{Q}}]$-modules.

(2) Under the isomorphism of (1), we have, for all $q|Q$:

$$H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \simeq H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}}) \otimes_{R_Q} H_Q(K).$$

Lemma 8.2.16. *Suppose Q is admissible with $\sigma(DQ)$ even, and K is an S-tidy level structure for* $\mathrm{GSpin}(V_{DQ})$. *Then for any $q|Q$, the ϖ-power-torsion of $H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)$ is contained in*

$$H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)^{\mathrm{Frob}_q=1} \simeq H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}})^{\mathrm{Frob}_q=1} \otimes_{R_Q} H_Q(K) \simeq H_Q(K)/(t_q).$$

(The element $t_q \in R_Q$ was defined in Lemma 8.2.8.)

Proof. By Lemma 8.2.8, we see that

$$H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}}) = R_Q/(t_q) \oplus R_Q(-1) \oplus M_1(-1)$$

as R_Q-modules with Frob_q-action, with $M_0 \oplus M_1$ the decomposition of (8.1) for M_Q^{univ}. In particular, $\mathrm{Frob}_q - 1$ acts invertibly on $R_Q(-1) \oplus M_1(-1)$. Since M_1 is free over R_Q and $H_Q(K)$ is ϖ-torsion-free by Theorem 2.7.5(2), it follows that the ϖ-torsion part of $H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}}) \otimes_{R_Q} H_Q(K)$ is contained in

$$\left(H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}}) \otimes_{R_Q} H_Q(K)\right)^{\mathrm{Frob}_q=1} = H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}})^{\mathrm{Frob}_q=1} \otimes H_Q(K) = H_Q(K)/(t_q),$$

as claimed.

□

Lemma 8.2.17. *Suppose Q is n-admissible with $\sigma(DQ)$ even, and K is an S-tidy level structure for* $\mathrm{GSpin}(V_{DQ})$. *Then for all $q|Q$, all $\alpha_0 \in \mathrm{Hom}_{R_Q}(H_Q(K)/(t_q), O/\varpi^n)$, and all $z \in \mathrm{SC}^2_K(V_{DQ}, O)$, we have*

$$\alpha_0 \circ \mathrm{res}_{\mathbb{Q}_q} \circ \partial_{\mathrm{AJ},\mathfrak{m}}(z) \in \partial_q(\kappa_n(Q; K)).$$

Here $\mathrm{res}_{\mathbb{Q}_q} \circ \partial \mathrm{AJ}_{\mathfrak{m}}(z)$ is viewed as an element of $H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)^{\mathrm{Frob}_q = 1}$, which we identify with $H_Q(K)/(t_q)$ by Lemma 8.2.16; O/ϖ^n is viewed as an R_Q-algebra via the map corresponding to $\rho_{\pi,n}$; and ∂_q was defined in Notation 4.2.9.

Proof. By construction, $T_{\pi,n}$ and $M_Q^{\mathrm{univ}} \otimes_{R_Q} O/\varpi^n$ are isomorphic as $O[G_{\mathbb{Q}}]$-modules. Given a map $\alpha_0 : H_Q(K)/(t_q) \to O/\varpi^n$, we obtain a corresponding map of Galois modules

$$\alpha = \mathrm{id} \otimes \alpha_0 : H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}} \simeq M_Q^{\mathrm{univ}} \otimes_{R_Q} H_Q(K) \to M_Q^{\mathrm{univ}} \otimes_{R_Q} O/\varpi^n \simeq T_{\pi,n}.$$

Let $\alpha_* : H^1\left(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQq})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \to H^1(\mathbb{Q}, T_{\pi,n})$ be the map induced by α. For any $z \in \mathrm{SC}^2_K(V_{DQq}, O)$, $\kappa_n(Q; K)$ contains $\alpha_*(\partial_{\mathrm{AJ},\mathfrak{m}})(z) \in H^1(\mathbb{Q}, T_{\pi,n})$. So the lemma follows from the commutative diagram

$$\begin{tikzcd}
H^1\left(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \arrow[r] \arrow[d, "\alpha_*"] & H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)^{\mathrm{Frob}_q=1} \arrow[r, "\sim"] \arrow[d] & H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}})^{\mathrm{Frob}_q=1} \otimes_{R_Q} H_Q(K) \arrow[r, "\sim"] \arrow[d, "\alpha_0"] & H_Q(K)/(t_q) \arrow[d, "\alpha_0"] \\
H^1(\mathbb{Q}, T_{\pi,n}) \arrow[r] & H^1(I_{\mathbb{Q}_q}, T_{\pi,n})^{\mathrm{Frob}_q=1} \arrow[r, "\sim"] & H^1(I_{\mathbb{Q}_q}, M_Q^{\mathrm{univ}})^{\mathrm{Frob}_q=1} \otimes_{R_Q} O/\varpi^n \arrow[r, "\sim"] & O/\varpi^n.
\end{tikzcd}$$

□

8.3. **Deformation theory: endoscopic case.** For this subsection, we assume π is endoscopic, associated to a pair (π_1, π_2) of automorphic representations of GL_2 with discrete series archimedean components of weights 2 and 4, in some order. In particular, we have $\rho_\pi = \rho_{\pi_1} \oplus \rho_{\pi_2}$.

Notation 8.3.1.

(1) Set $S_\pi := T_{\pi_1} \otimes T_{\pi_2}(-1)$ with the diagonal Galois action; for any $n \geq 1$, we also write $S_{\pi,n} := S_\pi \otimes_O O/\varpi^n$. Let

$$H^1_{\mathrm{cris}}(\mathbb{Q}_p, S_{\pi,n}) \subset H^1(\mathbb{Q}_p, S_{\pi,n}) \tag{8.3}$$

be the subspace of cocycles corresponding to extensions

$$0 \to T_{\pi_1,n} \to \mathcal{E} \to T_{\pi_2,n} \to 0$$

such that $\mathcal{E}$ is torsion crystalline with Hodge-Tate weights in $[-1, 2]$, cf. (1.5.4).

(2) For any squarefree $Q \geq 1$ with $p \nmid Q$, define

$$\mathrm{Sel}_{\mathcal{G}Q}(\mathbb{Q}, S_{\pi,n}) := \ker\left(H^1(\mathbb{Q}, S_{\pi,n}) \to \prod_{\ell \notin S \cup \mathrm{div}(Qp)} H^1(I_{\mathbb{Q}_\ell}, S_{\pi,n}) \times \frac{H^1(\mathbb{Q}_p, S_{\pi,n})}{H^1_{\mathrm{cris}}(\mathbb{Q}_p, S_{\pi,n})}\right).$$

When $Q = 1$ we drop it from the notation.

Lemma 8.3.2. *Suppose $Q \geq 1$ is admissible and let $n \geq 1$ be any integer. Then* $\mathrm{Sel}_{\mathcal{G}Q}(\mathbb{Q}, S_{\pi,n}) = \mathrm{Sel}_{\mathcal{G}}(\mathbb{Q}, S_{\pi,n})$.

Proof. Fix $q|Q$. Since q is admissible, the eigenvalues of Frob_q on $S_{\pi,n} \otimes O/\varpi = \overline{T}_{\pi_1} \otimes \overline{T}_{\pi_2}(-1)$ are of the form $\left\{\alpha, \alpha^{-1}, q\alpha^{-1}, \alpha q^{-1}\right\}$ for some $\alpha \in \overline{\mathbb{F}}_p^\times$ with $\alpha \neq 1, q, q^2, q^{-1}$. Thus $H^0(\mathbb{Q}_q, S_{\pi,n}) = H^0(\mathbb{Q}_q, S_{\pi,n}(1)) = 0$ for all n, so $H^1(\mathbb{Q}_q, S_{\pi,n}) = 0$ for all n by the local Euler characteristic formula, and the lemma follows. □

Notation 8.3.3. For an integer $n \geq 1$, recall the notion of pseudorepresentations of degree n from [116, Definition 2.1.1]. We use *op. cit.* as our basic reference for pseudodeformation theory, although some of the relevant results are due to Bellaïche and Chenevier [5].

(1) Let G be a group and R be a ring. If $\rho : G \to \mathrm{GL}_n(R)$ is any representation, we write $D_\rho : G \to R$ for the associated degree-n pseudorepresentation. A pseudorepresentation $D : G \to R$ of degree n is called *reducible* if it is equal to D_ρ for a reducible representation $\rho : G \to \mathrm{GL}_n(R)$.

(2) If $Q \geq 1$ is squarefree, let
$$\widetilde{\mathrm{PsDef}}_{\mathfrak{m}}^{Q} : \mathrm{CNL}_O \to \mathrm{Set}$$
be the functor defined by
$$A \mapsto \left\{\text{pseudorepresentations } D : G_{\mathbb{Q},S\cup\mathrm{div}(Qp)} \to A \text{ of degree } 4 \ : \ D \otimes_A k = D_{\overline{\rho}_\pi}\right\},$$
where $k = O/\varpi$ and $\overline{\rho}_\pi$ is viewed as valued in $\mathrm{GL}_4(k)$. Let $\mathrm{PsDef}_{\mathfrak{m}}^{Q} \subset \widetilde{\mathrm{PsDef}}_{\mathfrak{m}}^{Q}$ be the subfunctor of pseudorepresentations that are torsion crystalline at p with Hodge-Tate weights in $[-1, 2]$ in the sense of [116, Definition 2.5.4]. By Theorems 2.2.5 and 2.5.5 of *op. cit.*, $\widetilde{\mathrm{PsDef}}_{\mathfrak{m}}^{Q}$ and $\mathrm{PsDef}_{\mathfrak{m}}^{Q}$ are representable by universal pseudodeformation rings $\widetilde{R}_{\mathfrak{m}}^{Q}$ and $R_{\mathfrak{m}}^{Q}$, respectively.

(3) Let $\widetilde{J}_{\mathrm{red}}^{Q} \subset \widetilde{R}_{\mathfrak{m}}^{Q}$ be the reducibility ideal defined in [116, Proposition 4.2.2(2)], and $J_{\mathrm{red}}^{Q} \subset R_{\mathfrak{m}}^{Q}$ its image under the natural projection $\widetilde{R}_{\mathfrak{m}}^{Q} \twoheadrightarrow R_{\mathfrak{m}}^{Q}$.

Remark 8.3.4.

(1) The ideal $\widetilde{J}_{\mathrm{red}}^{Q}$ is characterized by the property that, for any morphism $f : \widetilde{R}_{\mathfrak{m}}^{Q} \to A$ in CNL_O corresponding to a pseudorepresentation $D : G_{\mathbb{Q},S\cup\mathrm{div}(Qp)} \to A$, D is reducible if and only if $f(\widetilde{J}_{\mathrm{red}}^{Q}) = 0$.

(2) In what follows, we we will apply the results of [116, §4]. Although the discussion there is carried out for residual representations which are a sum of two characters, as noted in Remark 4.3.6 of *op. cit.*, the results also apply for any residual representation which is multiplicity free with exactly 2 irreducible constituents, which includes $\overline{\rho}_\pi$ by Lemma 4.1.8.

For all squarefree $Q \geq 1$, let $p_\pi^Q : R_{\mathfrak{m}}^{Q} \to O$ be the augmentation corresponding to ρ_π.

Lemma 8.3.5. *Suppose the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, V_{\pi_1} \otimes V_{\pi_2}(-1))$ vanishes. Then there is a constant $C_{\mathrm{RS}} \geq 0$ such that, for any admissible $Q \geq 1$, there exists $j \in \mathrm{Ann}_{R_{\mathfrak{m}}^{Q}}(J_{\mathrm{red}}^{Q})$ with $p_\pi^Q(j) \not\equiv 0$ $\pmod{\varpi^C}$.*

Proof. We know $\mathrm{Fitt}_{R_{\mathfrak{m}}^{Q}}(J_{\mathrm{red}}^{Q}) \subset \mathrm{Ann}_{R_{\mathfrak{m}}^{Q}}(J_{\mathrm{red}}^{Q})$. Then since Fitting ideals are stable under base change, it suffices to show there exists C with $\varpi^C \in \mathrm{Fitt}_O(J_{\mathrm{red}}^{Q} \otimes_{p_\pi^Q} O)$ for all Q, or equivalently that $J_{\mathrm{red}}^{Q} \otimes_{p_\pi^Q} O$ is finite with uniformly bounded cardinality.

Let B^Q and C^Q be the finitely generated $R_{\mathfrak{m}}^{Q}$-modules appearing in [116, p. 38] for the deformation problem $\mathrm{PsDef}_{\mathfrak{m}}^{Q}$. By construction in [116, Proposition 4.2.2], we have a surjection $B^Q \otimes C^Q \twoheadrightarrow J_{\mathrm{red}}^{Q}$, so it suffices to show in turn that $B^Q \otimes_{p_\pi^Q} O$ and $C^Q \otimes_{p_\pi^Q} O$ are finite of uniformly bounded cardinality. Let M be a finitely generated O-module. Because p_π^Q corresponds to the reducible Galois representation ρ_π, we have $p_\pi^Q(J_{\mathrm{red}}^{Q}) = 0$, so [116, Theorem 4.3.5] gives canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_O(B^Q \otimes_{p_\pi^Q} O, M) &\cong \mathrm{Ext}^1_{O[G_{\mathbb{Q}}],\mathcal{C}^Q}(T_{\pi_2}, T_{\pi_1} \otimes_O M)\\ \mathrm{Hom}_O(C^Q \otimes_{q_\pi^Q} O, M) &\cong \mathrm{Ext}^1_{O[G_{\mathbb{Q}}],\mathcal{C}^Q}(T_{\pi_1}, T_{\pi_2} \otimes_O M).\end{aligned} \tag{8.4}$$

Here $\mathcal{C}^Q$ is the full subcategory of finitely generated $O[G_{\mathbb{Q}}]$-modules which are unramified outside $S \cup \mathrm{div}(Qp)$ and all of whose finite subquotients are torsion crystalline with Hodge-Tate weights in $[-1, 2]$.

In particular, for all $n \geq 1$,
$$\mathrm{Hom}_O(B^Q \otimes_{p_\pi^Q} O, O/\varpi^n) = \mathrm{Hom}_O(C^Q \otimes_{p_\pi^Q} O, O/\varpi^n) = \mathrm{Sel}_{\mathcal{G}^Q}(\mathbb{Q}, S_{\pi,n}).$$

We conclude that $B^Q \otimes_{p_\pi^Q} O$ and $C^Q \otimes_{p_\pi^Q} O$ are isomorphic, and by Lemma 8.3.2, they are also independent of Q; so take $Q = 1$ without loss of generality. Then by Proposition 1.5.5,

$$\mathrm{Hom}_O(B^1 \otimes_{p_\pi^1} O, O) \otimes \mathbb{Q}_p = \mathrm{Ext}^1_{O[G_\mathbb{Q}],\mathcal{C}^1}(T_{\pi_2}, T_{\pi_1}) \otimes \mathbb{Q}_p = H^1_f(\mathbb{Q}, V_{\pi_1} \otimes V_{\pi_2}(-1)) = 0.$$

Since $B^1 \otimes_{p_\pi^1} O$ is a finitely generated O-module, it follows that $B^1 \otimes_{p_\pi^1} O$ is finite, as desired. □

8.3.6. We now study deformations of each $\overline{\rho}_{\pi_i}$. As in the non-endoscopic case, we will apply the results and notations of Appendix B to ρ_{π_i}, which we view as valued in $\mathrm{GL}_2(O)$. In the same way as Lemma 8.2.1, we obtain:

Lemma 8.3.7. *For $i = 1, 2$, the representation ρ_{π_i} satisfies Assumptions B.1.3 and B.1.5 from Appendix B.*

8.3.8. For each prime ℓ, let $R_{\ell,i}$ be the local deformation ring for $\overline{\rho}_{\pi_i}$ as in Notation B.1.4. If q is BD-admissible for ρ_{π_i} (Definition 4.2.5), let $R^{\mathrm{ord}}_{q,i}$ be the Steinberg quotient of $R_{q,i}$ in the sense of [72, §2].

Lemma 8.3.9. *The ring $R^{\mathrm{ord}}_{q,i}$ is formally smooth over O of dimension 3.*

Proof. Immediate from [102, Proposition 5.5]. □

8.3.10. In light of Lemma 8.3.9, we take the "admissible" primes in Notation B.2.4 to be the BD-admissible ones for ρ_{π_i}. Then the notion of n-admissible in Definition B.2.5(3) coincides with the notion of n-BD-admissible from Definition 4.2.5; we will always say n-BD-admissible to avoid confusion.

Notation 8.3.11.

1. For $i = 1, 2$ and a squarefree integer $Q \geq 1$ coprime to p, let $\widetilde{R}^Q_{\mathfrak{m},i}$ be the global deformation ring of
$$\overline{\rho}_{\pi_i} : G_{\mathbb{Q},S\cup\mathrm{div}(Qp)} \to \mathrm{GL}_2(O/\varpi),$$
with fixed determinant χ_p^{cyc}.
2. Let R_i^Q and – when Q is BD-admissible for ρ_{π_i} – $R_{Q,i}$ be the quotients of $\widetilde{R}^Q_{\mathfrak{m},i}$ defined in Notation B.4.4(2,3). (We are identifying Q with $\mathrm{div}(Q)$ for notational convenience.)
3. Let $\rho^{\mathrm{univ}}_{Q,i} : G_{\mathbb{Q},S\cup\mathrm{div}(Qp)} \to \mathrm{GL}_2(R_{Q,i})$ be a framing of the universal deformation, with underlying $R_{Q,i}[G_\mathbb{Q}]$-module $M^{\mathrm{univ}}_{Q,i}$.

8.3.12. As in the non-endoscopic case, let $\mathrm{pr}_p : I_{\mathbb{Q}_q} \to \mathbb{Z}_p(1)$ be the maximal pro-p quotient. By the construction of $R^{\mathrm{ord}}_{q,i}$, we have:

Lemma 8.3.13. *Suppose $Q \geq 1$ is BD-admissible for ρ_{π_i}, and $q|Q$ is a prime. Then there exists a basis for $M^{\mathrm{univ}}_{Q,i}$ and an element $t_q \in R_{Q,i}$ such that*

$$\rho^{\mathrm{univ}}_{Q,i}|_{G_{\mathbb{Q}_q}} = \begin{pmatrix} \chi_p^{\mathrm{cyc}} & * \\ 0 & 1 \end{pmatrix},$$

and

$$\rho^{\mathrm{univ}}_{Q,i}(g) = \begin{pmatrix} 1 & \mathrm{pr}_p(g)t_q \\ 0 & 1 \end{pmatrix}, \ \forall g \in I_{\mathbb{Q}_q}.$$

Definition 8.3.14. If Q and $q \nmid Q$ are BD-admissible for ρ_{π_i}:

1. Define $P_{q,i}(T) = \det(\rho_Q^{\mathrm{univ}}(\mathrm{Frob}_q) - T) \in R_{Q,i}[T]$.
2. Define $R^{\mathrm{cong}}_{Q,q,i} := R_{Q,i} \otimes_{R^{Qq}_{\mathfrak{m},i}} R_{Qq,i}$.

The same proof as Lemma 8.2.10 shows:

Lemma 8.3.15. *Suppose Q and $q \nmid Q$ are BD-admissible for ρ_{π_i}. Then*

$$R^{\mathrm{cong}}_{Q,q,i} = R_{Q,i}/(P_{q,i}(q)) = R_{Qq,i}/(t_q).$$

Lemma 8.3.16. *Suppose q is n-BD-admissible for ρ_{π_i}. Then*

$$H^1(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi_i,n}) = H^1_{\mathrm{ord}}(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi_i,n}) \oplus H^1_{\mathrm{unr}}(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi_i,n}),$$

in which each factor is free of rank one over O/ϖ^n. In particular, q is standard in the sense of Definition B.4.7.

Proof. Let $\mathrm{Frob}_q \in G_{\mathbb{Q}_q}$ be a lift of Frobenius, and let $\tau_q \in I_{\mathbb{Q}_q}$ be an element such that $\mathrm{pr}_p(\tau_q) = 1$. Then, with respect to the basis in Lemma 8.3.13, $H^1(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi_i,n})$ is spanned by the following two cocycles:

$$\begin{aligned} &\mathrm{Frob}_q \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_q \mapsto 0 \\ &\mathrm{Frob}_q \mapsto 0, \qquad\qquad \tau_q \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The first generates $H^1_{\mathrm{unr}}(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi_i,n})$ and the second $H^1_{\mathrm{ord}}(\mathbb{Q}_q, \mathrm{ad}^0\,\rho_{\pi_i,n})$; also, both cocycles are clearly not ϖ^{n-1}-torsion, and the lemma follows. □

8.3.17. For the next two lemmas, we introduce some temporary notation. Let $Q \geq 1$ be admissible, let K be an S-tidy level structure for $\mathrm{GSpin}(V_{DQ})$, and abbreviate $\mathbb{T} := \mathbb{T}^{S \cup \mathrm{div}(Q)}_{K, V_{DQ}, O, \mathfrak{m}}$, which may be the zero ring. Also fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ inducing the prime $\mathfrak{p}$ of E_0. Then we write $\mathcal{T}$ for the set of relevant automorphic representations Π of $\mathrm{GSpin}(V_D)(\mathbb{A})$ such that $\Pi_f^K \neq 0$, and the Hecke action on $\iota^{-1}\Pi_f^K$ factors through $\mathbb{T}$. By Corollary 2.7.8, we have the embedding of $\mathbb{T}$-algebras

$$\mathbb{T} \hookrightarrow \bigoplus_{\Pi \in \mathcal{T}} \overline{\mathbb{Q}}_p(\Pi), \tag{8.5}$$

where $\overline{\mathbb{Q}}_p(\Pi)$ has Hecke action through the eigenvalues on $\iota^{-1}\Pi_f^K$. Then by [22, Corollary 1.14], there is a canonical pseudorepresentation

$$D_Q : G_{\mathbb{Q}, S \cup \mathrm{div}(Qp)} \to \mathbb{T}, \tag{8.6}$$

such that for all $\Pi \in \mathcal{T}$, the composite of D_Q with the character $\lambda_\Pi : \mathbb{T} \to \overline{\mathbb{Q}}_p(\Pi)$ is the pseudorepresentation associated to $\rho_{\Pi,\iota}$. Finally, let $\mathcal{T}_{\mathrm{end}} \subset \mathcal{T}$ be the subset of endoscopic representations, and let $\overline{\mathbb{T}}$ be the quotient of $\mathbb{T}$ defined by the actions on $\Pi \in \mathcal{T}_{\mathrm{end}}$.

Lemma 8.3.18.

1. *The pseudodeformation D_Q is induced by an O-algebra morphism $R^Q_{\mathfrak{m}} \to \mathbb{T}$.*
2. *The composite $R^Q_{\mathfrak{m}} \to \mathbb{T} \to \overline{\mathbb{T}}$ factors through $R^Q_{\mathfrak{m}}/J^Q_{\mathrm{red}}$.*

Proof. By definition, D_Q is induced by an O-algebra morphism $d : \widetilde{R}^Q_{\mathfrak{m}} \to \mathbb{T}$. By (8.5), to prove (1) it suffices to show each composite map $\widetilde{R}^Q_{\mathfrak{m}} \xrightarrow{d} \mathbb{T} \xrightarrow{\lambda_\Pi} \overline{\mathbb{Q}}_p$ factors through $R^Q_{\mathfrak{m}}$, with λ_Π as in (8.3.17); but this is clear because K_p is hyperspecial, so $\rho_{\Pi,\iota}$ is crystalline at p for all such Π. Similarly, for (2) it suffices to note that $\lambda_\Pi \circ d$ annihilates J^Q_{red} for all $\Pi \in \mathcal{T}_{\mathrm{end}}$, because $\rho_{\Pi,\iota}$ is reducible. □

8.3.19. Let

$$R^Q_{\mathfrak{m}}/J^Q_{\mathrm{red}} \to R_{Q_1,1} \otimes_O R_{Q_2,2} \tag{8.7}$$

be the canonical map, defined on moduli problems by sending a pair of deformations ρ_1, ρ_2 of $\overline{\rho}_{\pi_1}$ and $\overline{\rho}_{\pi_2}$ to the pseudorepresentation $D_{\rho_1 \oplus \rho_2}$. It follows from [116, Proposition 4.2.6] that (8.7) is surjective.

Lemma 8.3.20. *Write* $Q = Q_1 \cdot Q_2$ *such that each* $q|Q_i$ *is BD-admissible for* ρ_{π_i} *(Remark 4.2.6(2)). Then:*

(1) The map $R_{\mathfrak{m}}^Q/J_{\mathrm{red}}^Q \to \overline{\mathbb{T}}$ *induced by Lemma 8.3.18(2) factors through the surjection (8.7). In particular,* $\overline{\mathbb{T}}$ *is an* $R_{Q_i,i}$*-module for* $i = 1, 2$.

(2) Assume $\sigma(DQ)$ *is even. For any* $j \in \mathrm{Ann}_{R_{\mathfrak{m}}^Q}(J_{\mathrm{red}}^Q)$*, the* $\mathbb{T}$*-action on*

$$jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}$$

factors through $\overline{\mathbb{T}}$*, and as a* $\overline{\mathbb{T}}$*-module,* $jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}$ *is*

$$(\rho_{Q_1,1}^{\mathrm{univ}} \otimes_{R_{Q_1,1}} \overline{\mathbb{T}}, \rho_{Q_2,2}^{\mathrm{univ}} \otimes_{R_{Q_2,2}} \overline{\mathbb{T}})\textit{-typic.}$$

Proof. Arguing as in Lemma 8.3.18, for (1) it suffices to consider the pseudorepresentations attached to $\Pi \in \mathcal{T}_{\mathrm{end}}$. Suppose Π is associated to a pair (τ_1, τ_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ inducing $\mathfrak{p}$. Then $\rho_{\Pi,\iota} = \rho_{\tau_1,\iota} \oplus \rho_{\tau_2,\iota}$. We have $\overline{\rho}_{\tau_1,\iota} \oplus \overline{\rho}_{\tau_2,\iota} \cong \overline{\rho}_{\pi_1,\iota} \oplus \overline{\rho}_{\pi_2,\iota}$, and since $\overline{\rho}_{\pi_i,\iota}$ are both absolutely irreducible, without loss of generality we may assume $\overline{\rho}_{\tau_i,\iota} \cong \overline{\rho}_{\pi_i,\iota}$, $i = 1, 2$. By Lemma 2.2.9 and Fontaine-Laffaille theory, $\pi_{i,\infty}$ and $\tau_{i,\infty}$ have the same weight.

By Corollary 2.5.3(2) and Theorem 2.2.10(1), $(\rho_{\tau_1,\iota} \oplus \rho_{\tau_2,\iota})|_{G_{\mathbb{Q}_q}}$ is ramified for all $q|Q$. It then follows from [102, Propositions 5.3, 5.5] that $\rho_{\tau_i,\iota}|_{G_{\mathbb{Q}_q}}$ is ordinary for all $q|Q_i$, and unramified for all $q|Q/Q_i$. It is also clear from Lemma 4.3.2 and Theorem 2.2.10(3) that $\det \rho_{\tau_i,\iota} = \chi_p^{\mathrm{cyc}}$, and $\rho_{\pi_i,\iota}$ and $\rho_{\tau_i,\iota}$ have the same Hodge-Tate weights by Theorem 2.2.10(2). Hence $\rho_{\tau_i,\iota}$ arises from a deformation parametrized by $R_{Q_i,i}$, and this proves (1).

For (2), by Proposition 8.1.2(2) and Theorem 2.7.5(2), it suffices to show that $jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(2))_{\mathfrak{m}}$ is $(\rho_{Q_1,1}^{\mathrm{univ}} \otimes_{R_{Q_1,1}} \overline{\mathbb{T}}, \rho_{Q_2,2}^{\mathrm{univ}} \otimes_{R_{Q_2,2}} \overline{\mathbb{T}})$-typic.

For this, we use the decomposition of Corollary 2.7.7:

$$H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(2))_{\mathfrak{m}} = \bigoplus_{\Pi_f} \iota^{-1}\Pi_f^K \otimes \rho_{\Pi_f},$$

as Π_f ranges over finite parts of automorphic representations $\Pi \in \mathcal{T}$. This is a decomposition of $R_{\mathfrak{m}}^Q[G_{\mathbb{Q}}]$-modules, where $R_{\mathfrak{m}}^Q$ acts on the factor $\iota^{-1}\Pi_f^K \otimes \rho_{\Pi_f}$ via the map $\lambda_\Pi : R_{\mathfrak{m}}^Q \to \mathbb{T} \to \overline{\mathbb{Q}}_p(\Pi)$ corresponding to the pseudorepresentation of $\rho_{\Pi,\iota}$ (equivalently, to the Hecke eigenvalues of Π_f^K). In particular, $\lambda_\Pi(J_{\mathrm{red}}^Q) = 0$ if and only if $\rho_{\Pi,\iota}$ is reducible, which by Lemma 2.2.12 and Remark 4.1.2 occurs if and only if Π is endoscopic. Because the element $j \in R_{\mathfrak{m}}^Q$ annihilates J_{red}^Q, it then suffices to show that, for all relevant *endoscopic* automorphic representations $\Pi \in \mathcal{T}_{\mathrm{end}}$ associated to a pair (τ_1, τ_2), ρ_{Π_f} is either $\rho_{Q_1,1}^{\mathrm{univ}} \otimes_{R_{Q_1,1}} \overline{\mathbb{T}}$-typic or $\rho_{Q_2,2}^{\mathrm{univ}} \otimes_{R_{Q_2,2}} \overline{\mathbb{T}}$-typic as a $\overline{\mathbb{T}}[G_{\mathbb{Q}}]$-module. However, as $\rho_{\Pi_f} = \rho_{\tau_1,\iota}$ or $\rho_{\tau_2,\iota}$ by Corollary 2.7.7, this is clear from the construction of the map $R_{Q_1,1} \otimes_O R_{Q_2,2} \to \overline{\mathbb{T}}$. ☐

Definition 8.3.21. Let Q be admissible, with a factorization $Q = Q_1 \cdot Q_2$ such that $\sigma(DQ)$ is even and all $q|Q_i$ are BD-admissible for ρ_{π_i}. Fix an S-tidy level structure K for $\mathrm{GSpin}(V_{DQ})$, and an element $j \in \mathrm{Ann}_{R_{\mathfrak{m}}^Q}(J_{\mathrm{red}}^Q)$. Then we define

$$H_Q(K, j)^{(i)} = \mathrm{Hom}_{R_{Q_i,i}[G_{\mathbb{Q}}]}(M_{Q_i,i}^{\mathrm{univ}}, jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}})$$

for $i = 1, 2$.

Remark 8.3.22. In the context of Definition 8.3.21, by [98, Proposition 5.3] and Lemma 8.3.20(2), we have

$$jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}} \simeq M_{Q_1,1}^{\mathrm{univ}} \otimes_{R_{Q_1,1}} H_Q(K, j)^{(1)} \oplus M_{Q_2,2}^{\mathrm{univ}} \otimes_{R_{Q_2,2}} H_Q(K, j)^{(2)}.$$

Lemma 8.3.23. *In the context of Definition 8.3.21, let $q|Q_i$ be a prime. Then under the natural isomorphism*

$$H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \simeq H^1(I_{\mathbb{Q}_q}, M^{\text{univ}}_{Q_1,1}) \otimes_{R_{Q_1,1}} H_Q(K,j)^{(1)} \oplus \\ H^1(I_{\mathbb{Q}_q}, M^{\text{univ}}_{Q_2,2}) \otimes_{R_{Q_2,2}} H_Q(K,j)^{(2)},$$

the ϖ-power torsion of $H^1\left(I_{\mathbb{Q}_q}, jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)$ is contained in

$$H^1\left(I_{\mathbb{Q}_q}, jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)^{\mathrm{Frob}_q=1} \simeq H^1(I_{\mathbb{Q}_q}, M^{\text{univ}}_{Q_i,i})^{\mathrm{Frob}_q=1} \otimes_{R_{Q_i,i}} H_Q(K,j)^{(i)} \\ \simeq H_Q(K,j)^{(i)}/(t_q).$$

(The element $t_q \in R_{Q_i,i}$ was defined in Lemma 8.3.13.)

Proof. Without loss of generality, suppose $i = 1$. Then $\rho^{\text{univ}}_{Q_2,2}|_{G_{\mathbb{Q}_q}}$ is unramified, so by Lemma 8.3.13, we have

$$H^1(I_{\mathbb{Q}_q}, M^{\text{univ}}_{Q_i,i}) = \begin{cases} R_{Q_1,1}/(t_q) \oplus R_{Q_1,1}(-1), & i = 1, \\ M^{\text{univ}}_{Q_2,2}(-1), & i = 2. \end{cases}$$

Since $H_Q(K,j)^{(1)}$ and $H_Q(K,j)^{(2)}$ are ϖ-torsion-free by Theorem 2.7.5(2), the lemma follows as in the proof of Lemma 8.2.16. ☐

Lemma 8.3.24. *In the context of Definition 8.3.21, for all $j \in \mathrm{Ann}_{R^Q_{\mathfrak{m}}}(J^Q_{\text{red}})$, all $z \in \mathrm{SC}^2_K(V_{DQ}, O)$, all $q|Q_i$, and all $\alpha_0 \in \mathrm{Hom}_{R_{Q_i,i}}(H_Q(K,j)^{(i)}/(t_q), O/\varpi^n)$, we have*

$$\alpha_0 \circ j_* \circ \mathrm{res}_{\mathbb{Q}_q} \circ \partial_{\mathrm{AJ},\mathfrak{m}}(z) \in \partial_q(\kappa_n(Q;K)).$$

Here,

$$j_* : H^1(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}) \to H^1(I_{\mathbb{Q}_q}, jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}) \tag{8.8}$$

is the natural map, and O/ϖ^n is viewed as an $R_{Q_i,i}$-algebra through the map corresponding to $\rho_{\pi_i,n}$.

Proof. For any $\alpha_0 \in \mathrm{Hom}_{R_{Q_i,i}}(H_Q(K;j)^{(i)}/(t_q), O/\varpi^n)$, we obtain a corresponding induced map of Galois modules

$$\alpha = \mathrm{id} \otimes \alpha_0 : M^{\text{univ}}_{Q_i,i} \otimes_{R_{Q_i,i}} H_Q(K;j)^{(i)} \to M^{\text{univ}}_{Q_i,i} \otimes_{R_{Qi,i}} O/\varpi^n = T_{\pi_i,n}.$$

Then by the decomposition from Remark 8.3.22, we can also view α as a map of Galois modules

$$jH^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}} \to T_{\pi_i,n}.$$

Let $(\alpha \circ j)_*$ be the induced map

$$H^1(\mathbb{Q}, H^3_{\text{ét}}(\mathrm{Sh}_K(V_{DQ})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}) \to H^1(\mathbb{Q}, T_{\pi_i,n}).$$

For any $z \in \mathrm{SC}^2_K(V_{DQ}, O)$, $\kappa^D_n(Q;K)$ contains

$$(\alpha \circ j)_*(\partial_{\mathrm{AJ},\mathfrak{m}}(z)) \in H^1(\mathbb{Q}, T_{\pi_i,n}) \hookrightarrow H^1(\mathbb{Q}, T_{\pi,n}).$$

The lemma now follows as in the proof of Lemma 8.2.17. ☐

8.4. **A test function calculation.** For this subsection, we fix the following additional data:

- An integer $n \geq 1$.
- A squarefree integer $D \geq 1$ with $\operatorname{div}(D) \subset S$, and an n-admissible $Q \geq 1$ coprime to D, such that $\sigma(DQ)$ is odd.
- An n-admissible prime $q \nmid Q$.
- An S-level structure K for $\mathrm{GSpin}(V_{DQ})$. Let $L \subset V_{DQ} \otimes \mathbb{Q}_q$ be the unique self-dual lattice stabilized by K_q.
- A $\mathbb{Z}_q$-basis $\{v_0, v_1, v_2, v_1^*, v_2^*\}$ for L as in (5.1.2), which identifies $V_{DQ} \otimes \mathbb{Q}_q$ with the standard split five-dimensional quadratic space over $\mathbb{Q}_q$.

We will apply the results and notations of §7, with the D therein always replaced by DQq. However, we do not yet specify the choice of q-adic uniformization datum for V_{DQq}. The goal of this subsection is the crucial Lemma 8.4.6.

8.4.1. Let $\varphi_q^{(0)}, \varphi_q^{(1)}, \varphi_q^{\star}, \varphi_q^{\mathrm{tot}} \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, \mathbb{Z})$ be as in (7.7.1). We also let $\overline{\varphi}_q^{?} \in \mathcal{S}(V_{DQ}^2 \otimes \mathbb{Q}_q, \overline{\mathbb{F}}_p)$ be the reduction of $\varphi_q^?$ for $? = (0)$, (1), $\star$, tot.

Notation 8.4.2. Without loss of generality, we write the almost level-raising generic character χ from Proposition 5.6.2 as

$$\chi = |\cdot|^{1/2} \boxtimes \alpha : (\mathbb{Q}_q^\times)^2 \to \overline{\mathbb{F}}_p^\times, \tag{8.9}$$

where $\alpha^2 \neq |\cdot|^{\pm 1}$.

Then we can consider condition (C_χ) from Definition 5.4.3, for $\overline{\varphi}_q^{\mathrm{tot}}$:

(C_χ) There exists $g \in \mathrm{Mp}_4(\mathbb{Q}_q)$ such that $f_\chi(\overline{\omega_\psi(1,g)\overline{\varphi}_q^{\mathrm{tot}}}) \neq 0$, where

$$f_\chi : \left((|\cdot|^{-\frac{1}{2}})^{\boxtimes 2} \boxtimes \chi_\psi \cdot (|\cdot|^{\frac{1}{2}})^{\boxtimes 2}\right) \otimes \mathcal{S}(\mathbb{Q}_q^2, \overline{\mathbb{F}}_p) \to \chi \boxtimes \chi_\psi \cdot \chi^{-1}$$

is the unique projection deduced from Lemma 5.2.1.

Lemma 8.4.3. *The test function $\overline{\varphi}_q^{\mathrm{tot}}$ satisfies condition (C_χ).*

Proof. Let $g \in \mathrm{Mp}_4(\mathbb{Q}_q)$ be a lift of the Weyl element

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}_q)$$

in the standard basis $\{e_1, e_2, e_1^*, e_2^*\}$. Then for some unit $u \in \overline{\mathbb{F}}_p^\times$, we have

$$\omega_\psi(1,g)\varphi(x,y) = u \int_{z \in V_{DQ} \otimes \mathbb{Q}_q} \varphi(z,y)\psi(x \cdot z)\mathrm{d}z$$

for all

$$\varphi \in S(V_{DQ}^2 \otimes \mathbb{Q}_q, \overline{\mathbb{F}}_p).$$

Notice that, since $\chi = |\cdot|^{\frac{1}{2}} \boxtimes \alpha$, the projection

$$f_\chi : \left((|\cdot|^{-\frac{1}{2}})^{\boxtimes 2} \boxtimes \chi_\psi \cdot (|\cdot|^{\frac{1}{2}})^{\boxtimes 2}\right) \otimes \mathcal{S}(\mathbb{Q}_q^2, \overline{\mathbb{F}}_p) \to \chi \boxtimes \chi_\psi \cdot \chi^{-1}$$

is the composite of the integration map

$$\begin{aligned}\mathrm{int} : \mathcal{S}(\mathbb{Q}_q^2, \overline{\mathbb{F}}_p) &\to \mathcal{S}(\mathbb{Q}_q, \overline{\mathbb{F}}_p)\\ \varphi &\mapsto \left(t \mapsto \int \varphi(t_1, t)\mathrm{d}t_1\right)\end{aligned}$$

and the projection

$$f_{\alpha|\cdot|^{1/2}} : \mathcal{S}(\mathbb{Q}_q, \overline{\mathbb{F}}_p) \to \alpha|\cdot|^{1/2} \boxtimes \alpha^{-1}|\cdot|^{-1/2}.$$

Abbreviate

$$s^{?} = u^{-1}\,\mathrm{int}\left(\overline{\omega_\psi(1,g)\overline{\varphi}_q^{?}}\right) \in \mathcal{S}(\mathbb{Q}_q, \overline{\mathbb{F}}_p)$$

for $? = (0), (1), \star, \mathrm{tot}$. We will compute s^{tot} explicitly to show $f_{\alpha|\cdot|^{1/2}}(s^{\mathrm{tot}}) \neq 0$, which will prove the lemma. By definition we have

$$s^{?}(t) = \int_{t_1 \in \mathbb{Q}_q} \int_{a \in \mathbb{Q}_q} \int_{z \in V_{DQ} \otimes \mathbb{Q}_q} \overline{\varphi}_q^{?}(z, tv_2 + av_1)\psi(t_1 z \cdot v_1)\mathrm{d}z\mathrm{d}a\mathrm{d}t_1.$$

Note that $\overline{\varphi}_q^{?}(z, tv_2 + av_1)$ depends on z modulo q^2L only, so the inner integral is nonzero only on the set $\{t_1 \in q^{-2}\mathbb{Z}_q\}$. We may therefore reorder the integrals and obtain

$$\begin{aligned}s^{?}(t) &= \int_{z \in L} \int_{a \in q^{-1}\mathbb{Z}_q} \int_{t_1 \in q^{-2}\mathbb{Z}_q} \overline{\varphi}_q^{?}(z, tv_2 + av_1)\psi(t_1 z \cdot v_1)\mathrm{d}t_1\mathrm{d}a\mathrm{d}z\\ &= q^2 \int_{z \in L} \int_{a \in q^{-1}\mathbb{Z}_q} \overline{\varphi}_q^{?}(z, tv_2 + av_1) \cdot \mathbb{1}_{z \cdot v_1 \in q^2\mathbb{Z}_q}\mathrm{d}a\mathrm{d}z.\end{aligned}$$

For the inner integral to be nonzero, we must have $z \cdot tv_2 \in \mathbb{Z}_q^\times$ and $t \in q^{-1}\mathbb{Z}_q$. In particular, since z and v_2 lie in L, $s^{?}(t)$ is supported on $q^{-1}\mathbb{Z}_q^\times \sqcup \mathbb{Z}_q^\times$. At this point, we are ready to compute the following table of values for $s^{?}(t)$:

	$s^{(0)}(t)$	$s^{(1)}(t)$	$s^{\star}(t)$	$s^{\mathrm{tot}}(t)$
$q^{-1}\mathbb{Z}_q^\times$	0	$\frac{q-1}{q^4}$	$\frac{(q^2-1)(q-1)}{q^4}$	$\frac{(q-1)^2}{q^3}$
$\mathbb{Z}_q^\times$	$\frac{q-1}{q^2}$	0	$\frac{(q-1)^2}{q^2}$	0

The fourth column is determined by the first three by

$$s^{\mathrm{tot}} = s^{\star} + (1-q)(s^{(0)} + s^{(1)}).$$

On the other hand, the given values for $s^{\mathrm{tot}}(t)$ immediately imply $f_{\alpha|\cdot|^{1/2}}(s^{\mathrm{tot}}) \neq 0$ for any $\alpha \neq |\cdot|^{-\frac{1}{2}}$. It remains to explain the calculation of the first three columns. First, since $\overline{\varphi}_q^{(0)}(z, tv_2 + av_1) = 0$ unless $tv_2 + av_1 \in L$, we have $s^{(0)}(t) = 0$ for $t \in q^{-1}\mathbb{Z}_q^\times$. For $t \in \mathbb{Z}_q^\times$, we have

$$s^{(0)}(t) = q^2\,\mathrm{Vol}\left\{z \in L - qL \,:\, z \cdot v_2 \in \mathbb{Z}_q^\times,\, z \cdot v_1 \in q^2\mathbb{Z}_q, z \cdot z \in q\mathbb{Z}_q\right\}. \tag{8.10}$$

Label the set in (8.10) by $S^{(0)}$ and let $\overline{S}^{(0)}$ be its image in L/qL. Write $\overline{z}, \overline{v}_1, \overline{v}_2$ for the reductions in L/qL.

Then $\overline{S}^{(0)}$ is the set of $\overline{z} \in L/qL$ that are isotropic, and orthogonal to $\overline{v}_1$ but not $\overline{v}_2$. Now, there are $q^3 = q^2 \cdot q$ isotropic vectors in L/qL orthogonal to $\overline{v}_1$ since $\overline{v}_1^\perp/\overline{v}_1$ is a split quadratic space over $\mathbb{F}_q$ of dimension three. Of these, q^2 are also orthogonal to $\overline{v}_2$; these are just the vectors in $\mathrm{span}_{\mathbb{F}_q}\{\overline{v}_1, \overline{v}_2\}$, since the latter is a maximal isotropic subspace. So

$$\#\overline{S}^{(0)} = q^3 - q^2.$$

On the other hand, given any $\overline{z}_0 \in \overline{S}^{(0)}$, we have

$$\mathrm{Vol}\left\{z \in S^{(0)} \,:\, \overline{z} = \overline{z}_0\right\} = \frac{1}{q^6}.$$

(It is almost the full coset $\overline{z}_0+qL$, which has volume $\frac{1}{q^5}$, except that we must ensure that z remains orthogonal to v_1 modulo q^2; this cuts down the volume by a factor of q.) So

$$\operatorname{Vol}(S^{(0)})=\frac{q^3-q^2}{q^6}=\frac{q-1}{q^4},$$

and

$$s^{(0)}(t)=q^2\operatorname{Vol}(S^{(0)})=\frac{q-1}{q^2}\text{ for }t\in\mathbb{Z}_q^\times.$$

Next let us consider $s^{(1)}(t)$. Since $\varphi_q^{(1)}(z,tv_2+av_1)\cdot\mathbb{1}_{z\cdot v_1\in q^2\mathbb{Z}_q}$, when restricted to $a\in q^{-1}\mathbb{Z}_q$, is supported on $z\in qL$ such that $z\cdot tv_2\in\mathbb{Z}_q^\times$, we see that $s^{(1)}(t)=0$ for $t\in\mathbb{Z}_q^\times$. For $t\in q^{-1}\mathbb{Z}_q^\times$, we have

$$s^{(1)}(t)=q^2\operatorname{Vol}\left\{z\in qL,a\in q^{-1}\mathbb{Z}_q\;:\;z\cdot v_2\in q\mathbb{Z}_q^\times,\;z\cdot v_1\in q^2\mathbb{Z}_q\right\}.$$

(Note that the condition $z\cdot z\in q\mathbb{Z}_q$ in the definition of X (7.27) is automatic from $z\in qL$.) Then replacing z with $\frac{z}{q}$, we have

$$\begin{aligned}s^{(1)}(t)&=\frac{q^3}{q^5}\operatorname{Vol}\left\{z\in L\;:\;z\cdot v_2\in\mathbb{Z}_q^\times,\;z\cdot v_1\in q\mathbb{Z}_q\right\}\\&=\frac{q^3}{q^{10}}\#\left\{\overline{z}\in L/qL\;:\;\overline{z}\perp\overline{v}_1,\;\overline{z}\not\perp\overline{v}_2\right\}\\&=\frac{q^3}{q^{10}}\cdot(q^4-q^3)=\frac{q-1}{q^4}.\end{aligned}$$

Finally, consider $s^\star(t)$. For $t\in\mathbb{Z}_q^\times$, we have

$$\begin{aligned}s^\star(t)&=q^2\operatorname{Vol}\left\{z\in L-qL,\;a\in q^{-1}\mathbb{Z}_q^\times\;:\;z\cdot v_2\in\mathbb{Z}_q^\times,\;z\cdot v_1\in q^2\mathbb{Z}_q\right\}\\&=q^2(q-1)\operatorname{Vol}\left\{z\in L\;:\;z\cdot z\in q\mathbb{Z}_q,\;z\cdot v_2\in\mathbb{Z}_q^\times,\;z\cdot v_1\in q^2\mathbb{Z}_q\right\}.\end{aligned}$$

This is the same set that appeared for $s^{(0)}(t)$, so we have

$$s^\star(t)=(q-1)s^{(0)}(t)=\frac{(q-1)^2}{q^2}\text{ for }t\in\mathbb{Z}_q^\times.$$

For $t\in q^{-1}\mathbb{Z}_q^\times$, we have

$$\begin{aligned}s^\star(t)&=q^2\operatorname{Vol}\left\{z\in L-qL,\;a\in q^{-1}\mathbb{Z}_q\;:\;z\cdot z\in q\mathbb{Z}_q,\;z\cdot v_2\in q\mathbb{Z}_q^\times,\;z\cdot v_1\in q^2\mathbb{Z}_q\right\}\\&=q^3\operatorname{Vol}\left\{z\in L-qL\;:\;z\cdot z\in q\mathbb{Z}_q,\;z\cdot v_2\in q\mathbb{Z}_q^\times,\;z\cdot v_1\in q^2\mathbb{Z}_q\right\}.\end{aligned}\tag{8.11}$$

To compute this, we use the same technique as for $s^{(0)}$. Let $S^\star$ be the set in the second line of (8.11) and let $\overline{S}^\star$ be its image in L/qL. Then $\overline{S}^\star$ consists of nonzero isotropic vectors in L/qL orthogonal to $\overline{v}_2$ and $\overline{v}_1$, of which there are q^2-1. On the other hand, for $\overline{z}_0\in\overline{S}^\star$, we have

$$\operatorname{Vol}\left\{z\in S^\star:\overline{z}=\overline{z}_0\right\}=\frac{q-1}{q^7}:$$

this is because, out of the coset $\overline{z}_0+qL$, we must take only those vectors with $z\cdot v_2\in q\mathbb{Z}_q^\times$ and $z\cdot v_1\in q^2\mathbb{Z}_q$. So

$$s^\star(t)=(q^2-1)\cdot\frac{q-1}{q^7}\cdot q^3=\frac{(q-1)(q^2-1)}{q^4}\text{ for }t\in q^{-1}\mathbb{Z}_q^\times,$$

as desired. ☐

8.4.4. Now we return to the geometric setting of §7. We will write $\mathfrak{m} := \mathfrak{m}_{\pi,\mathfrak{p}}^{S\cup\mathrm{div}(Qq)} \subset \mathbb{T}_O^{S\cup\mathrm{div}(Qq)}$.

Lemma 8.4.5. *The maximal ideal* $\mathfrak{m} \subset \mathbb{T}_O^{S\cup\mathrm{div}(Qq)}$ *is generic and non-Eisenstein, and weakly* q*-generic (Definition 7.3.9).*

Proof. That $\mathfrak{m}$ is generic and non-Eisenstein follows from Lemma 4.1.7. Then by Corollary 2.7.8, it suffices to show that $\langle q\rangle T_{q,2}^2 - 4q^2(q+1)^2 \notin \mathfrak{m}_{\pi,\mathfrak{p}}^S \subset \mathbb{T}_O^S$. Indeed, this holds by Remark 7.3.10, because the admissibility of q implies $\mathrm{tr}(\mathrm{Frob}_q\,|\overline{\rho}_\pi) \neq \pm 2(q+1)$. ☐

In particular, we can consider, for any choice of q-adic uniformization datum for V_{DQq}, the map

$$\xi := \nabla\circ\zeta : M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \twoheadrightarrow \frac{O\left[\mathrm{Sh}_{K^qK_q}(V_{DQ})\right]_{\mathfrak{m}}}{(\mathtt{T}_{\mathrm{lr}})} \tag{8.12}$$

from Theorem 7.4.7; here we use that $\langle q\rangle = 1$ on $O\left[\mathrm{Sh}_{K^qK_q}(V_{D/q})\right]_{\mathfrak{m}}$ by Lemma 4.3.2. (The map ξ, and the identification of K^q with a compact open subgroup of $\mathrm{GSpin}(V_{DQq})(\mathbb{A}_f^q)$, both depend on the choice of uniformization datum.)

The following is the crucial lemma for the proof of Theorems 8.5.1 and 8.5.2.

Lemma 8.4.6. *Suppose* Qq *is* n*-admissible. Then there exists a test function* $\alpha \in \mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$*, a q-adic uniformization datum for* V_{DQq}*, and a special cycle* $z \in \mathrm{SC}_{K^qK_q^{\mathrm{ram}}}(V_{DQq}, O)$ *such that*

$$\alpha\circ\xi\circ\mathrm{res}_{\mathbb{Q}_q}(\partial_{\mathrm{AJ},\mathfrak{m}}(z)) \in O/\varpi^n$$

generates $\lambda_n(Q;K)$.

Here we are using Theorem 7.1.11 to apply ξ to $\mathrm{res}_{\mathbb{Q}_q}(\partial_{\mathrm{AJ},\mathfrak{m}}(z))$.

Proof. By Corollary 5.6.3 and Lemma 8.4.3, we conclude

$$\lambda_n^D(Q;K) = \lambda_n^D(Q, \varphi^q\otimes\varphi_q^{\mathrm{tot}}; K)$$

for some $\varphi^q \in \mathcal{S}(V_{DQ}^2\otimes\mathbb{A}_f^q, O)^{K^q}$. Then by definition, there exists a test vector

$$\alpha \in \mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$$

such that $\alpha(Z(T, \varphi^q\otimes\varphi_q^{\mathrm{tot}})_K)$ generates $\lambda_n(Q;K)$, for some $T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$.

Now note that, for any choice of uniformization datum, $\alpha\circ\xi$ gives a well-defined map

$$M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \twoheadrightarrow \frac{O\left[\mathrm{Sh}_{K^qK_q}(V_{DQ})\right]_{\mathfrak{m}}}{(\mathtt{T}_{\mathrm{lr}})} \twoheadrightarrow O/\varpi^n$$

because $\alpha(\mathtt{T}_{\mathrm{lr}}) \subset (\varpi^n)$ by Remark 7.4.6.

Then the lemma is immediate from Theorem 7.7.2.

☐

8.5. **Conclusions.** Finally, we are ready to prove the main results for this section. We start with the non-endoscopic case.

Theorem 8.5.1. *Suppose* π *is not endoscopic. Fix an integer* $m \geq 1$*, and let* $n_0 = n_0(m, \rho_\pi)$ *satisfy the conclusion of Lemma B.4.5. Suppose* $Q \geq$ *is* n*-admissible and* $q \nmid Q$ *is an* n*-admissible prime, where* $n \geq \max\{3m, n_0\}$*, such that* $\sigma(DQ)$ *is odd.*

(1) Suppose $\overline{\mathrm{Sel}}_{\mathcal{F}(Q)}(\mathbb{Q}, \mathrm{ad}^0\rho_m) = 0$*. Then*

$$\partial_q\kappa_n^D(Qq) \supset \lambda_n^D(Q)\cdot(\varpi^C),$$

where

$$C = 2\,\mathrm{lg}_O\,\mathrm{Sel}_{\mathcal{F}(Q)^{\mathrm{rel}}}(\mathbb{Q}, \mathrm{ad}^0\rho_{n-m+1}) + m - 1.$$

(2) *Suppose there exists* $q'|Q$ *which is* not $(n+1)$*-admissible, such that* Q/q' *is* $(n+m)$*-admissible and*

$$\overline{\mathrm{Sel}}_{\mathcal{F}(Q/q')}(\mathbb{Q}, \mathrm{ad}^0\rho_m) = \overline{\mathrm{Sel}}_{\mathcal{F}(Qq)}(\mathbb{Q}, \mathrm{ad}^0\rho_m) = 0,$$

but

$$\overline{\mathrm{Sel}}_{\mathcal{F}(Q)}(\mathbb{Q}, \mathrm{ad}^0\rho_{2m-1}) \neq 0.$$

Then

$$\partial_q\kappa_n^D(Qq) \supset \lambda_n^D(Q)\cdot\varpi^C,$$

where $C = 2(m-1) + \mathrm{lg}_{\mathcal{O}}\mathrm{Sel}_{\mathcal{F}(Qq)^{\mathrm{rel}}}(\mathbb{Q}, \mathrm{ad}^0\rho_{n-m+1})$.

Proof. Choose an S-tidy level structure K for $\mathrm{GSpin}(V_{DQ})$ such that $\lambda_n^D(Q;K) = \lambda_n^D(Q)$ (possible by Lemma 4.4.7), and fix a q-adic uniformization datum for V_{DQq}, a special cycle $z \in \mathrm{SC}^2_{K^qK_q^{\mathrm{ram}}}(V_{DQq})$, and a test function $\alpha \in \mathrm{Test}_K(V_{DQ}, \pi, O/\varpi^n)$ satisfying the conclusion of Lemma 8.4.6; in particular, we have

$$\big(\alpha\circ\xi\circ\mathrm{res}_{\mathbb{Q}_q}(\partial_{\mathrm{AJ},\mathfrak{m}}(z))\big) = \lambda_n^D(Q). \tag{8.13}$$

Now note that

$$M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right)$$

is ϖ-power-torsion because $H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(2))_{\mathfrak{m}}$ is pure as a $G_{\mathbb{Q}_q}$ representation by Corollary 2.7.7 and Theorem 2.2.10(1). Hence by Lemma 8.2.16 and Theorem 7.4.7, we have a diagram:

$$\begin{tikzcd} M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \arrow[r, hook] \arrow[d, two heads, "\xi"] & H_{Qq}/(t_q) \\ H_Q/(\mathtt{T}_q^{\mathrm{lr}}), & \end{tikzcd} \tag{8.14}$$

where we set

$$H_Q := O\left[\mathrm{Sh}_K(V_{DQ})\right]_{\mathfrak{m}}, \quad H_{Qq} := H_{Qq}(K^qK_q^{\mathrm{ram}})$$

(Definition 8.2.14). By Remark 7.4.6, $(\mathtt{T}_q^{\mathrm{lr}}) = (P_q(q))$ as ideals of $\mathbb{T}^{S\cup\mathrm{div}(Q)}_{K,V_{DQ},\mathfrak{m}}$. Hence by Lemmas 8.2.10 and 8.2.13, the diagram (8.14) is a diagram of $R^{\mathrm{cong}}_{Q,q}$-modules. Let $Q' = Q$ in case (1) and $Q' = Qq$ in case (2), and define an element $a \in R_{Q'}$ as follows. Let $\tau_{Q'} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(O)$ be the representation constructed by Theorem B.2.12, and let $I_{Q'} \subset R_{Q'}$ be the kernel of the corresponding homomorphism $f_{Q'} : R_{Q'} \to O$. By Lemma B.4.5, we may fix an element $a \in \mathrm{Ann}_{R_{Q'}}(I_{Q'})$ such that

$$\mathrm{ord}_{\varpi} f_{Q'}(a) \leq \mathrm{lg}_O\,\mathrm{Sel}_{\mathcal{F}(Q')^{\mathrm{rel}}}(\mathbb{Q}, \mathrm{ad}^0\,\rho_{n-m+1}). \tag{8.15}$$

By the definition of C in each case of the theorem, we may assume without loss of generality that

$$\mathrm{lg}_O\,\mathrm{Sel}_{\mathcal{F}(Q')^{\mathrm{rel}}}(\mathbb{Q}, \mathrm{ad}^0\,\rho_{n-m+1}) < n-m+1.$$

Let $f_\pi : R^Q_{\mathfrak{m}} \to O$ be the map corresponding to ρ_π. Since $f_{Q'} \equiv f_\pi \pmod{\varpi^{n-m+1}}$ by Theorem B.2.12(1), we then have

$$\mathrm{ord}_{\varpi} f_\pi(a) \leq \mathrm{lg}_O\,\mathrm{Sel}_{\mathcal{F}(Q')^{\mathrm{rel}}}(k, \mathrm{ad}^0\,\rho_n). \tag{8.16}$$

Applying a to the diagram (8.14) of $R_{Q'}$-modules, we obtain a diagram

$$\begin{tikzcd} a\cdot M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}\right) \arrow[r, hook] \arrow[d, "\xi"] & a\cdot(H_{Qq}/(t_q)) \\ a\cdot\left(O\left[\mathrm{Sh}_K(V_{DQ})\right]_{\mathfrak{m}}/(\mathtt{T}_q^{\mathrm{lr}})\right) = aH_Q/(aH_Q\cap f_{Q'}(\mathtt{T}_q^{\mathrm{lr}})H_Q). & \end{tikzcd} \tag{8.17}$$

Suppose first we are in case (1). Because a annihilates I_Q, Lemma 8.2.10 implies that (8.17) is a diagram of $R^{\mathrm{cong}}_{Q,q} \otimes_{R_{Q,f_Q}} \mathcal{O} = \mathcal{O}/f_Q(\mathtt{T}_q^{\mathrm{lr}})$-modules. Note that

$$\frac{aH_Q}{af_Q(\mathtt{T}_q^{\mathrm{lr}})H_Q}$$

is free over $\mathcal{O}/f_Q(\mathtt{T}_q^{\mathrm{lr}})$ because H_Q, hence aH_Q, is ϖ-torsion-free. Since the natural surjection

$$\frac{aH_Q}{af_Q(\mathtt{T}_q^{\mathrm{lr}})H_Q} \twoheadrightarrow \frac{aH_Q}{aH_Q \cap f_Q(\mathtt{T}_q^{\mathrm{lr}})H_Q} \tag{8.18}$$

has kernel annihilated by $f_Q(a)$, we conclude that

$$f_Q(a) \cdot \mathrm{Ext}^1_{\mathcal{O}/f_Q(\mathtt{T}_q^{\mathrm{lr}})}(-, aH_Q/(aH_Q \cap f_Q(\mathtt{T}_q^{\mathrm{lr}})) = 0. \tag{8.19}$$

In particular, by (8.17), there exists a map $\widetilde{\xi} : a \cdot (H_{Qq}/(t_q)) \to aH_Q/(aH_Q \cap f_Q(\mathtt{T}_q^{\mathrm{lr}})H_Q)$ fitting into the following commutative diagram.

$$\begin{tikzcd}
a \cdot M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, \mathcal{O}(2))_{\mathfrak{m}}\right) \arrow[r, hook] \arrow[d, "f_Q(a)\cdot\xi"] & a \cdot (H_{Qq}/(t_q)) \arrow[dl, "\widetilde{\xi}"] \\
aH_Q/(aH_Q \cap \mathtt{T}_q^{\mathrm{lr}}H_Q) &
\end{tikzcd}$$

Recall the test function $\alpha \in \mathrm{Test}_K(V_{DQ}, \pi, \mathcal{O}/\varpi^n)$ fixed above, and let β denote the composite map

$$H_{Qq}/(t_q) \xrightarrow{a} a \cdot (H_{Qq}/(t_q)) \xrightarrow{\widetilde{\xi}} aH_Q/(aH_Q \cap \mathtt{T}_q^{\mathrm{lr}}H_Q) \xrightarrow{\alpha} \mathcal{O}/\varpi^n.$$

By Lemma 8.2.17 and (8.13), we conclude

$$\partial_q\kappa_n^D(Qq) \supset (\beta(\partial_{\mathrm{AJ},\mathfrak{m}}(z))) = \left(\alpha \circ a^2\xi \circ \mathrm{res}_{\mathbb{Q}_q}(\partial_{\mathrm{AJ},\mathfrak{m}}(z))\right) = f_\pi(a)^2\lambda_n^D(Q). \tag{8.20}$$

Then the theorem follows from (8.16).

For case (2), (8.17) is a diagram of $R^{\mathrm{cong}}_{Q,q} \otimes_{R_{Qq},f_{Qq}} \mathcal{O} = \mathcal{O}/f_{Qq}(t_q)$-modules. By Lemma 8.2.11, all admissible primes are standard in the sense of Definition B.4.7, so by Lemma B.4.8, we have

$$f_{Qq}(t_q) \not\equiv 0 \pmod{\varpi^{n+m}}. \tag{8.21}$$

In particular, (8.17) is a commutative diagram of $\mathcal{O}/\varpi^{n+m-1}$-modules. Because

$$\varpi^{m-1}\mathrm{Ext}^1_{\mathcal{O}/\varpi^{n+m-1}}(-, \mathcal{O}/\varpi^n) = 0,$$

we conclude that there exists a map $\widetilde{\alpha} : a \cdot (H_{Qq}/(t_q)) \to \mathcal{O}/\varpi^n$ fitting into the following commutative diagram:

$$\begin{tikzcd}
a \cdot M_{-1}H^1\left(I_{\mathbb{Q}_q}, H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}}, \mathcal{O}(2))_{\mathfrak{m}}\right) \arrow[r, hook] \arrow[d, "\alpha\circ\xi\circ\varpi^{m-1}"] & a \cdot (H_{Qq}/(t_q)) \arrow[dl, "\widetilde{\alpha}"] \\
\mathcal{O}/\varpi^n &
\end{tikzcd}$$

A priori, $\widetilde{\alpha}$ is only equivariant with respect to f_{Qq}, but $f_\pi \equiv f_{Qq} \pmod{\varpi^{n-m+1}}$. Multiplying by ϖ^{m-1}, we then obtain an f_π-equivariant map

$$\varpi^{m-1}(\widetilde{\alpha} \circ a) : H_{Qq}/(t_q) \to \mathcal{O}/\varpi^n.$$

Hence by Lemma 8.2.17 and (8.13), we conclude

$$\begin{aligned}\partial_q\kappa_n^D(Qq)\supset\varpi^{m-1}\left(\widetilde{\alpha}\circ a\circ\mathrm{res}_{\mathbb{Q}_q}(\partial_{\mathrm{AJ},\mathfrak{m}}(z))\right)&=\varpi^{2(m-1)}\left(\alpha\circ\xi\left(a\cdot\mathrm{res}_{\mathbb{Q}_q}\partial_{\mathrm{AJ},\mathfrak{m}}(z)\right)\right)\\&=\varpi^{2(m-1)}f_\pi(a)\left(\alpha\circ\xi\circ\mathrm{res}_{\mathbb{Q}_q}\partial_{\mathrm{AJ},\mathfrak{m}}(z)\right)\\&=\varpi^{2(m-1)}f_\pi(a)\lambda_n^D(Q).\end{aligned}$$

Combined with (8.16), this completes the proof in case (2). ☐

Theorem 8.5.2. *Suppose π is endoscopic associated to a pair (π_1,π_2). Assume $H^1_f(\mathbb{Q},V_{\pi_1,\mathfrak{p}}\otimes V_{\pi_2,\mathfrak{p}}(-1))=0$, and let C_{RS} be as in Lemma 8.3.5.*

Fix an integer $m\geq 1$, let $n_0=n_0(m,\rho_{\pi_1})$ satisfy the conclusion of Lemma B.4.5, and let $n\geq\max\{3m,n_0\}$ be an integer. Suppose $Q=Q_1\cdot Q_2$ is n-admissible such that Q_1 is BD-admissible for ρ_{π_1}, Q_2 is BD-admissible for ρ_{π_2}, and $\sigma(DQ)$ is odd. Let $q\nmid Q$ be an n-admissible prime, BD-admissible for ρ_{π_i}.

(1) Suppose $\overline{\mathrm{Sel}}_{\mathcal{F}(Q_i)}(\mathbb{Q},\mathrm{ad}^0\rho_{\pi_i,m})=0$. Then

$$\partial_q\kappa_n^D(Qq)\supset\lambda_n^D(Q)\cdot\varpi^C,$$

where

$$C=2\mathrm{lg}_O\mathrm{Sel}_{\mathcal{F}(Q_i)^{\mathrm{rel}}}(\mathbb{Q},\mathrm{ad}^0\rho_{\pi_i,n-m+1})+2C_{\mathrm{RS}}+m-1.$$

(2) Suppose there exists a prime $q'|Q_i$ which is not *$(n+1)$-admissible such that Q_i/q' is $(n+m)$-admissible and*

$$\overline{\mathrm{Sel}}_{\mathcal{F}(Q_i/q')}(\mathbb{Q},\mathrm{ad}^0\rho_{\pi_i,m})=\overline{\mathrm{Sel}}_{\mathcal{F}(Q_iq)}(\mathbb{Q},\mathrm{ad}^0\rho_{\pi_i,m})=0$$

but

$$\overline{\mathrm{Sel}}_{\mathcal{F}(Q_i)}(\mathbb{Q},\mathrm{ad}^0\rho_{\pi_i,2m-1})\neq 0.$$

Then

$$\partial_q\kappa_n^D(Qq)\supset\lambda_n^D(Q)\cdot\varpi^C,$$

where

$$C=\mathrm{lg}_O\mathrm{Sel}_{\mathcal{F}(Q_iq)^{\mathrm{rel}}}(\mathbb{Q},\mathrm{ad}^0\rho_{\pi_i,n-m+1})+C_{\mathrm{RS}}+2(m-1).$$

Proof. First, choose the S-tidy level structure K for $\mathrm{GSpin}(V_{DQ})$ such that $\lambda_n^D(Q;K)=\lambda_n^D(Q)$ (possible by Lemma 4.4.7). Then fix a q-adic uniformization datum for V_{DQq}, a special cycle $z\in\mathrm{SC}^2_{K^qK_q^{\mathrm{ram}}}(V_{DQq})$, and a test function $\alpha\in\mathrm{Test}_K(V_{DQ},\pi,O/\varpi^n)$ satisfying the conclusion of Lemma 8.4.6; in particular, we have

$$\left(\alpha\circ\xi\circ\mathrm{res}_{\mathbb{Q}_q}(\partial_{\mathrm{AJ},\mathfrak{m}}(z))\right)=\lambda_n^D(Q).$$

We also fix $j\in\mathrm{Ann}_{R_{\mathfrak{m}}^{Qq}}(J_{\mathrm{red}}^{Qq})$ satisfying the conclusion of Lemma 8.3.5.

As in the proof of Theorem 8.5.1, $M_{-1}H^1\left(I_{\mathbb{Q}_q},H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}},O(2))_{\mathfrak{m}}\right)$ is ϖ-power-torsion, hence $j_*M_{-1}H^1\left(I_{\mathbb{Q}_q},H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}},O(2))_{\mathfrak{m}}\right)$ is as well, where j_* is as in (8.8). Since the kernel of (8.8) is j-torsion, we obtain the following diagram arising from Theorem 7.4.7 and Lemma 8.3.23:

$$\begin{tikzcd}
j_*M_{-1}H^1(I_{\mathbb{Q}_q},H^3_{\text{ét}}(\mathrm{Sh}_{K^qK_q^{\mathrm{ram}}}(V_{DQq})_{\overline{\mathbb{Q}}},O(2))_{\mathfrak{m}}) \arrow[r, hook] \arrow[d, two heads, "\xi"] & H^1(I_{\mathbb{Q}_q},M^{\mathrm{univ}}_{Q_iq,1})^{\mathrm{Frob}_q=1}\otimes_{R_{Q_iq,1}}H^{(i)}_{Qq}\simeq H^{(i)}_{Qq}/(t_q) \\
j\cdot\left(H_Q/(\mathtt{T}_q^{\mathrm{lr}})\right)=jH_Q/(jH_Q\cap\mathtt{T}_q^{\mathrm{lr}}(H_Q) &
\end{tikzcd}$$

where we abbreviate

$$H_Q := O\left[\mathrm{Sh}_K(V_{DQ})\right]_{\mathfrak{m}},\ H_{Qq}^{(i)} := H_{Qq}(K,j)^{(i)}.$$

From here, the argument is entirely analogous to Theorem 8.5.1, replacing Lemma 8.2.17 with Lemma 8.3.24.

☐

9. MAIN RESULT: RANK ZERO CASE

9.1. **Chebotarev primes and proof of the main result.**

9.1.1. Throughout this section, we let π, S, and E_0 be as in Notation 4.0.1, and fix for now a prime $\mathfrak{p}$ of E_0.

Lemma 9.1.2. *Suppose that π is non-endoscopic, that $\mathfrak{p}$ satisfies Assumption 4.1.1(3), and that there exist admissible primes for $\rho_\pi = \rho_{\pi,\mathfrak{p}}$. Let $C \geq 0$ be the constant from Corollary C.2.8 applied to T_π. Then for all integers $m \geq n \geq 1$ and for any cocycle $c \in H^1(\mathbb{Q}, T_{\pi,n})$, there are infinitely many m-admissible primes q such that*

$$\mathrm{ord}_\varpi \operatorname{loc}_q c \geq \mathrm{ord}_\varpi c - C.$$

Recall here that loc_q was defined in Notation 4.2.9.

Proof. Let $g \in G_\mathbb{Q}$ be an admissible element for ρ_π, which is possible by Lemma 4.2.3. By Corollary C.2.8, we have $\varpi^C H^1(\mathbb{Q}(T_\pi)/\mathbb{Q}, T_{\pi,n}) = 0$, so by inflation-restriction there exists an element $h \in G_{\mathbb{Q}(T_\pi)}$ such that $\mathrm{ord}_\varpi c(h) \geq \mathrm{ord}_\varpi c - C$. Because $\overline{T}_\pi$ is absolutely irreducible, we can assume without loss of generality that the component of $c(h)$ in the 1-eigenspace for g is nonzero modulo $\varpi^{n-\mathrm{ord}_\varpi c + C + 1}$. Then since

$$c(gh) = gc(h) + c(g),$$

after possibly replacing g by gh we may assume without loss of generality that the same is true for the component of $c(g)$ in the 1-eigenspace for g (which is independent of the choice of cocycle representative). Then any prime $q \notin S \cup \{p\}$ with Frobenius conjugate to g in $\mathrm{Gal}(\mathbb{Q}(T_{\pi,m}, c))$ satisfies the conclusion of the lemma. ☐

The following theorem is a corollary of the work of Newton-Thorne [83] and Thorne [110].

Theorem 9.1.3. *Suppose π is non-endoscopic, and $\mathfrak{p}$ is a prime of E_0 of residue characteristic $p > 3$ such that π_p is unramified. Then*

$$H^1_f(\mathbb{Q}, \mathrm{ad}^0 \rho_{\pi,\mathfrak{p}}) = 0.$$

Proof. By [110, Theorem 6.2], it suffices to show

$$V_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}(\mu_{p^\infty})}} \text{ is absolutely irreducible.} \tag{9.1}$$

By Lemmas 2.2.12 and C.2.5, we can write

$$V_{\pi,\mathfrak{p}} \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} V_0$$

for a finite extension $K/\mathbb{Q}$, where V_0 is a strongly irreducible representation of G_K. By [87, Lemma 2.2.9] and the assumption that π_p is unramified, we conclude K is unramified at p; hence (9.1) follows from Lemma C.2.9. ☐

Theorem 9.1.4. *Let π be non-endoscopic. Suppose $\mathfrak{p}$ satisfies Assumption 4.1.1, and that there exist admissible primes for $\rho_{\pi,\mathfrak{p}}$. Suppose as well that there exists a prime ℓ_0 such that π_{ℓ_0} is transferrable (Definition 2.4.5). Then*

$$L(\pi, \mathrm{spin}, 1/2) \neq 0 \implies H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = 0.$$

Proof. Set $D := \ell_0$, so that, by Theorem 2.4.6, π_f^D can be completed to an automorphic representation of $\mathrm{GSpin}(V_D)(\mathbb{A})$. By Proposition 4.4.5, if $L(\pi, \mathrm{spin}, 1/2) \neq 0$ then we have $\lambda^D(1) \neq 0$, so there exists a constant $C_0 \geq 0$ such that

$$\mathrm{ord}_\varpi \lambda_m^D(1) \geq m - C_0, \ \ \forall m \geq 1. \tag{9.2}$$

Suppose for contradiction that there exists a non-torsion element $c \in H^1_f(\mathbb{Q}, T_\pi)$, and let c_n be the image of c in $H^1_f(\mathbb{Q}, T_{\pi,n})$ for all $n \geq 1$. We fix a large integer N to be specified later. Let $M \geq N$ be the integer of Lemma 1.6.3(3) for $M = T_\pi$ and $n = N$. By Lemma 9.1.2 and Lemma 4.1.6(2), we may choose an M-admissible prime q such that

$$\mathrm{ord}_\varpi \, \mathrm{loc}_q \, c_N \geq N - C_1 \tag{9.3}$$

for a constant $C_1 \geq 0$. Now by Theorem 9.1.3 and Lemma B.3.6 (which applies to ρ_π by Lemma 8.2.1), for some $m_0 \geq 1$ we have

$$\overline{\mathrm{Sel}}_{\mathcal{F}}(\mathbb{Q}, \mathrm{ad}^0 \rho_{\pi,m_0}) = 0.$$

Moreover, by Corollary B.4.3, $\mathrm{lg}_{\mathcal{O}} \mathrm{Sel}_{\mathcal{F}^{\mathrm{rel}}}(\mathbb{Q}, \mathrm{ad}^0 \rho_{\pi,n})$ is uniformly bounded in n. Hence by Theorem 8.5.1(1), as long as M is sufficiently large depending on m_0 – which we can ensure by choosing N sufficiently large – there exists a constant $C_2 \geq 0$ and an element $\kappa_M^D(q)_0 \in \kappa_M^D(q)$ such that

$$\mathrm{ord}_\varpi \partial_q \kappa_M^D(q)_0 \geq \mathrm{ord}_\varpi \lambda_M^D(1) - C_2 \geq M - C_2 - C_0. \tag{9.4}$$

Let $\kappa_N^D(q)_0$ be the image of $\kappa_M^D(q)_0$ in $H^1(\mathbb{Q}, T_{\pi_1,N})$. We now consider the global Tate pairing

$$0 = \langle \kappa_N^D(q)_0, c_N \rangle = \sum_v \langle \kappa_N^D(q)_0, c_N \rangle_v. \tag{9.5}$$

For $v \notin S \cup \{q\}$, the local Tate pairing vanishes by Proposition 4.4.6(1) and Lemma 1.6.3(3) – recall here that the local Tate pairing of two unramified classes is always trivial. By the same argument as [66, Lemma 4.3(1)], we may also pick a constant $C_3 \geq 0$ independent of N such that, for all $v \in S$, $\varpi^{C_3} H^1(\mathbb{Q}_v, T_{\pi_1,N}) = 0$; hence

$$\mathrm{ord}_\varpi \langle \kappa_N^D(q)_0, c_N \rangle_v \leq C_3, \ \forall v \in S.$$

It then follows from (9.5) that

$$\mathrm{ord}_\varpi \langle \kappa_N^D(q)_0, c_N \rangle_q \leq C_3.$$

On the other hand, by Proposition 4.2.8, (9.3) and (9.4) together imply

$$\mathrm{ord}_\varpi \langle \kappa_N^D(q)_0, c_N \rangle_q \geq N - C_0 - C_1 - C_2,$$

so we obtain a contradiction when

$$N > C_0 + C_1 + C_2 + C_3.$$

□

9.2. **The endoscopic case.**

9.2.1. For completeness, we include an analogue of Theorem 9.1.4 in the endoscopic case. First, we require an analogue of Lemma 9.1.2.

Lemma 9.2.2. *Suppose π is endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, and fix $i = 1$ or 2. Let $\mathfrak{p}$ be a prime of E_0 such that there exist admissible primes for $\rho_\pi = \rho_{\pi,\mathfrak{p}}$ which are BD-admissible for $\rho_{\pi_i} = \rho_{\pi_i,\mathfrak{p}}$. Then there is a constant C with the following property.*

For all integers $m \geq n \geq 0$ and for any cocycle $c \in H^1(\mathbb{Q}, T_{\pi_i})$, there are infinitely many m-admissible primes q, BD-admissible for ρ_{π_i}, such that

$$\mathrm{ord}_\varpi \, \mathrm{loc}_q \, c \geq \mathrm{ord}_\varpi c - C.$$

Proof. Without loss of generality, we may assume $i = 1$ (after possibly relabeling π_1 and π_2).

Claim. There exists a constant $C \geq 0$ such that $\varpi^C H^1(\mathbb{Q}(\rho_\pi)/\mathbb{Q}, T_{\pi_1,n}) = 0$ for all $n \geq 1$.

Proof of claim. By Corollary C.2.8 and inflation-restriction, it suffices to show

$$\mathrm{Hom}_{G_\mathbb{Q}}(\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\rho_{\pi_1})), T_{\pi_1,n})$$

is uniformly bounded in n. Note that, unless π_2 has CM by an imaginary quadratic subfield K of $\mathbb{Q}(\rho_{\pi_1})$, the abelianization of $\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\rho_{\pi_1}))$ is finite. Indeed, if π_2 is CM, this can be checked by hand; otherwise one can use Theorem C.3.2 and Corollary C.1.2(2) to see that no closed normal subgroup of $\mathrm{Gal}(\mathbb{Q}(\rho_{\pi_2})/\mathbb{Q}(\mu_{p^\infty}))$ can have an infinite abelian quotient.

So we assume without loss of generality that π_2 has CM by $K \subset \mathbb{Q}(\rho_{\pi_1})$. In this case, one can calculate directly that complex conjugation acts on $\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\rho_{\pi_1}))$ by -1, so the image of any Galois-invariant group homomorphism $\mathrm{Gal}(\mathbb{Q}(\rho_\pi,)/\mathbb{Q}(\rho_{\pi_1})) \to T_{\pi_1,n}$ lies in the -1 eigenspace of complex conjugation; since $\overline{T}_{\pi_1}$ is absolutely irreducible and odd, we conclude that all such homomorphisms vanish. ☐

Let $g \in G_\mathbb{Q}$ be an element which is admissible for ρ_π and BD-admissible for ρ_{π_1}, which is possible by (the argument of) Lemma 4.2.3. By the claim and inflation-restriction, there exists $h \in G_{\mathbb{Q}(T_\pi)}$ such that $\mathrm{ord}_\varpi c(h) \geq \mathrm{ord}_\varpi c - C$; as in the proof of Lemma 9.1.2 above, after possibly replacing g with gh, we may assume without loss of generality that the component of $c(g)$ in the 1-eigenspace for g is nonzero modulo $\varpi^{n-\mathrm{ord}_\varpi c + C + 1}$. Then any $q \notin S \cup \{p\}$ with Frobenius conjugate to g in $\mathrm{Gal}(\mathbb{Q}(T_{\pi,m}, c_n))$ satisfies the conclusion of the lemma. ☐

The same proof of Theorem 9.1.3 also shows:

Proposition 9.2.3. *Suppose π is endoscopic associated to a pair (π_1, π_2). Then if $\mathfrak{p}$ is any prime of E_0 of residue characteristic p such that π_p is unramified,*

$$H^1_f(\mathbb{Q}, \mathrm{ad}^0 \rho_{\pi_1,\mathfrak{p}}) = H^1_f(\mathbb{Q}, \mathrm{ad}^0 \rho_{\pi_2,\mathfrak{p}}) = 0.$$

Theorem 9.2.4. *Let π be endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, which we order so that $\pi_{1,\infty}$ and $\pi_{2,\infty}$ have weights 2 and 4, respectively. Assume there exists a prime ℓ_0 such that π_{1,ℓ_0} is discrete series.*

Fix $i = 1$ or 2. Then for any prime $\mathfrak{p}$ satisfying Assumption 4.1.1, such that there exist admissible primes for $\rho_{\pi,\mathfrak{p}}$ which are BD-admissible for $\rho_{\pi_i,\mathfrak{p}}$, if $H^1_f(\mathbb{Q}, V_{\pi_1,\mathfrak{p}} \otimes V_{\pi_2,\mathfrak{p}}(-1)) = 0$ then we have

$$L(\pi, \mathrm{spin}, 1/2) \neq 0 \implies H^1_f(\mathbb{Q}, V_{\pi_i,\mathfrak{p}}) = 0.$$

Remark 9.2.5. Note that $L(\pi, \mathrm{spin}, 1/2)$ is the product of central L-values for π_1 and π_2; in particular, Theorem 9.2.4 recovers (with extra conditions) the result of Kato [48].

Proof. The proof of Theorem 9.1.4 applies almost verbatim, with the following substitutions: Theorem 2.5.2 for Theorem 2.4.6; Lemma 9.2.2 for Lemma 9.1.2; Proposition 9.2.3 for Theorem 9.1.3; and Theorem 8.5.2(1) for Theorem 8.5.1(1). ☐

9.3. **Rigidity and $\mathfrak{p}$-integral vanishing of the Selmer group.**

9.3.1. We can also give a more precise result on the vanishing of the dual Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}/T_{\pi,\mathfrak{p}})$, under some stronger conditions. Assume for this subsection that π is not endoscopic.

We consider the following additional assumptions on $\mathfrak{p}$ (the R stands for "rigid").

(R1) The image of the $G_\mathbb{Q}$ action on $\overline{T}_{\pi,\mathfrak{p}}$ contains a nontrivial scalar.
(R2) We have $\mathrm{Sel}_{\mathcal{F}^{\mathrm{rel}}}(\mathbb{Q}, \mathrm{ad}^0 \overline{\rho}) = 0$, with notation as in Definition B.4.1.
(R3) For all $\ell \in S$, $H^1(\mathbb{Q}_\ell, \overline{T}_{\pi,\mathfrak{p}}) = 0$.

It is proved in Theorem C.4.9 that (R1) holds for cofinitely many $\mathfrak{p}$. We will show in Proposition 9.3.5 below that the same is true of (R2) and (R3).

9.3.2. Under Assumption 4.1.1(3), let $R_{\mathfrak{p}}^{\mathrm{univ}}$ be the universal GSp_4-valued deformation ring of $\overline{\rho}_{\pi,\mathfrak{p}}$ denoted R^1 in Notation 8.2.6, with corresponding universal deformation $\rho_{\mathfrak{p}}^{\mathrm{univ}} : G_{\mathbb{Q},S\cup\{p\}} \to \mathrm{GSp}_4(R_{\mathfrak{p}}^{\mathrm{univ}})$.

Lemma 9.3.3. *Let* $(R_{\mathfrak{p}}^{\mathrm{univ}})^0 \subset R_{\mathfrak{p}}^{\mathrm{univ}}$ *be the subring generated by the coefficients of the characteristic polynomials of elements of* $G_{\mathbb{Q}}$ *under the composite*

$$\rho_{\mathfrak{p}}^{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(R_{\mathfrak{p}}^{\mathrm{univ}}) \hookrightarrow \mathrm{GL}_4(R_{\mathfrak{p}}^{\mathrm{univ}}).$$

Then

$$(R_{\mathfrak{p}}^{\mathrm{univ}})^0 = R_{\mathfrak{p}}^{\mathrm{univ}}.$$

Proof. Recall the category CNL_O from (1.1.3). By the universal property of $\rho_{\mathfrak{p}}^{\mathrm{univ}}$, it suffices to show the following: given two morphisms

$$f_1, f_2 : R_{\mathfrak{p}}^{\mathrm{univ}} \to A$$

in CNL_O, if $f_1 = f_2$ on $(R_{\mathfrak{p}}^{\mathrm{univ}})^0$, then the associated deformations

$$\rho_1, \rho_2 : G_{\mathbb{Q}} \to \mathrm{GSp}_4(A)$$

are $\mathrm{GSp}_4(A)$-conjugate. By [20, Théorème 1], ρ_1 and ρ_2 are $\mathrm{GL}_4(A)$-conjugate, say by a matrix $a \in \mathrm{GL}_4(A)$. Then

$$\rho_1(g) = a \cdot \rho_2(g) \cdot a^{-1} \ \ \forall g \in G_{\mathbb{Q}}. \tag{9.6}$$

On the other hand, because ρ_1 and ρ_2 are valued in GSp_4 with similitude character χ_p^{cyc}, we also have

$$\rho_i(g) = -\chi_{p,\mathrm{cyc}}(g) \cdot \Omega \cdot \rho_i(g)^{-t} \cdot \Omega \ \ \forall g \in G_{\mathbb{Q}}, \ i = 1, 2, \tag{9.7}$$

where Ω is the matrix from (1.1.4) with $n = 2$. Combining (9.6) and (9.7) and using Schur's lemma, it follows that $\Omega a^t \Omega a$ is scalar, or equivalently $a \in \mathrm{GSp}_4(A)$, and this proves the lemma. ☐

Lemma 9.3.4. *For all but finitely many primes* $\mathfrak{p}$ *of* E_0, $\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}(\zeta_p)}}$ *is absolutely irreducible.*

Proof. First, recall that $\overline{\rho}_{\pi,\mathfrak{p}}$ is absolutely irreducible for all but finitely many $\mathfrak{p}$ by Lemma 4.1.5. We restrict our attention to these $\mathfrak{p}$; the argument is based on [65, Proposition 4.5], but we are able to take advantage of the low-dimensionality of $\overline{\rho}_{\pi,\mathfrak{p}}$. The representation $\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}(\zeta_p)}}$ is semisimple, so, after possibly extending scalars, we may write

$$\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}(\zeta_p)}} = \bigoplus_{i=1}^{M} \rho_i^{\oplus m_i}$$

for distinct absolutely irreducible representations ρ_i of $G_{\mathbb{Q}(\zeta_p)}$ and multiplicities $m_i \geq 1$. The same argument as [38, Lemma 2.1] (which applies here because $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ has degree coprime to p) shows that the integers $m_i = m$ are all equal, and all ρ_i have the same dimension M, such that $m^2 M$ is the number of characters ν of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ such that $\overline{\rho}_{\pi,\mathfrak{p}} \cong \overline{\rho}_{\pi,\mathfrak{p}} \otimes \nu$. Considering the self-duality of $\overline{\rho}_{\pi,\mathfrak{p}}$, we see that this can occur only if $\nu^2 = 1$, so that $m^2 M \leq 2$. In particular, $m = 1$, so $\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}(\zeta_p)}}$ is multiplicity-free, and the result now follows by the same argument as [65, Proposition 4.5]. ☐

Proposition 9.3.5. *All but finitely many primes* $\mathfrak{p}$ *of* E_0 *satisfy (R2) and (R3).*

Proof. First, choose an imaginary quadratic field K such that all $\ell \in S$ are split in K, and restrict attention to the cofinitely many $\mathfrak{p}$ such that the underlying rational prime p is unramified in K and $\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible. Then $\mathbb{Q}(\rho_{\pi,\mathfrak{p}}) \cap K$ is unramified at all finite primes, hence equal to $\mathbb{Q}$, and so

$$\rho_{\pi,\mathfrak{p}}(G_K) = \rho_{\pi,\mathfrak{p}}(G_{\mathbb{Q}}). \tag{9.8}$$

Note as well that $\mathrm{BC}(\pi) \otimes \omega_{K/\mathbb{Q}} \ncong \mathrm{BC}(\pi)$ (with $\mathrm{BC}(\pi)$ as in Lemma 2.2.17), for otherwise we would have $\rho_{\pi,\mathfrak{p}} \otimes \omega_{K/\mathbb{Q}} \cong \rho_{\pi,\mathfrak{p}}$, and it is easy to check using Schur's lemma that this implies $\rho_{\pi,\mathfrak{p}}|_{G_K}$ is not absolutely irreducible for any $\mathfrak{p}$, contradicting (9.8) and Lemma 2.2.12 as long as $p > 3$. In particular, the base change Π of $\mathrm{BC}(\pi)$ to $\mathrm{GL}_4(\mathbb{A}_K)$ is a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_K)$ by [3, Chapter 3, Theorems 4.2(a), 5.1]. It is also clear that E_0 is a strong coefficient field for Π.

Recall the universal deformation $\rho_{\mathfrak{p}}^{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(R_{\mathfrak{p}})$ from (9.3.2). Note that $K \cap \mathbb{Q}(\rho_{\mathfrak{p}}^{\mathrm{univ}})$ is unramified at all finite primes, hence equal to $\mathbb{Q}$; in particular, there exists $\gamma \in G_{\mathbb{Q}}$ such that $\rho_{\mathfrak{p}}^{\mathrm{univ}}(\gamma) = 1$ and γ acts nontrivially on K.

Recall the group scheme

$$\mathscr{G} := (\mathrm{GL}_4 \times \mathrm{GL}_1) \rtimes \{1, \mathfrak{c}\}$$

from [25, §2.1]. By the recipe of [65, Lemma 2.3(2)], $\rho_{\mathfrak{p}}^{\mathrm{univ}}$ gives rise to a Galois representation

$$\widetilde{\rho}_{\mathfrak{p}}^{\mathrm{univ}} : G_{\mathbb{Q}} \to \mathscr{G}(R_{\mathfrak{p}}^{\mathrm{univ}}), \tag{9.9}$$

with residual representation $\widetilde{\overline{\rho}}_{\mathfrak{p}} : G_{\mathbb{Q}} \to \mathscr{G}(O_{\mathfrak{p}}/\varpi_{\mathfrak{p}})$. Note

$$\widetilde{\rho}_{\mathfrak{p}}^{\mathrm{univ}}|_{G_K} = (\rho_{\mathfrak{p}}^{\mathrm{univ}}|_{G_K}, \chi_{p,\mathrm{cyc}}, 1) : G_K \to (\mathrm{GL}_4 \times \mathrm{GL}_1)(R_{\mathfrak{p}}^{\mathrm{univ}}) \rtimes \{1, \mathfrak{c}\}\,, \tag{9.10}$$

where p is the residue characteristic of $\mathfrak{p}$. By the argument of [65, Theorem 4.8] applied to Π, we have for all but finitely many $\mathfrak{p}$:

(1) $\widetilde{\overline{\rho}}_{\mathfrak{p}} : G_{\mathbb{Q}} \to \mathscr{G}(O_{\mathfrak{p}}/\varpi_{\mathfrak{p}})$ satisfies [65, Definition 3.36] for the pair $(S, \emptyset)$.

(2) There are no regular algebraic conjugate self-dual cuspidal automorphic representations Π' of $\mathrm{GL}_N(\mathbb{A}_K)$ such that Π' is unramified outside primes above S, Π' has the same archimedean weights as Π, and there is a congruence of associated Hecke eigensystems with respect to any isomorphism $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ inducing $\mathfrak{p}$.

For all but finitely many $\mathfrak{p}$, we also have:

(3) $\widetilde{\overline{\rho}}_{\pi,\mathfrak{p}}|_{G_{K(\zeta_p)}}$ is absolutely irreducible

by Lemma 9.3.4 and (9.8).

Restrict to the $\mathfrak{p}$ satisfying the above properties with $p \notin S$, and such that $\rho_{\mathfrak{p}}^{\mathrm{univ}}|_{G_{\mathbb{Q}_p}}$ is Fontaine-Lafaille, i.e. $p \geq 5$. Let $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$ be the universal Fontaine-Laffaille deformation ring of $\widetilde{\overline{\rho}}_{\mathfrak{p}}$ considered in [65, p. 1630] with $\Sigma^+_{\min} = S$ and $\Sigma^+_{\mathrm{lr}} = \emptyset$; more explicitly, $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}}$ classifies deformations that are unramified outside primes above $S \cup \{p\}$ and Fontaine-Laffaille at primes above p. It follows from the definition that (9.9) corresponds to an $O_{\mathfrak{p}}$-algebra morphism

$$\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}} \to R_{\mathfrak{p}}^{\mathrm{univ}}. \tag{9.11}$$

Now let V be the unique four-dimensional Hermitian space over K which is positive definite and split at all finite places. Then [65, Theorem 3.38] applied to V, together with properties (1)-(3) above, shows that $\mathsf{R}_{\mathscr{S}}^{\mathrm{univ}} = O_{\mathfrak{p}}$ for all but finitely many $\mathfrak{p}$. We claim that these $\mathfrak{p}$ satisfy (R2) and (R3).

Now, by Remark B.4.6, (R2) is equivalent to requiring that the map $R_{\mathfrak{p}}^{\mathrm{univ}} \to O_{\mathfrak{p}}$ arising from $\rho_{\pi,\mathfrak{p}}$ is an isomorphism, and so it suffices to show that the morphism (9.11) is surjective. By (9.10), the image of (9.11) contains the coefficients of the characteristic polynomials of $\rho_{\mathfrak{p}}^{\mathrm{univ}}(g)$ for $g \in G_K$, and so the desired surjectivity follows from Lemma 9.3.3 and the analogue of (9.8) for $\rho_{\mathfrak{p}}^{\mathrm{univ}}$.

For (R3), the local deformation ring of

$$\widetilde{\overline{\rho}}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}_\ell}} : G_{\mathbb{Q}_\ell} \to \mathscr{G}(O_{\mathfrak{p}}/\varpi_{\mathfrak{p}})$$

is formally smooth of relative dimension 16 over $O_{\mathfrak{p}}$ for all $\ell \in S$: this follows from [65, Proposition 3.33(3)], where Definition 3.36(1) of *op. cit.* is satisfied by property (1) above. In fact, because ℓ splits in K, this is just the local deformation ring of

$$\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}_\ell}} : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_4(O_{\mathfrak{p}}/\varpi_{\mathfrak{p}})$$

(without any self-duality condition). The usual tangent space computation together with the formal smoothness then implies

$$16 = \dim_{O_{\mathfrak{p}}/\varpi_{\mathfrak{p}}} Z^1(\mathbb{Q}_\ell, \operatorname{End} \overline{T}_\pi) = 16 - \dim_{O_{\mathfrak{p}}/\varpi_{\mathfrak{p}}} H^0(\mathbb{Q}_\ell, \operatorname{End} \overline{T}_\pi) + H^1(\mathbb{Q}_\ell, \operatorname{End} \overline{T}_\pi).$$

In particular, by the local Euler characteristic formula and local Poitou-Tate duality, we conclude

$$H^0(\mathbb{Q}_\ell, \operatorname{End} \overline{T}_\pi(1)) = 0. \tag{9.12}$$

We then claim that $H^0(\mathbb{Q}_\ell, \overline{T}_\pi) = 0$: otherwise $\overline{\rho}_{\pi,\mathfrak{p}}$ has a $G_{\mathbb{Q}_\ell}$-invariant line, so by duality, it also has a quotient on which $G_{\mathbb{Q}_\ell}$ acts by $\chi_{p,\mathrm{cyc}}$, and so there is a line in $\operatorname{End} T_\pi$ on which $G_{\mathbb{Q}_\ell}$ acts by $\chi_{p,\mathrm{cyc}}^{-1}$, contradicting (9.12).

On the other hand, again by the local Euler characteristic formula and local duality, we see that $\dim H^1(\mathbb{Q}_\ell, \overline{T}_\pi) = 2 \dim H^0(\mathbb{Q}_\ell, \overline{T}_\pi)$, so (R3) follows.

☐

Proposition 9.3.6. *Suppose* $\mathfrak{p}$ *satisfies Assumption 4.1.1(3) and (R1). Then for any nonzero class* $c \in H^1(\mathbb{Q}, \overline{T}_{\pi,\mathfrak{p}})$ *and any* $N \geq 0$*, there exist infinitely many* N*-admissible primes* q *such that* $\operatorname{loc}_q c \neq 0$.

Proof. By the same argument as [40, Proposition 9.1], assumption (R1) implies that $H^1(\mathbb{Q}(\overline{T}_\pi)/\mathbb{Q}, \overline{T}_\pi) = 0$, so by inflation-restriction

$$c|_{G_{\mathbb{Q}(\overline{T}_\pi)}} : G_{\mathbb{Q}(\overline{T}_\pi)} \to \overline{T}_\pi$$

is nontrivial. Let $g \in G_{\mathbb{Q}}$ be an admissible element for ρ_π. Arguing as in Lemma 9.1.2, there exists $h \in G_{\mathbb{Q}}$ such that h acts as g on $\overline{T}_\pi$ and $c(h)$ has nonzero component in the 1-eigenspace for h.

Claim. We have $\mathbb{Q}(T_{\pi,N}) \cap \mathbb{Q}(\overline{T}_\pi, c) = \mathbb{Q}(\overline{T}_\pi)$.

Proof of claim. It suffices to show any group G with $G_{\mathbb{Q}}$-action which is a quotient of both the Galois groups $\operatorname{Gal}(\mathbb{Q}(\overline{T}_\pi, c)/\mathbb{Q}(\overline{T}_\pi))$ and $\operatorname{Gal}(\mathbb{Q}(T_{\pi,N})/\mathbb{Q}(\overline{T}_\pi))$, must be trivial. Note that $\operatorname{Gal}(\mathbb{Q}(\overline{T}_\pi, c)/\mathbb{Q}(\overline{T}_\pi))$ is an $\mathbb{F}_p[G_{\mathbb{Q}}]$-submodule of $\overline{T}_\pi$, so any element $z \in G_{\mathbb{Q}}$ that acts as a nontrivial scalar on $\overline{T}_\pi$ acts nontrivially on G unless $G = 1$.

On the other hand, the group $\operatorname{Gal}(\mathbb{Q}(T_{\pi,N})/\mathbb{Q}(\overline{T}_\pi))$ has a $G_{\mathbb{Q}}$-stable filtration in which each quotient is abelian and isomorphic to an $\mathbb{F}_p[G_{\mathbb{Q}}]$-subquotient of $\operatorname{ad}^0 \overline{\rho}_\pi$. In particular, z acts trivially on G, which is an abelian quotient of $\operatorname{Gal}(\mathbb{Q}(T_{\pi,N})/\mathbb{Q}(\overline{T}_\pi))$. We conclude G is trivial. ☐

By the claim, there exists $\widetilde{g} \in G_{\mathbb{Q}}$ that acts as g on $T_{\pi,N}$ and has image h in $\operatorname{Gal}(\mathbb{Q}(\overline{T}_\pi, c)/\mathbb{Q}(\overline{T}_\pi))$. Any prime $q \notin S \cup \{p\}$ with Frobenius conjugate to $\widetilde{g}$ in $\operatorname{Gal}(\mathbb{Q}(T_{\pi,N}, c)/\mathbb{Q})$ satisfies the conclusion of the lemma.

☐

Definition 9.3.7. Let $\mathfrak{p}$ be a prime of E_0 such that $\overline{T}_{\pi,\mathfrak{p}}$ is absolutely irreducible (so $T_{\pi,\mathfrak{p}}$ is well-defined up to scalars).

(1) We write

$$W_{\pi,\mathfrak{p}} = V_{\pi,\mathfrak{p}}/T_{\pi,\mathfrak{p}}.$$

As usual, we drop the subscript $\mathfrak{p}$ when clarity permits.

(2) For each rational prime ℓ, let $H^1_f(\mathbb{Q}_\ell, W_\pi)$ be the annihilator of $H^1_f(\mathbb{Q}_\ell, T_\pi)$ under the perfect local Tate pairing

$$H^1(\mathbb{Q}_\ell, W_\pi) \times H^1(\mathbb{Q}_\ell, T_\pi) \to E/O,$$

and let

$$H^1_f(\mathbb{Q}, W_\pi) := \ker\left(H^1(\mathbb{Q}, W_\pi) \to \prod_\ell \frac{H^1(\mathbb{Q}_\ell, W_\pi)}{H^1_f(\mathbb{Q}_\ell, W_\pi)}\right).$$

Theorem 9.3.8. *Suppose π is non-endoscopic, and $\mathfrak{p}$ is a prime of E_0 such that:*

1. *Assumption 4.1.1 and (R1)-(R3) hold for $\mathfrak{p}$, and there exist admissible primes for $\rho_{\pi,\mathfrak{p}}$.*
2. *There exists a squarefree $D > 1$ with $\nu(D)$ odd, such that π_ℓ is transferrable for all $\ell | D$ (Definition 2.4.5) and*

$$\lambda^D(1)_{\mathfrak{p}} \not\equiv 0 \pmod{\mathfrak{p}}.$$

Then

$$H^1_f(\mathbb{Q}, W_{\pi,\mathfrak{p}}) = 0.$$

Proof. We follow the proof of Theorem 9.1.4, but with some integral refinements. Fix $\mathfrak{p}$ and D satisfying (1) and (2), and omit $\mathfrak{p}$ from the notation for the rest of the proof. Suppose for contradiction that $H^1_f(\mathbb{Q}, W_\pi) \neq 0$. By the long exact sequence in Galois cohomology associated to

$$0 \to \overline{T}_\pi \to W_\pi \to W_\pi \to 0,$$

we have

$$H^1(\mathbb{Q}, \overline{T}_\pi) = H^1(\mathbb{Q}, W_\pi)[\varpi];$$

in particular, there exists a class $0 \neq c \in H^1(\mathbb{Q}, \overline{T}_\pi)$ with the following property:

$$\operatorname{Res} c_v \in H^1(\mathbb{Q}_v, \overline{T}_\pi) \text{ has trivial image in } \frac{H^1(\mathbb{Q}_v, W_\pi)}{H^1_f(\mathbb{Q}_v, W_\pi)} \text{ for all primes } v. \tag{9.13}$$

We fix an integer $N \geq 3$ sufficiently large to satisfy the conclusion of Lemma 1.6.3(2) for $n = 1$ and $M = T_\pi$, and greater than the number $n_0(1, \rho_\pi)$ from Theorem B.2.12.[10]

By Proposition 9.3.6 and the assumption (R1), we may choose an N-admissible prime $q \notin S$ such that

$$\operatorname{loc}_q c \neq 0. \tag{9.14}$$

Then by Theorem 8.5.1(1) and the assumption (R2), we have an element $\kappa^D_N(q)_0 \in \kappa^D_N(q)$ such that its image $\kappa^D_1(q)_0$ in $H^1(\mathbb{Q}, \overline{T}_\pi)$ satisfies

$$\partial_q \kappa^D_1(q)_0 \neq 0. \tag{9.15}$$

We now consider the global Tate pairing

$$0 = \sum_v \langle c, \kappa^D_1(q)_0 \rangle_v. \tag{9.16}$$

By the assumption (R3), the local terms vanish for all $v \neq q, p$. The local term at $v = q$ is nonzero by Proposition 4.2.8 combined with (9.14), (9.15). So to obtain a contradiction with (9.16), it suffices to show the local pairing at p vanishes. Indeed, the maps

$$\alpha : H^1(\mathbb{Q}_p, T_\pi) \to H^1(\mathbb{Q}_p, \overline{T}_\pi), \quad \beta : H^1(\mathbb{Q}_p, \overline{T}_\pi) \to H^1(\mathbb{Q}_p, W_\pi)$$

are adjoint with respect to the local Tate pairings, and by Lemma 1.6.3(2) combined with Proposition 4.4.6, there exists $d \in H^1_f(\mathbb{Q}_p, T_\pi)$ such that $\alpha(d) = \operatorname{Res}_p \kappa^D_1(q)_0$. Hence indeed

$$\langle \kappa^D_1(q)_0, c \rangle_p = \langle d, \beta(\operatorname{Res}_p c) \rangle_p = 0$$

by (9.13). ☐

Corollary 9.3.9. *Suppose π is relevant and non-endoscopic, and there exists a place ℓ_0 such that π_{ℓ_0} is transferrable (Definition 2.4.5). If $L(\pi, \mathrm{spin}, 1/2) \neq 0$, then for all but finitely many primes $\mathfrak{p}$ such that admissible primes exist for $\rho_{\pi,\mathfrak{p}}$, $H^1_f(\mathbb{Q}, W_{\pi,\mathfrak{p}}) = 0$.*

---

[10] For cofinitely many $\mathfrak{p}$, all the local deformation rings of $\overline{\rho}_{\pi,\mathfrak{p}}|_{G_{\mathbb{Q}_\ell}}$ for $\ell \in S \cup \{p\}$ will be formally smooth (by the same argument of Proposition 9.3.5 for $\ell \in S$ and by [10, Theorem A] for $\ell = p$), so one can take $n_0(1, \rho_{\pi,\mathfrak{p}}) = 1$. The assumption $N \geq 3$ is to conform to the statement of Theorem 8.5.1 with $m = 1$, but in reality it is not needed since $N \geq 3m$ is used only in Theorem 8.5.1(2). In particular, the argument would work with $N = 1$ in practice.

Proof. By Theorem 2.4.6, $\pi_f^{\ell_0}$ can be completed to an automorphic representation of $\mathrm{GSpin}(V_{\ell_0})(\mathbb{A})$. Thus the corollary follows from Theorem 9.3.8 combined with Proposition 4.4.5, Lemma 4.1.5, Theorem C.4.9, and Proposition 9.3.5. ☐

Conditions are given in Theorem C.4.11 under which admissible primes exist for $\rho_{\pi,\mathfrak{p}}$ for cofinitely many $\mathfrak{p}$. In particular:

Corollary 9.3.10. *Suppose π is relevant and non-endoscopic, and there exists a place ℓ such that π_ℓ is type IIa in the sense of* [95]. *If $L(\pi, \mathrm{spin}, 1/2) \neq 0$, then for all but finitely many primes $\mathfrak{p}$, $H^1_f(\mathbb{Q}, W_{\pi,\mathfrak{p}}) = 0$.*

9.4. **Applications to automorphic inductions.** In this section, we give some corollaries of Theorem 9.1.4 which may be of independent interest. Both of them could be upgraded to statements about dual Selmer groups using Theorem 9.3.8; we omit the details only for concision.

In the next two corollaries, when π is a cuspidal unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that π_∞ is discrete series of weight $k \geq 2$ and $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ is an isomorphism, the associated p-adic Galois representations ρ_{π,ι_p} are normalized so that $\det \rho_{\pi,\iota_p} = \chi_{p,\mathrm{cyc}}^{k-1}\omega_\pi$, where ω_π is the central character. We also have the usual p-adic Galois representation χ_{ι_p} associated to any algebraic automorphic character χ of $\mathbb{A}_K^\times$, with $K/\mathbb{Q}$ a number field. We write the (semisimplified) reductions mod p as $\overline{\rho}_{\pi,\iota_p}$ and $\overline{\chi}_{\iota_p}$.

Corollary 9.4.1. *Let π be a non-CM cuspidal unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with π_∞ discrete series of weight 3, and with central character ω_π. Let K be an imaginary quadratic field and $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ an automorphic character of infinity type $(-1, 0)$ such that $\chi|_{\mathbb{A}_\mathbb{Q}^\times} = |\cdot|\omega_\pi^{-1}$.*

Fix an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and assume:

(1) p splits in K and is coprime to the conductor of f and χ.
(2) For some inert nonarchimedean place v of K, $\mathrm{WD}(\rho_{\pi,\iota_p}|_{G_{K_v}})$ is indecomposable.
(3) $\overline{\rho} := \overline{\rho}_{\pi,\iota_p} \otimes \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{\chi}_{\iota,p}$ satisfies:
  (a) $\overline{\rho}$ is absolutely irreducible and generic (Definition 2.7.3).
  (b) There exists a prime q such that $\overline{\rho}|_{G_{\mathbb{Q}_q}}$ is unramified, $q^4 \not\equiv 1 \pmod{p}$, and $\overline{\rho}(\mathrm{Frob}_q)$ has eigenvalues $\{q, 1, \alpha, q/\alpha\}$ with $\alpha \notin \{\pm 1, \pm q, q^2, q^{-1}\}$.

Then

$$L(f, \chi, 1/2) \neq 0 \implies H^1_f(K, \rho_{\pi,\iota_p} \otimes \chi_{\iota_p}) = 0.$$

Moreover, the conditions in (3) hold for all but finitely many p split in K and all choices of ι_p.

Remark 9.4.2. The condition that p split in K is actually necessary for (3b) to hold.

Proof. Let $\mathrm{BC}_K(\pi)$ be the base change to $\mathrm{GL}_2(\mathbb{A}_K)$. By [3, Chapter 3, Theorems 4.2(e), 5.1], there is a (strong) automorphic induction Π of $\mathrm{BC}_K(\pi) \otimes \chi$ to $\mathrm{GL}_4(\mathbb{A})$. Then by [92, Theorem C], there is an automorphic representation $\widetilde{\pi}$ of $\mathrm{GSp}_4(\mathbb{A})$ with trivial central character, such that Π is the base change of $\widetilde{\pi}$ as in Lemma 2.2.17. Note that $\widetilde{\pi}$ is relevant by a direct calculation with archimedean L-parameters. We have

$$\rho_{\widetilde{\pi},\iota_p} = \rho_{\pi,\iota_p} \otimes \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \chi_{\iota_p},$$

where the symplectic structure is by viewing ρ_{π,ι_p} as symplectic and $\mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \chi_{\iota_p}$ as orthogonal. The "moreover" assertion of the corollary therefore follows from Lemma C.4.7 combined with Lemma 4.1.5.

For the rest, by Shapiro's Lemma we have $H^1_f(\mathbb{Q}, \rho_{\widetilde{\pi},\iota_p}) = H^1_f(K, \rho_{\pi,\iota_p} \otimes \chi_{\iota_p})$. So by Theorem 9.1.4 applied to $\widetilde{\pi}$, it suffices to show that there exists a prime ℓ such that $\widetilde{\pi}_\ell$ is transferrable.

Let v be the place in (2), and let ℓ be the rational prime underlying v. Comparing Definition 2.4.5 with the explicit local Langlands parameters in [95, Table A.7] (and using as well Theorem 2.2.10(1) to see that

$\widetilde{\pi}_\ell$ is tempered), we see that it suffices to show the associated Weil-Deligne representation

$$\tau_\ell : W_{\mathbb{Q}_\ell} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$$

of $\rho_{\widetilde{\pi},\iota_p}|_{G_{\mathbb{Q}_\ell}}$ does not factor through a Siegel parabolic subgroup; equivalently, τ_ℓ does not stabilize an isotropic plane. Let W be the underlying two-dimensional complex symplectic space of the Weil-Deligne representation $\tau_{\pi,\ell} : W_{\mathbb{Q}_\ell} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ corresponding to $\rho_{\pi,\iota_p}|_{G_{\mathbb{Q}_\ell}}$, and let $V = \mathrm{Ind}_{W_{K_v}}^{W_{\mathbb{Q}_\ell}} \tau_{\chi,v}$, where $\tau_{\chi,v} : W_{K_v} \to \mathbb{C}^\times$ is the character corresponding to $\chi_{\iota_p}|_{G_{K_v}}$.

In particular, there is an isotropic basis $\{e_1, e_2\}$ for V stable (as a set) under W_{K_v}. Suppose for contradiction that $I \subset W \otimes V$ is a $W_{\mathbb{Q}_\ell} \times \mathrm{SL}(2, \mathbb{C})$-stable isotropic plane. Because v is inert, $I \neq W \otimes e_1, W \otimes e_2$. In particular, it follows that

$$I = \{w \otimes e_1 + g(w) \otimes e_2,\ w \in W\}$$

for some $g \in \mathrm{GL}_2(\mathbb{C})$. Then g commutes with $\tau_{\pi,\ell}(W_{K_v} \times \mathrm{SL}(2, \mathbb{C}))$, hence is scalar by (2); but clearly such an I is not isotropic, so we have obtained a contradiction. □

Corollary 9.4.3. *Let K be a real quadratic field, and let π be a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_K)$ with π_v discrete series of weights 2 and 4, in some order, for the two places $v|\infty$ of K.*

Let p be a prime and let $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ be an isomorphism such that:

1. *p is unramified in K and coprime to the conductor of π.*
2. *π_v is discrete series for some nonarchimedean place v of K, and if v is split, then $\pi_v \not\cong \pi_{\overline{v}}$.*
3. *$\overline{\rho} := \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \overline{\rho}_{\pi,\iota_p}$ satisfies:*
    - (a) *$\overline{\rho}$ is absolutely irreducible and generic (Definition 2.7.3).*
    - (b) *There exists a prime q such that $\overline{\rho}|_{G_{\mathbb{Q}_q}}$ is unramified, $q^4 \not\equiv 1 \pmod p$, and $\overline{\rho}(\mathrm{Frob}_q)$ has eigenvalues $\{q, 1, \alpha, q/\alpha\}$ with $\alpha \notin \{\pm 1, \pm q, q^2, q^{-1}\}$.*

Then

$$L(\pi, 1/2) \neq 0 \implies H^1_f(K, \rho_{\pi,\iota_p}) = 0.$$

Moreover:

- *If π is non-CM and not Galois-conjugate to a twist of $\pi \circ \tau$, where $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ is a generator, then the conditions in (3) hold for all but finitely many p and all choices of ι_p.*
- *If π is non-CM and Galois-conjugate to a twist of $\pi \circ \tau$, then then the conditions in (3) hold for all but finitely many p split in K.*
- *If π has CM by a totally imaginary quadratic extension F/K, then the conditions in (3) hold for all but finitely many p split completely in F.*

Proof. By the same argument as for Corollary 9.4.1, there exists an automorphic representation $\widetilde{\pi}$ of $\mathrm{GSp}_4(\mathbb{A})$ such that the base change of $\widetilde{\pi}$ to $\mathrm{GL}_4(\mathbb{A})$ is the automorphic induction of π. Again it suffices to show that $\widetilde{\pi}_\ell$ is transferrable, where ℓ is the rational prime underlying v from (2); in this case, the "moreover" assertions follow from Lemmas C.4.6 and C.4.8, combined with Lemma 4.1.5.

Now, let W be the underlying symplectic space of the associated Weil-Deligne representation

$$\tau_\ell : W_{\mathbb{Q}_\ell} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$$

to $\rho_{\widetilde{\pi},\iota_p}|_{G_{\mathbb{Q}_\ell}}$. As in the proof of Corollary 9.4.1, we wish to show that τ_ℓ does not factor through a Siegel parabolic subgroup, so suppose for contradiction that W contains a $W_{\mathbb{Q}_\ell} \times \mathrm{SL}(2, \mathbb{C})$-stable isotropic plane I. We have a $W_{K_v} \times \mathrm{SL}(2, \mathbb{C})$-stable decomposition

$$W = W_1 \oplus W_2,$$

where $W_{K_v} \times \mathrm{SL}(2, \mathbb{C})$ acts on W_1 through the Weil-Deligne representation corresponding to $\rho_{\pi_v,\iota_p}|_{\mathbb{Q}_\ell}$ – in particular, irreducibly because π_v is discrete series. The symplectic form is nondegenerate on each of W_1 and W_2, so we conclude that

$$I = \{w + \ell(w)\ :\ w \in W_1\}$$

for some linear isomorphism $\ell : W_1 \xrightarrow{\sim} W_2$. This ℓ is necessarily $W_{K_v} \times \mathrm{SL}(2, \mathbb{C})$-intertwining, so the assumption in (2) means v is an inert prime. Hence $W = W_1 \otimes V$ where $V = \mathrm{Ind}_{W_{K_v}}^{W_{G_{\mathbb{Q}_\ell}}} \mathbb{C}$, and the same argument as in Corollary 9.4.1 shows again that no such I can exist. $\square$

10. The second explicit reciprocity law: geometric inputs

10.1. **Setup and notation.**

10.1.1. Let $D \geq 1$ be squarefree with $\sigma(D)$ even, and recall the quadratic space V_D from (1.1.6). For this section, we suppose fixed a matrix $T \in \mathrm{Sym}^2(\mathbb{Q})_{>0}$ satisfying:

(T1) $T_{11} \in \mathbb{Q}^\times \setminus (\mathbb{Q}^\times)^2$.

(T2) The two-dimensional quadratic space defined by T has nontrivial local Hasse invariant for some prime $\ell \nmid D$.

10.1.2. Choose a base point $(e_1^T, e_2^T) \in \Omega_{T,V_D}(\mathbb{Q})$ (Construction 3.1.2(2)), and let

$$V_D^\circ := (e_1^T)^\perp \subset V_D, \;\; V_T := \mathrm{span}_{\mathbb{Q}} \left\{ e_1^T, e_2^T \right\} \subset V_D, \;\; V_D^\diamondsuit := V_T^\perp \subset V_D^\circ. \tag{10.1}$$

Then V_D° is a four-dimensional quadratic space with discriminant field $F := \mathbb{Q}(\sqrt{T_{11}})$.

10.1.3. Let $K = \prod K_v \subset \mathrm{GSpin}(V_D)(\mathbb{A}_f)$ be a neat compact open subgroup, and fix, throughout this section, an element $g_0 = \prod_v g_{0,v} \in \mathrm{GSpin}(V_D)(\mathbb{A}_f)$. We write

$$\begin{cases} K_v^\diamondsuit = g_{0,v} K_v g_{0,v}^{-1} \cap \mathrm{GSpin}(V_D^\diamondsuit)(\mathbb{Q}_v) \\ K_v^\circ = g_{0,v} K_v g_{0,v}^{-1} \cap \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_v) \end{cases}$$

for all finite places v of $\mathbb{Q}$, and $K^? := \prod K_v^?$ for $? = \diamondsuit, \circ$. The special cycle $Z(g_0, V_T, V_D)_K$ factors as:

$$\mathrm{Sh}_{K^\diamondsuit}(V_D^\diamondsuit) \to \mathrm{Sh}_{K^\circ}(V_D^\circ) \xrightarrow{\cdot g_0} \mathrm{Sh}_K(V_D). \tag{10.2}$$

10.2. **Integral models at good primes.**

Fix a prime $q \nmid D$ satisfying the following:

Assumption 10.2.1.

(1) T lies in $\mathrm{GL}_2(\mathbb{Z}_{(q)}) \subset M_2(\mathbb{Q})$.

(2) T_{11} lies in $(\mathbb{Z}_q^\times \setminus (\mathbb{Z}_q^\times)^2) \cap \mathbb{Q}$.

(3) K_q is hyperspecial and $g_q \in \mathrm{GSpin}(V_D)(\mathbb{Q}_q)$ lies in K_q.

Notation 10.2.2.

(1) Let $O_D \subset B_D$ be a maximal $\mathbb{Z}_{(q)}$-order.

(2) The lattice $\mathrm{Span}_{\mathbb{Z}_{(q)}} \left\{ e_1^T, e_2^T \right\}$ defines a maximal $\mathbb{Z}_{(q)}$-order $\mathcal{O}_T \subset C(V_T)$, with the natural positive nebentype involution.

(3) Let $\mathcal{O}_F \subset \mathcal{O}_T$ be the subalgebra generated by e_1^T, which is the unique maximal $\mathbb{Z}_{(q)}$-order in F.

(4) Fix an arbitrary positive involution $*$ of O_D (necessarily nebentype). The Clifford involution $*$ is positive and nebentype on $\mathcal{O}_T$, and stabilizes $\mathcal{O}_F$.

10.2.3. Under Notation 10.2.2, we have the chain of embeddings of $\mathbb{Z}_{(q)}$-algebras with positive involutions:

$$O_D \hookrightarrow O_D \otimes \mathcal{O}_F \hookrightarrow O_D \otimes \mathcal{O}_T. \tag{10.3}$$

We now use (10.3) to describe q-integral models for the cycles (10.2).

Construction 10.2.4. Using Corollary 1.3.5, we fix a four-dimensional abelian scheme A_0 over $\breve{\mathbb{Z}}_q$ of supersingular reduction, equipped with:

(1) An embedding $\iota_0^\diamondsuit : O_D \otimes_{\mathbb{Z}_{(q)}} \mathcal{O}_T \hookrightarrow \mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_{(q)}$.

(2) A prime-to-q quasi-polarization $\lambda_0 : A_0 \to A_0^\vee$ such that

$$\iota_0^\diamondsuit(\alpha)^\vee \circ \lambda_0 = \lambda_0 \circ \iota_0^\diamondsuit(\alpha^*), \quad \forall \alpha \in O_D \otimes \mathcal{O}_T.$$

By restricting along (10.3), we also have $\iota_0 : O_D \hookrightarrow \mathrm{End}(A_0) \otimes \mathbb{Z}_{(q)}$ and $\iota_0^\circ : O_D \otimes_{\mathbb{Z}_{(q)}} \mathcal{O}_F \hookrightarrow \mathrm{End}(A_0) \otimes \mathbb{Z}_{(q)}$. Note that $(A_0, \iota_0, \lambda_0)$ is an $(O_D, *)$-triple in the sense of Definition 1.4.2. Using Remark 1.4.5(1), we extend this to a q-adic uniformization datum $(A_0, \iota_0, \lambda_0, i_D, i_{Dq})$ for $(O_D, *)$. Consider the three PEL data with self-dual q-integral refinements:

$$\begin{aligned} \mathcal{D}^\diamondsuit &= (B_D \otimes_{\mathbb{Q}} C(V_T), *, H, \psi), & \mathscr{D}^\diamondsuit &= (O_D \otimes_{\mathbb{Z}_{(q)}} \mathcal{O}_T, *, \Lambda, \psi) \\ \mathcal{D}^\circ &= (B_D \otimes_{\mathbb{Q}} F, *, H, \psi), & \mathscr{D}^\circ &= (O_D \otimes_{\mathbb{Z}_{(q)}} \mathcal{O}_F, *, \Lambda, \psi) \\ \mathcal{D} &= (B_D, *, H, \psi), & \mathscr{D} &= (O_D, *, \Lambda, \psi) \end{aligned}$$

arising from A_0. Also write $K^{q?} := \prod_{v \neq q} K_v^?$ for $? = \diamondsuit, \circ$.

For $? = \diamondsuit, \circ$, or $\emptyset$, let $X^?$ be the smooth quasiprojective scheme over $\mathbb{Z}_{(q)}$ representing the moduli functor $\mathcal{M}^?_{K^{q?}}$ associated to $\mathscr{D}^?$ at level $K^{q?}$.

Lemma 10.2.5. *The scheme $X^\diamondsuit$ is proper over $\mathrm{Spec}\,\mathbb{Z}_{(q)}$.*

Proof. The biquaternion algebra $B_D \otimes C(V_T)$ is nonsplit by (T2). If $B_D \otimes C(V_T) = M_2(B_d)$ for some squarefree $d > 1$, then $\sigma(d)$ is necessarily even. By Corollary 1.3.4 combined with Propositions 1.2.4 and 1.2.12, $X^\diamondsuit$ can be identified with the canonical (smooth) integral model of the Shimura curve attached to B_d at level $K^\diamondsuit$. The latter is well-known to be proper. ☐

10.2.6. We have natural finite maps

$$X^\diamondsuit \to X^\circ \to X, \tag{10.4}$$

defined on the level of moduli problems by

$$\begin{aligned} (A, \iota^\diamondsuit, \lambda, \eta) &\mapsto (A, \iota^\diamondsuit|_{O_D \otimes_{\mathbb{Z}_{(q)}} \mathcal{O}_F}, \lambda, \eta) \\ (A, \iota^\circ, \lambda, \eta) &\mapsto (A, \iota^\circ|_{O_D}, \lambda, g_0^q \cdot \eta). \end{aligned}$$

Let $X^?_{\mathbb{Q}}$ denote the generic fiber of $X^?$, for $? = \diamondsuit, \circ$, or $\emptyset$, and let $X^?_{\mathbb{F}_q}$ denote the special fiber.

Proposition 10.2.7. *There are isomorphisms*

$$X^?_{\mathbb{Q}} \cong \mathrm{Sh}_{K^?}(V_D^?)$$

for $? = \diamondsuit, \circ$, or $\emptyset$, such that the generic fiber of (10.4) recovers (10.2).

Proof. This follows from the discussion in [56, §2]; note the isomorphisms depend on our choice of q-adic uniformization datum in Construction 10.2.4. ☐

10.2.8. We let O be the ring of integers of a finite extension of $\mathbb{Q}_p$, where $q \neq p$, and let $\varpi \in O$ be a uniformizer.

Lemma 10.2.9. *For all i and for $? = \diamondsuit, \circ$, or $\emptyset$, there are canonical $G_{\mathbb{Q}_q}$-equivariant isomorphisms*

$$\begin{aligned} H^i_{\text{ét}}(\mathrm{Sh}_{K^?}(V_D^?)_{\overline{\mathbb{Q}}}, O) &\cong H^i_{\text{ét}}(X^?_{\overline{\mathbb{F}}_q}, O) \\ H^i_{\text{ét},c}(\mathrm{Sh}_{K^?}(V_D^?)_{\overline{\mathbb{Q}}}, O) &\cong H^i_{\text{ét},c}(X^?_{\overline{\mathbb{F}}_q}, O). \end{aligned} \tag{$\mathrm{BC}_{X^?}$}$$

These isomorphisms commute with the actions of prime-to-q Hecke correspondences and with the pullback and pushforward maps induced by (10.4).

Proof. Let $R\Psi_{X^?}O$ denote the nearby cycles complex on $X^?_{\overline{\mathbb{F}}_q}$. Since $X^?$ is smooth over $\mathbb{Z}_{(q)}$, the natural map $O \to R\Psi_{X^?}O$ is an isomorphism. On the other hand, by [57, Corollary 5.20], the base change map

$$R\Gamma_{\text{ét}}(X^?_{\overline{\mathbb{Q}}}, O) \to R\Gamma_{\text{ét}}(X^?_{\overline{\mathbb{F}}_q}, R\Psi_{X^?}O)$$

is also an isomorphism, and the lemma follows. □

10.3. **Unramified Rapoport-Zink spaces.**

Notation 10.3.1.

(1) Recall from Remark 1.4.5(2) that our choice of q-adic uniformization datum in Construction 10.2.4 entails a choice of isomorphism

$$\operatorname{End}^0(\overline{A}_0, \overline{\iota}_0)^{\dagger=1,\mathrm{tr}=0} \xrightarrow{\sim} V_{Dq},$$

hence an inclusion $V_T \hookrightarrow V_{Dq}$. Let

$$V_{Dq}^{\diamondsuit} := V_T^{\perp} \subset V_{Dq}$$

and

$$V_{Dq}^{\circ} = (e_1^T)^{\perp} \subset V_{Dq}.$$

(2) For each $? = \diamondsuit, \circ$, or $\emptyset$, let $\mathcal{N}^?$ denote the Rapoport-Zink space over Spf $\breve{\mathbb{Z}}_q$ parametrizing framed polarized deformations $(X, \iota^?, \lambda, \rho)$ of $(\overline{A}_0[q^\infty], \overline{\iota}_0^? \otimes \mathbb{Z}_q, \lambda_0)$, where $\overline{\iota}_0^{\diamondsuit} \otimes \mathbb{Z}_q : O_D \otimes \mathcal{O}_T \otimes \mathbb{Z}_q \hookrightarrow \operatorname{End}^0(\overline{A}_0[q^\infty])$ is the induced embedding, and likewise for $\overline{\iota}_0^{\circ}, \overline{\iota}_0$. We let $\mathcal{M}^?$ denote the underlying reduced scheme of $\mathcal{N}^?$.

(3) Let $\varphi : \mathcal{N}^? \xrightarrow{\sim} \sigma^*\mathcal{N}^?$ be the natural Weil descent datum as in (6.1.4).

10.3.2. We have natural closed embeddings

$$\mathcal{N}^{\diamondsuit} \hookrightarrow \mathcal{N}^{\circ} \hookrightarrow \mathcal{N} \tag{10.5}$$

compatible with the actions of

$$\operatorname{GSpin}(V_{Dq}^{\diamondsuit})(\mathbb{Q}_q) \subset \operatorname{GSpin}(V_{Dq}^{\circ})(\mathbb{Q}_q) \subset \operatorname{GSpin}(V_{Dq})(\mathbb{Q}_q).$$

From the Rapoport-Zink uniformization theorem, we deduce:

Proposition 10.3.3. *For each $? = \diamondsuit, \circ$, or $\emptyset$, let $X^{ss?}_{\overline{\mathbb{F}}_q}$ denote the supersingular locus. Then there is a canonical isomorphism*

$$X^{ss?}_{\overline{\mathbb{F}}_q} \cong \operatorname{GSpin}(V^?_{Dq})(\mathbb{Q}) \backslash \operatorname{GSpin}(V^?_{Dq})(\mathbb{A}_f^q) \times \mathcal{M}^?/K^{q?},$$

compatible with prime-to-q Hecke correspondences, Frobenius action, and the maps arising from (10.4), (10.5).

Here $K^{q?}$ is viewed as a subgroup of $\operatorname{GSpin}(V^?_{Dq})(\mathbb{A}_f^q)$ by Remark 1.4.5(2).

Proposition 10.3.4.

(1) Each irreducible component of $\mathcal{M}^{\circ}$ or $\mathcal{M}$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$; in particular, $\mathcal{M}^{\circ}$ is a union of irreducible components of $\mathcal{M}$.

(2) The group $\operatorname{GSpin}(V_{Dq})(\mathbb{Q}_q)$ acts transitively on the set of irreducible components of $\mathcal{M}$, and the stabilizer of each component is a paramodular subgroup.

(3) There are two $\operatorname{GSpin}(V^{\circ}_{Dq})(\mathbb{Q}_q)$-orbits of irreducible components of $\mathcal{M}$ interchanged by φ, and the stabilizer of each component is a hyperspecial subgroup.

(4) For any irreducible component $A \subset \mathcal{M}$, we have $\varphi^2(A) = (\sigma^2)^* (\langle q \rangle \cdot A)$.

Proof. See [56, §4] and [55, §4] for the structure of $\mathcal{M}$ and $\mathcal{M}^\circ$, respectively. ☐

Notation 10.3.5.

1. Fix an irreducible component $\mathcal{M}(1) \subset \mathcal{M}^\circ$ as a basepoint, and let $K_q^{\mathrm{Pa}} \subset \mathrm{GSpin}(V_{Dq})(\mathbb{Q}_q)$ be the stabilizer of $\mathcal{M}(1)$.
2. For all $g \in \mathrm{GSpin}(V_{Dq})(\mathbb{Q}_q)/K_q^{\mathrm{Pa}}$, define
$$\mathcal{M}(g) \coloneqq g \cdot \mathcal{M}(1) \subset \mathcal{M}.$$
By Proposition 10.3.4(2), this defines a bijection between $\mathrm{GSpin}(V_{Dq})(\mathbb{Q}_q)/K_q^{\mathrm{Pa}}$ and the irreducible components of $\mathcal{M}$.
3. Let $F \in \mathrm{GSpin}(V_{Dq})(\mathbb{Q}_q)$ be an element normalizing K_q^{Pa} such that $F^2 = \langle q \rangle$ and $\varphi(\mathcal{M}(g)) = \sigma^*\mathcal{M}(gF)$ for all $g \in \mathrm{GSpin}(V_{Dq})(\mathbb{Q}_q)$; such an F exists by Proposition 10.3.4(4).
4. Let $K_q^\circ \coloneqq K_q^{\mathrm{Pa}} \cap \mathrm{GSpin}(V_{Dq}^\circ)(\mathbb{Q}_q)$, which is hyperspecial by Proposition 10.3.4(3).[11] We also set $K^\circ = K^{q\circ}K_q^\circ \subset \mathrm{GSpin}(V_{Dq}^\circ)(\mathbb{A}_f)$.

Remark 10.3.6. By Proposition 10.3.4(3), under Notation 10.3.5 the irreducible components of $\mathcal{M}^\circ$ are labeled by $\mathcal{M}(g)$ and $\mathcal{M}(gF)$ for $g \in \mathrm{GSpin}(V_{Dq}^\circ)(\mathbb{Q}_q)/K_q^\circ$.

10.4. Tate classes on $X^\circ_{\overline{\mathbb{F}}_q}$.

Definition 10.4.1.

1. We label the irreducible components of $X^{ss\circ}_{\overline{\mathbb{F}}_q}$ as $B^\circ_\delta(g)$ for
$$(g, \delta) \in \mathrm{Sh}_{K^\circ}(V_{Dq}^\circ) \times \{0, 1\},$$
by defining $B^\circ_\delta(g)$ to be the image of $(g^q, \mathcal{M}(g_q F^\delta))$ under the uniformization of Proposition 10.3.3.
2. We define the incidence map
$$\begin{aligned} \mathrm{inc}^{\circ*} : H^2(X^\circ_{\overline{\mathbb{F}}_q}, O(1)) \to & \bigoplus_{(g,\delta)\in \mathrm{Sh}_{K^\circ}(V_{Dq}^\circ)\times\{0,1\}} H^2(B^\circ_\delta(g), O(1)) \\ & \cong O[\mathrm{Sh}_{K^\circ}(V_{Dq}^\circ)]^{\oplus 2} \end{aligned} \tag{10.6}$$
and dually
$$\begin{aligned} \mathrm{inc}^\circ_* : O[\mathrm{Sh}_{K^\circ}(V_{Dq}^\circ)]^{\oplus 2} \cong & \bigoplus_{(g,\delta)\in \mathrm{Sh}_{K^\circ}(V_{Dq}^\circ)\times\{0,1\}} H^2(B^\circ_\delta(g), O(1)) \\ & \to H^2_c(X^\circ_{\overline{\mathbb{F}}_q}, O(1)). \end{aligned} \tag{10.7}$$

10.4.2. Let S° be the set of primes ℓ such that K_ℓ° is not hyperspecial. For the rest of this section, we shall apply the results and notations of Appendix A, with the added superscript $\circ$ for consistency. In particular, we obtain a compact open subgroup $\widetilde{K}^\circ = \prod \widetilde{K}^\circ_\ell \subset C^+(V_D^\circ)^\times(\mathbb{A}_f)$, where $\widetilde{K}^\circ_\ell$ is hyperspecial for $\ell \notin S^\circ$.

Proposition 10.4.3. *The integral model X° for $\mathrm{Sh}_{K^\circ}(V_D^\circ)$ extends to a smooth canonical model $\widetilde{X}^\circ$ over $\operatorname{Spec}\mathbb{Z}_{(q)}$ for $\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_D^\circ)$, with an open and closed embedding*
$$X^\circ \hookrightarrow \widetilde{X}^\circ \tag{10.8}$$
extending the map on generic fibers. Moreover, the universal abelian variety on X extends naturally to $\widetilde{X}$.

---

[11] Since the subgroup $K_q^\circ \subset \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_q)$ from (10.1.3) is also hyperspecial, we hope this will not produce any confusion.

Proof. Under the condition (3) from Appendix A, this follows from the construction of the embedding on generic fibers in [68, Proposition 2.10, Remark 2.11]. ☐

Lemma 10.4.4. *For each i, there are canonical $G_{\mathbb{Q}_q}$-equivariant isomorphisms*

$$\begin{aligned} H^i_{\text{ét}}(\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_D^\circ)_{\overline{\mathbb{Q}}}, O) &\cong H^i_{\text{ét}}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O) \\ H^i_{\text{ét},c}(\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_D^\circ))_{\overline{\mathbb{Q}}}, O) &\cong H^i_{\text{ét},c}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O). \end{aligned} \tag{$\mathrm{BC}_{\widetilde{X}^\circ}$}$$

compatible with those of Lemma 10.2.9.

Proof. See [68, Lemma 4.4]. ☐

10.4.5. The Clifford algebra $C^+(V_{Dq}^\circ)$ is a totally definite quaternion algebra over F, whose local invariants coincide with those of $C^+(V_D^\circ)$ at all finite places; let $Q^\times$ be the $\mathbb{Q}$-algebraic group of units of $C^+(V_{Dq}^\circ)$. Then by [68, Theorem 3.13], we have a uniformization

$$\widetilde{X}^{ss\circ}_{\overline{\mathbb{F}}_q} \simeq Q^\times(\mathbb{Q})\backslash Q^\times(\mathbb{A}_f^q) \times \mathcal{M}^\circ/\widetilde{K}^{q\circ} \tag{10.9}$$

compatible with Proposition 10.3.3 for $? = \circ$. Here $\widetilde{K}^{q\circ}$ is viewed as a compact open subgroup of $Q^\times(\mathbb{A}_f^q)$ using the isomorphism

$$V_{Dq}^\circ \otimes_{\mathbb{Q}} \mathbb{A}_f^q \simeq V_D^\circ \otimes_{\mathbb{Q}} \mathbb{A}_f^q$$

that follows from the choice of q-adic uniformization datum, cf. Notation 10.3.1(1).

We may therefore extend $\mathrm{inc}^{\circ *}$ to a map $\widetilde{\mathrm{inc}}^{\circ *}$ fitting into the following commutative diagram:

$$\begin{tikzcd} H^2_{\text{ét}}(X^\circ_{\overline{\mathbb{F}}_q}, O(1)) \arrow[r, "\mathrm{inc}^{\circ *}"] \arrow[d, hook] & O\left[\mathrm{Sh}_{K^\circ}(V_{Dq}^\circ)\right]^{\oplus 2} \arrow[d, hook] \\ H^2_{\text{ét}}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O(1)) \arrow[r, "\widetilde{\mathrm{inc}}^{\circ *}"] & O\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_{Dq}^\circ)\right]^{\oplus 2}. \end{tikzcd} \tag{10.10}$$

On the bottom right, the level subgroup is $\widetilde{K}^\circ = \widetilde{K}^{q\circ}\widetilde{K}^\circ_q$, where $\widetilde{K}^\circ_q$ is the unique hyperspecial subgroup of $Q^\times(\mathbb{Q}_q)$ containing K_q°, and by definition

$$\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_{Dq}^\circ) = Q^\times(\mathbb{Q})\backslash Q^\times(\mathbb{A}_f)/\widetilde{K}^\circ.$$

Similarly, we have a map $\widetilde{\mathrm{inc}}^\circ_*$ fitting into a commutative diagram

$$\begin{tikzcd} O\left[\mathrm{Sh}_{K^\circ}(V_{Dq}^\circ)\right]^{\oplus 2} \arrow[r, "\mathrm{inc}^\circ_*"] \arrow[d, hook] & H^2_{\text{ét},c}(X^\circ_{\overline{\mathbb{F}}_q}, O(1)) \arrow[d, hook] \\ O\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_{Dq}^\circ)\right]^{\oplus 2} \arrow[r, "\widetilde{\mathrm{inc}}^\circ_*"] & H^2_{\text{ét},c}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O(1)). \end{tikzcd} \tag{10.11}$$

10.4.6. *Hecke actions.* Recall the local and global Hecke algebras $\mathbb{T}_\ell^\circ$, $\widetilde{\mathbb{T}}_\ell^\circ$, $\mathbb{T}^{\circ S^\circ}$, and $\widetilde{\mathbb{T}}^{\circ S^\circ}$ from (A.1.3). We define actions of the local Hecke algebras $\mathbb{T}_q^\circ \cong \widetilde{\mathbb{T}}_q^\circ$ on $H^i_{\text{ét}}(X^\circ_{\overline{\mathbb{F}}_q}, O)$, $H^i_{\text{ét},c}(X^\circ_{\overline{\mathbb{F}}_q}, O)$, $H^i_{\text{ét}}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O)$, and $H^i_{\text{ét},c}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O)$ via the isomorphisms of Lemmas 10.2.9 and 10.4.4.

Lemma 10.4.7.

(1) *The maps* $\mathrm{inc}^{\circ *}$ *and* $\widetilde{\mathrm{inc}}^{\circ *}$ *are equivariant for* $\mathbb{T}^{\circ S^\circ}$ *and* $\widetilde{\mathbb{T}}^{\circ S^\circ}$*, respectively.*

(2) *The maps* inc°_* *and* $\widetilde{\mathrm{inc}}^\circ_*$ *are equivariant for* $\mathbb{T}^{\circ S^\circ}$ *and* $\widetilde{\mathbb{T}}^{\circ S^\circ}$*, respectively, after extending scalars to* $\overline{\mathbb{Q}}_p$*.*

Proof. It is clear that $\mathrm{inc}^{\circ *}$, $\widetilde{\mathrm{inc}}^{\circ *}$, inc°_*, and $\widetilde{\mathrm{inc}}^\circ_*$ are equivariant for all prime-to-q Hecke operators, so it suffices to consider the action of $\mathbb{T}^\circ_q \cong \widetilde{\mathbb{T}}^\circ_q$. Also, by the commutative diagrams (10.10) and (10.11), it suffices to consider $\widetilde{\mathrm{inc}}^{\circ *}$ and $\widetilde{\mathrm{inc}}^\circ_*$. The final reduction is that we may prove both statements of the lemma after extending scalars to $\overline{\mathbb{Q}}_p$, since the target of $\widetilde{\mathrm{inc}}^{\circ *}$ is O-torsion-free.

Applying Lemma 10.4.4 and the étale comparison theorem, for (1) it therefore suffices to show that any map

$$H^2(\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V_D)(\mathbb{C}), \mathbb{C}) \to \mathbb{C}\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right]$$

which is equivariant for $\widetilde{\mathbb{T}}^{\circ S^\circ \cup \{q\}}$ is also equivariant for $\widetilde{\mathbb{T}}^\circ_q$; but this is clear from the Jacquet-Langlands correspondence and strong multiplicity one for GL_2. The proof of (2) is the same. ☐

Definition 10.4.8. Let $T^\circ_q \in \widetilde{\mathbb{T}}^\circ_q \cong \mathbb{T}^\circ_q$ be the double coset operator represented by $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ in any basis such that $\widetilde{K}^\circ_q = \mathrm{GL}_2(\mathbb{Z}_{q^2})$.

Lemma 10.4.9. (1) *The composite maps*

$$O\left[\mathrm{Sh}_{K^\circ}(V^\circ_{Dq})\right]^{\oplus 2} \xrightarrow{\mathrm{inc}^\circ_*} H^2_{\text{ét},c}(X^\circ_{\overline{\mathbb{F}}_q}, O(1)) \to H^2_{\text{ét}}(X^\circ_{\overline{\mathbb{F}}_q}, O(1)) \xrightarrow{\mathrm{inc}^{\circ *}} O\left[\mathrm{Sh}_{K^\circ}(V^\circ_{Dq})\right]^{\oplus 2}$$

and

$$O\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right]^{\oplus 2} \xrightarrow{\widetilde{\mathrm{inc}}^\circ_*} H^2_{\text{ét},c}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O(1)) \to H^2_{\text{ét}}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O(1)) \xrightarrow{\widetilde{\mathrm{inc}}^{\circ *}} O\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right]^{\oplus 2}$$

are both given by the matrix

$$\begin{pmatrix} -2q & T^\circ_q \langle q\rangle^{-1} \\ T^\circ_q & -2q \end{pmatrix}.$$

(2) *The restricted map*

$$\mathrm{inc}^{\circ *} : (T^{\circ 2}_q \langle q\rangle^{-1} - 4q^2) H^2_{\text{ét},!}(X^\circ_{\overline{\mathbb{F}}_q}, O(1))^{\mathrm{Frob}^2_q = \langle q\rangle} \to O\left[\mathrm{Sh}_{K^\circ}(V^\circ_{Dq})\right]^{\oplus 2}$$

has ϖ-power-torsion kernel.

Proof. For (1), see [66, Proposition 2.21(4)]; the off-diagonal entries follow from the intersection combinatorics of $\mathcal{M}^\circ$ described in [55, §4]. For (2), it suffices to prove the analogous statement for $\widetilde{\mathrm{inc}}^{\circ *}$ due to the commutative diagram (10.10). Let us fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Extending scalars to $\overline{\mathbb{Q}}_p$ and applying Lemma 10.4.7 and Proposition A.1.4, it suffices to show that

$$\widetilde{\mathrm{inc}}^{\circ *} : H^2_{\text{ét},!}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_p(1))^{\mathrm{Frob}^2_q = \langle q\rangle}[\iota^{-1}\tau_f] \to \overline{\mathbb{Q}}_p\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right][\iota^{-1}\tau_f]^{\oplus 2}$$

is injective for all discrete automorphic representations τ of $C^+(V^\circ_D)^\times(\mathbb{A})$ such that

$$T^{\circ 2}_q \langle q\rangle^{-1} - 4q^2 \neq 0 \text{ on } \tau_f^{\widetilde{K}^\circ}. \tag{Hecke_q}$$

Now note that $\mathrm{Frob}^2_q = \langle q\rangle$ on the image of

$$\widetilde{\mathrm{inc}}^\circ_* : O\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right]^{\oplus 2} \to H^2_{\text{ét},c}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, O(1));$$

this follows from Proposition 10.3.4(4). In particular, we have a well-defined composite map

$$\overline{\mathbb{Q}}_p\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right]^{\oplus 2}[\iota^{-1}\tau_f] \xrightarrow{\widetilde{\mathrm{inc}}^\circ_*} H^2_{\text{ét},!}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_p(1))^{\mathrm{Frob}^2_q = \langle q\rangle}[\iota^{-1}\tau_f] \xrightarrow{\widetilde{\mathrm{inc}}^{\circ *}} \overline{\mathbb{Q}}_p\left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right]^{\oplus 2}[\iota^{-1}\tau_f],$$

given by the matrix in part (1), which is invertible by the assumption (Hecke_q).

It therefore suffices to show that

$$\dim_{\overline{\mathbb{Q}}_p} H^2_{\text{ét},!}(\widetilde{X}^\circ_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_p(1))^{\mathrm{Frob}_q^2=\langle q\rangle}[\iota^{-1}\tau_f] \leq 2\dim \mathbb{Q}_p \left[\widetilde{\mathrm{Sh}}_{\widetilde{K}^\circ}(V^\circ_{Dq})\right][\iota^{-1}\tau_f]$$

for all τ satisfying (Hecke_q). This dimension count follows from [111, Proposition 2.25] when τ is cuspidal; when τ is not cuspidal, it is clear from Proposition A.1.4. ☐

10.4.10. Consider the algebraic cycle class

$$[X^\diamondsuit_{\overline{\mathbb{F}}_q}] \in H^2_{\text{ét},c}(X^\circ_{\overline{\mathbb{F}}_q}, O(1)). \tag{10.12}$$

which makes sense by Lemma 10.2.5.

Lemma 10.4.11. *There exists a special cycle $Z \in \mathrm{SC}^1_{K^\circ}(V^\circ_{Dq})$ such that*

$$\mathrm{inc}^{\circ *}([X^\diamondsuit_{\overline{\mathbb{F}}_q}]) = (Z, Z) \in O\left[\mathrm{Sh}_{K^\circ}(V^\circ_{Dq})\right].$$

Proof. Since $[X^\diamondsuit_{\overline{\mathbb{F}}_q}]$ is Frobenius-invariant, it suffices to consider the first coordinate of $\mathrm{inc}^{\circ *}([X^\diamondsuit_{\overline{\mathbb{F}}_q}])$, which we write as $\mathrm{inc}_1^{\circ *}([X^\diamondsuit_{\overline{\mathbb{F}}_q}])$. Since $\mathcal{M}^\diamondsuit$ is zero-dimensional, all the intersections of $X^\diamondsuit_{\overline{\mathbb{F}}_q}$ with supersingular curves on $X^\circ_{\overline{\mathbb{F}}_q}$ are proper. Hence $\mathrm{inc}_1^{\circ *}([X^\diamondsuit_{\overline{\mathbb{F}}_q}])$ is computed by

$$\sum_{[(g^q, g_q)] \in \mathrm{GSpin}(V^\diamondsuit_{Dq})(\mathbb{Q}) \backslash \mathrm{GSpin}(V^\diamondsuit_{Dq})(\mathbb{A}_f^q) \times \mathrm{GSpin}(V^\circ_{Dq})(\mathbb{Q}_q)/K^{\diamondsuit q} \times K^\circ_q} m(g_q)[(g^q, g_q)], \tag{10.13}$$

where, for $g_q \in \mathrm{GSpin}(V^\circ_{Dq})(\mathbb{Q}_q)$, $m(g_q)$ is the degree of the divisor $\mathcal{N}^\diamondsuit \cap \mathcal{M}(g_q)$ on $\mathcal{M}(g_q) \cong \mathbb{P}^1_{\overline{\mathbb{F}}_q}$. In particular, $m(g_q)$ depends only on the $\mathrm{GSpin}(V^\diamondsuit_{Dq})(\mathbb{Q}_q)$-orbit of g_q. Moreover, $g_q \mapsto m(g_q)$ is a compactly supported function on $\mathrm{GSpin}(V^\diamondsuit_{Dq})(\mathbb{Q}_q) \backslash \mathrm{GSpin}(V^\circ_{Dq})(\mathbb{Q}_q)$ since each point of $\mathcal{M}$ lies on only finitely many irreducible components.[12] Hence (10.13) is a finite linear combination of special cycles

$$Z(h_q^{(i)}, V_T \cap V^\circ_{Dq}, V^\circ_{Dq})_{K^\circ} \in \mathbb{Z}\left[\mathrm{Sh}_{K^\circ}(V^\circ_{Dq})\right]$$

for some elements

$$h_q^{(i)} \in \mathrm{GSpin}(V^\diamondsuit_{Dq})(\mathbb{Q}_q) \backslash \mathrm{GSpin}(V^\circ_{Dq})(\mathbb{Q}_q)/K^\circ_q,$$

which completes the proof. ☐

Remark 10.4.12. It would not be difficult to make Lemma 10.4.11 more explicit, but it is unnecessary for the main results.

Theorem 10.4.13. *For any $h \in \mathbb{T}_O^{\circ S^\circ}$, there exists a cycle $Z^\circ_{Dq} \in \mathrm{SC}^1_{K^\circ}(V^\circ_{Dq}, O)$ such that*

$$\mathrm{inc}^\circ_*[(Z^\circ_{Dq}, Z^\circ_{Dq})] - (T_q^{\circ 2} - 4q^2\langle q\rangle)^2 h[X^\diamondsuit_{\overline{\mathbb{F}}_q}] \in H^2_{\text{ét},c}(X^\circ_{\overline{\mathbb{F}}_q}, O(1))$$

has ϖ-power-torsion image in $H^2_{\text{ét},!}(X^\circ_{\overline{\mathbb{F}}_q}, O(1))$.

Proof. Let $Z \in \mathrm{SC}^1_{K^\circ}(V^\circ_{Dq})$ be the special cycle in Lemma 10.4.11, so that

$$\mathrm{inc}^{\circ *}([X^\diamondsuit_{\overline{\mathbb{F}}_q}]) = (Z, Z),$$

and define

$$Z^\circ_{Dq} := (T_q^{\circ 2} - 4q^2\langle q\rangle) \cdot (T^\circ_q + 2q) \cdot h \cdot Z \in \mathrm{SC}^1_{K^\circ}(V^\circ_{Dq}, O).$$

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[12] One can see this using that $\mathcal{M}$ is locally of finite type over $\overline{\mathbb{F}}_q$, or more concretely from [55, §4].

By Lemma 10.4.9(2), it suffices to show that

$$\mathrm{inc}^{\circ *}\,\mathrm{inc}^{\circ}_{*}[(Z^{\circ}_{Dq}, Z^{\circ}_{Dq})] = \mathrm{inc}^{\circ *}(T_q^{\circ 2} - 4q^2\langle q\rangle)^2 h[X^{\diamondsuit}_{\overline{\mathbb{F}}_q}]. \tag{10.14}$$

By Lemmas 10.4.9(1) and 10.4.7, the left-hand side of (10.14) is

$$(T_q^{\circ 2} - 4q^2\langle q\rangle)\cdot(T_q^{\circ} + 2q)\cdot h\cdot\begin{pmatrix}-2q & T_q^{\circ}\langle q\rangle^{-1}\\ T_q^{\circ} & -2q\end{pmatrix}\cdot\begin{pmatrix}Z\\ Z\end{pmatrix} = (T_q^{\circ 2} - 4q^2\langle q\rangle)^2\cdot h\cdot\begin{pmatrix}Z\\ Z\end{pmatrix}$$

because the Hecke operator $\langle q\rangle$ acts trivially on special cycles. This coincides with the right-hand side of (10.14) by Lemma 10.4.7 and the definition of Z, so the proof is complete. ☐

10.5. **Pushing forward from X° to X.**

10.5.1. Now let S be a finite set of primes of $\mathbb{Q}$ such that K_ℓ is hyperspecial for $\ell \notin S$, and let $\mathfrak{m} \subset \mathbb{T}_O^{S\cup\{q\}}$ be a generic, non-Eisenstein maximal ideal.

Notation 10.5.2. We denote by j the closed embedding $X^{\circ} \hookrightarrow X$ of (10.4).

Notation 10.5.3. Let

$$\partial_{\mathrm{AJ},\mathfrak{m},\mathbb{F}_q} : \mathrm{CH}^2(X_{\mathbb{F}_q}) \to H^1(\mathbb{F}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}})$$

denote the local Abel-Jacobi map constructed analogously to (4.4.2).

Lemma 10.5.4. *The composite map*

$$\mathrm{CH}^1(X^{\circ}_{\mathbb{F}_q}) \xrightarrow{j_*} \mathrm{CH}^2(X_{\mathbb{F}_q}) \xrightarrow{\partial_{\mathrm{AJ},\mathfrak{m},\mathbb{F}_q}} H^1(\mathbb{F}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}})$$

factors through the specialization map

$$H^2(X^{\circ}_{\mathbb{F}_q}, O(1)) \to H^2(X^{\circ}_{\overline{\mathbb{F}}_q}, O(1))^{\mathrm{Frob}_q=1}.$$

Proof. For any variety Y defined over $\mathbb{F}_q$, we have the Hochschild-Serre spectral sequence:

$$E_2^{i,j} = H^i(\mathbb{F}_q, H^j_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, O(n)) \implies H^{i+j}_{\text{ét}}(Y, O(n)),\ \ \forall n\in\mathbb{Z}. \tag{10.15}$$

Since $\mathbb{F}_q$ has cohomological dimension one, it follows immediately that the map

$$H^i(Y, O(n)) \to H^i(Y_{\overline{\mathbb{F}}_q}, O(n))^{\mathrm{Frob}_q=1}$$

is surjective. Let $H^i(Y, O(n))^0 \subset H^i(Y, O(n))$ be the kernel of this map, and let

$$\partial : H^i(Y, O(n))^0 \to H^1(\mathbb{F}_q, H^{i-1}(Y_{\overline{\mathbb{F}}_q}, O(n))$$

be the edge map from (10.15).

It then suffices to show that the map

$$H^2(X^{\circ}_{\mathbb{F}_q}, O(1)) \xrightarrow{j_*} H^4(X_{\mathbb{F}_q}, O(2)) \xrightarrow{\partial} H^1(\mathbb{F}_q, H^3(X_{\overline{\mathbb{F}}_q}, O(2))_{\mathfrak{m}})$$

factors through the surjection $H^2(X^{\circ}_{\mathbb{F}_q}, O(1)) \twoheadrightarrow H^2(X^{\circ}_{\overline{\mathbb{F}}_q}, O(1))^{\mathrm{Frob}_q=1}$, i.e. is trivial on $H^2(X^{\circ}_{\mathbb{F}_q}, O(1))^0$.

Consider the commutative diagram arising from the functoriality of (10.15):

$$\begin{tikzcd}
H^2_{\text{ét}}(X^\circ_{\mathbb{F}_q}, O(1))^0 \arrow[r, "\partial"] \arrow[d, "j_*"] & H^1(\mathbb{F}_q, H^1_{\text{ét}}(X^\circ_{\overline{\mathbb{F}}_q}, O(1))) \arrow[d, "j_*"] \\
H^4_{\text{ét}}(X_{\mathbb{F}_q}, O(2))^0 \arrow[r, "\partial"] \arrow[d, "\mathrm{loc}_\mathfrak{m}"] & H^1(\mathbb{F}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(2))) \arrow[d, "\mathrm{loc}_\mathfrak{m}"] \\
H^4_{\text{ét}}(X_{\mathbb{F}_q}, O(2))^0_\mathfrak{m} \arrow[r, "\partial"] & H^1(\mathbb{F}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(2))_\mathfrak{m}).
\end{tikzcd}$$

Now note that the composite map

$$H^1_{\text{ét}}(X^\circ_{\overline{\mathbb{F}}_q}, O(1)) \xrightarrow{j_*} H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(1)) \xrightarrow{\mathrm{loc}_\mathfrak{m}} H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(1))_\mathfrak{m}$$

is identically zero: indeed, the source is ϖ-power-torsion by [73, Chapter IX, Corollary 7.15(iii)], and the target is ϖ-torsion-free by Theorem 2.7.5(2). In particular, the composite from the top left to the bottom right of the commutative diagram vanishes, which proves the lemma. ☐

10.5.5. We now construct a map

$$\partial_{ss,\mathfrak{m}} : O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]_\mathfrak{m} \to H^1_{\mathrm{unr}}(\mathbb{Q}_q, H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_\mathfrak{m})$$

in several steps.

Construction 10.5.6. Assume $\langle q \rangle = 1$ in $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$.

1. For $g \in \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})$, let $B(g) \subset X^{ss}_{\overline{\mathbb{F}}_q}$ be the image of $(g^q, \mathcal{M}(g_q))$ under the uniformization of Proposition 10.3.3.
2. Define an action of Frob_q on $\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})$ by $g \mapsto gF$. Then using Proposition 10.3.4(4), we obtain a map
$$O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]^{\mathrm{Frob}_q = 1} \to \mathrm{CH}^2(X_{\mathbb{F}_q}, O).$$
3. By our assumption $\langle q \rangle = 1$, the natural map gives an isomorphism
$$O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]^{\mathrm{Frob}_q^2 = 1}_\mathfrak{m} \xrightarrow{\sim} O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]_\mathfrak{m}.$$
4. Finally, we define the map
$$\partial_{ss,\mathfrak{m}} : O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]_\mathfrak{m} \to H^1(\mathbb{F}_q, H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_\mathfrak{m})$$
to be the composite
$$O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]_\mathfrak{m} \xrightarrow{(3)^{-1}} O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]^{\mathrm{Frob}_q^2 = 1}_\mathfrak{m} \xrightarrow{[g] \mapsto [g] + [gF]} O[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})]^{\mathrm{Frob}_q = 1}_\mathfrak{m}$$
$$\xrightarrow{(2)} \mathrm{CH}^2(X_{\mathbb{F}_q}, O)_\mathfrak{m} \xrightarrow{\partial_{\mathrm{AJ}, \mathbb{F}_q, \mathfrak{m}}} H^1(\mathbb{F}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, O(2))_\mathfrak{m}) \xrightarrow[\mathrm{BC}_X]{\sim} H^1_{\mathrm{unr}}(\mathbb{Q}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, O(2))_\mathfrak{m}).$$

Theorem 10.5.7. *Suppose* $\mathfrak{m} \subset \mathbb{T}_O^{S \cup \{q\}}$ *satisfies:*

(1) $\mathfrak{m}$ *is non-Eisenstein and generic.*
(2) $\langle q \rangle = 1$ *in* $\mathbb{T}^S_{K,V_D,O,\mathfrak{m}}$.

Let $h \in \mathbb{T}_O^{\circ S^\circ}$ *be any element, and let* $C \geq 0$ *be an integer such that* ϖ^C *annihilates the* ϖ*-power-torsion in* $H^2(X^\circ_{\overline{\mathbb{Q}}}, O(1))$*. Then there exists a special cycle*

$$Z_{Dq} \in \mathrm{SC}^2_K(V_{Dq}, O)$$

such that

$$\varpi^C \cdot \mathrm{Res}_{\mathbb{Q}_q} \partial_{\mathrm{AJ},\mathfrak{m}} \left(j_* \left((T_q^{\circ 2} - 4q^2\langle q\rangle)^2 \cdot h \cdot [X^{\diamondsuit}] \right) \right) = \varpi^C \cdot \partial_{ss,\mathfrak{m}}(Z_{Dq}) \in H^1_{\mathrm{unr}}(\mathbb{Q}_q, H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}).$$

Proof. Let

$$\bar{j} : \mathrm{Sh}_{K^\circ}(V^\circ_{Dq}) \xrightarrow{\cdot g_0^q} \mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})$$

be the natural map, induced by the embedding $V^\circ_{Dq} \hookrightarrow V_{Dq}$ from Notation 10.3.1(1). In particular, we have

$$j(B^\circ_\delta(g)) = B(\bar{j}(g)F^\delta), \quad (g, \delta) \in \mathrm{Sh}_{K^\circ}(V^\circ_{Dq}) \times \{0, 1\}\,. \tag{10.16}$$

Then we take

$$Z^\circ_{Dq} = \sum_{g \in \mathrm{Sh}_{K^\circ}(V^\circ_{Dq})} n(g)[g] \in \mathrm{SC}^1_{K^\circ}(V^\circ_{Dq}, O)$$

to be the special cycle provided by Theorem 10.4.13, and let

$$Z_{Dq} = \bar{j}\left(Z^\circ_{Dq}\right) \in \mathrm{SC}^2_K(V_{Dq}, O)$$

be the pushforward. If we let

$$\mathrm{cl}^\circ_{ss}(Z^\circ_{Dq}) = \sum_{g \in \mathrm{Sh}_{K^\circ}(V^\circ_{Dq})} n(g)\left([B^\circ_0(g)] + [B^\circ_1(g)]\right) \in \mathrm{CH}^1(X^\circ_{\mathbb{F}_q}, O),$$

then by (10.16),

$$\partial_{ss,\mathfrak{m}}(Z_{Dq}) = \mathrm{BC}_X\left(\partial_{\mathrm{AJ},\mathfrak{m},\mathbb{F}_q} j_* \,\mathrm{cl}^\circ_{ss}(Z^\circ_{Dq})\right) \in H^1_{\mathrm{unr}}(\mathbb{Q}_q, H^3(X_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}).$$

In light of Lemma 10.5.4 and the local-global compatibility of the Abel-Jacobi map, it then suffices to show that

$$\mathrm{cl}^\circ_{ss}(Z^\circ_{Dq}) - (T_q^{\circ 2} - 4q^2\langle q\rangle)^2 \cdot h \cdot [X^{\diamondsuit}_{\mathbb{F}_q}] \in \mathrm{CH}^1(X^\circ_{\mathbb{F}_q})$$

has ϖ-power-torsion image in $H^2_{\text{ét}}(X^\circ_{\overline{\mathbb{F}}_q}, O(1))$. But this is precisely the content of Theorem 10.4.13. □

10.5.8. Now let π, S, and E_0 be as in Notation 4.0.1, and let $\mathfrak{p}$ be a prime of E_0 satisfying Assumption 4.1.1. We set $\mathfrak{m} := \mathfrak{m}_{\pi,\mathfrak{p}}$ as usual.

Corollary 10.5.9. *Suppose $q \nmid D$ is n-admissible and K is an S-tidy level structure for* $\mathrm{GSpin}(V_D)$*. Then for any $\alpha \in \mathrm{Test}_K(V_D, \pi, O/\varpi^n)$ and any $h \in \mathbb{T}^{\circ S^\circ}_O$,*

$$\mathrm{ord}_\varpi \lambda^D_n(q) \geq \mathrm{ord}_\varpi \mathrm{loc}_q \circ \alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}} \left(j_* \left((T_q^{\circ 2} - 4q^2\langle q\rangle)^2 \cdot h \cdot [X^{\diamondsuit}] \right) \right) - C,$$

where C is a constant independent of n and q.

Proof. Recall the conditions on $\mathfrak{m}$ from Theorem 10.5.7 are satisfied by Lemmas 4.1.7 and 4.3.2. Let α' be the composite map

$$O\left[\mathrm{Sh}_{K^q K_q^{\mathrm{Pa}}}(V_{Dq})\right]_{\mathfrak{m}} \xrightarrow{\partial_{ss,\mathfrak{m}}} H^1_{\mathrm{unr}}(\mathbb{Q}_q, H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_{\mathfrak{m}}) \xrightarrow{\alpha_*} H^1_{\mathrm{unr}}(\mathbb{Q}_q, T_{\pi,n}) \simeq O/\varpi^n,$$

where the last isomorphism is from Proposition 4.2.8. Then $\lambda^D_n(q)$ contains $\alpha'(Z_{Dq})$, where $Z_{Dq} \in \mathrm{SC}^2_K(V_{Dq}, O)$ is the cycle from Theorem 10.5.7, and the corollary follows. □

11. SECOND EXPLICIT RECIPROCITY LAW

The goal of this section is to prove Theorem 11.2.6 below.

11.1. **Setup and notation.** Let π, S, and E_0 be as in Notation 4.0.1, and fix the following data:

- A prime $\mathfrak{p}$ of E_0 satisfying Assumption 4.1.1, with residue characteristic p.
- A squarefree product $D \geq 1$ of primes in S, with $\sigma(D)$ even.
- An S-tidy level structure K for $\mathrm{GSpin}(V_D)$.
- A Schwartz function $\varphi = \bigotimes'_\ell \varphi_\ell \in \mathcal{S}(V_D^2 \otimes \mathbb{A}_f, O)^K$.

11.1.1. For all $\ell \notin S$, let $L_\ell \subset V_D \otimes \mathbb{Q}_\ell$ be the unique self-dual lattice stabilized by K_ℓ.

11.2. **Modifying the test function.** In several steps, we now construct a new Schwartz function φ' which coincides with φ at cofinitely many primes.

Construction 11.2.1.

(1) Label the finitely many imaginary quadratic fields contained in $\rho_\pi = \rho_{\pi,\mathfrak{p}}$ as $E_1, \ldots, E_s$ for some $s \geq 0$.
(2) For each $1 \leq i \leq s$, fix an odd prime $\ell_i \notin S \cup \{p, \ell_1, \ldots, \ell_{i-1}\}$ inert in E_i such that $\rho_\pi(\mathrm{Frob}_{\ell_i})$ has distinct eigenvalues (possible *a fortiori* by Assumption 4.1.1(2)).
(3) Let

$$X_{\ell_i} = \left\{(x,y) \in V_D^2 \otimes \mathbb{Q}_{\ell_i} \,:\, x \cdot x \in (\mathbb{Z}_{\ell_i}^\times)^2,\, x \cdot y \in \mathbb{Z}_{\ell_i},\, y \cdot y \in \mathbb{Z}_{\ell_i}^\times - (\mathbb{Z}_{\ell_i}^\times)^2\right\}.$$

(4) We define a test function

$$\varphi'_{\ell_i} = (\ell_i + 1) \cdot \mathbb{1}_{\left\{(x,y) \in L_{\ell_i}^2 \cap X_{\ell_i}\right\}} + \mathbb{1}_{\left\{(x,y) \in L_{\ell_i} \times (\ell_i^{-1} L - L_{\ell_i}) \cap X_{\ell_i}\right\}} \in \mathcal{S}(V_D^2 \otimes \mathbb{Q}_{\ell_i}, \mathbb{Z})^{K_{\ell_i}}.$$

(5) Now fix a prime $\ell_0 \notin S \cup \{p, \ell_1, \ldots, \ell_s\}$ such that $\rho_\pi(\mathrm{Frob}_{\ell_0})$ has distinct eigenvalues.
(6) Let

$$\varphi'_{\ell_0} \in \mathcal{S}(V_D^2 \otimes \mathbb{Q}_{\ell_0}, \mathbb{Z})^{K_{\ell_0}}$$

be the indicator function of the set

$$\left\{(x,y) \in L_{\ell_0}^2 \,:\, x \cdot x \in \mathbb{Z}_{\ell_0}^\times - (\mathbb{Z}_{\ell_0}^\times)^2,\, x \cdot y \in \ell_0 \mathbb{Z}_{\ell_0},\, y \cdot y \in \ell_0 \mathbb{Z}_{\ell_0}^\times\right\}.$$

(7) Finally, we define

$$\varphi' \coloneqq \bigotimes_{\ell \neq \ell_i} \varphi_\ell \otimes \bigotimes_{i=0}^{s} \varphi'_{\ell_i} \in \mathcal{S}(V_D^2 \otimes \mathbb{A}_f, O)^K.$$

Lemma 11.2.2. *Suppose $\kappa^D(1, \varphi; K) \neq 0$. Then there exists a test function $\alpha \in \mathrm{Test}_K(V_D, \pi, O)$ and a matrix $T \in \mathrm{Sym}_2(\mathbb{Q})_{>0}$ satisfying (T1) and (T2) of (10.1.1) such that*

$$\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}} \left(Z(T, \varphi')_K\right) \neq 0.$$

Moreover, for all $1 \leq i \leq s$, ℓ_i is split in the quadratic field $F \coloneqq \mathbb{Q}(\sqrt{T_{11}})$ and T lies in $\mathrm{GL}_2(\mathbb{Z}_{(\ell_i)})$.

Proof. Repeatedly applying Corollary 5.6.6, we conclude $\kappa^D(1, \varphi'; K) \neq 0$; i.e., there exists $T \in \mathrm{Sym}_2(\mathbb{Q})_{\geq 0}$ and $\alpha \in \mathrm{Test}_K(V_D, \pi, O)$ such that

$$\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}} \left(Z(T, \varphi')_K\right) \neq 0.$$

By definition of $Z(T, \varphi')_K$, it follows that, for all $0 \leq i \leq s$, there exists $x, y \in \Omega_{T,V_D}(\mathbb{Q}_{\ell_i})$ such that $\varphi'_{\ell_i}(x,y) \neq 0$. Now the choice of φ'_{ℓ_0} implies that T is positive definite, T_{11} is not a rational square, and the quadratic space defined by T has nontrivial Hasse invariant at $\ell_0 \nmid D$, and this proves the first claim of the lemma. Similarly, for each $1 \leq i \leq s$, ℓ_i is split in F and T lies in $\mathrm{GL}_2(\mathbb{Z}_{(\ell_i)})$ by the choice of φ'_{ℓ_i}. $\square$

11.2.3. We will now assume $\kappa^D(1, \varphi; K) \neq 0$, and fix α and T satisfying the conclusion of Lemma 11.2.2. We choose a base point $(e_1^T, e_2^T) \in \Omega_{T,V_D}(\mathbb{Q})$ such that $e_1^T, e_2^T \in L_{\ell_i}$ for all $1 \leq i \leq s$ (possible by the last point of Lemma 11.2.2), and we adopt the setup of §10 for this choice of T, (e_1^T, e_2^T), and K; the choice of g_0 is postponed to Proposition 11.2.5.

Definition 11.2.4. For each ℓ split in F and each hyperspecial subgroup $K_\ell^\circ \subset \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_\ell)$, we define a Hecke operator $T_\ell^\circ \in \mathbb{T}_\ell^\circ$ as follows. Choose an isomorphism

$$\mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_\ell) \simeq G_\ell \coloneqq \{(g_1, g_2) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \times \mathrm{GL}_2(\mathbb{Q}_\ell) \ : \ \det g_1 = \det g_2\}$$

mapping K_ℓ° to $G_\ell \cap (\mathrm{GL}_2(\mathbb{Z}_\ell) \times \mathrm{GL}_2(\mathbb{Z}_\ell))$. Then T_ℓ° is the double coset operator represented by

$$\left(\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \right) \in G_\ell.$$

Proposition 11.2.5. *There exists an element* $g_0 \in \mathrm{GSpin}(V_D)(\mathbb{A}_f^{\{\ell_1, \ldots, \ell_s\}})$ *such that, adopting the notation of (10.1.3) for this choice of* g_0*:*

1. *The compact open subgroups* $K_{\ell_i}^\circ \subset \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i})$ *are hyperspecial for* $1 \leq i \leq s$.
2. *We have*

$$\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left(\prod_{i=1}^{s} T_{\ell_i}^\circ \cdot [\mathrm{Sh}_{K^\diamond}(V_D^\diamond)] \right) \neq 0,$$

where $[\mathrm{Sh}_{K^\diamond}(V_D^\diamond)] \in \mathrm{CH}^1(\mathrm{Sh}_{K^\circ}(V_D^\circ))$ *is the class of the cycle from (10.2).*

Proof. By definition of $Z(T, \varphi')_K$ and the assumption $\alpha \circ \partial_{\mathrm{AJ},\mathfrak{m}}(Z(T, \varphi')_K) \neq 0$, there exists $g_0 \in \mathrm{GSpin}(V_D)(\mathbb{A}_f^{\{\ell_1, \ldots, \ell_s\}})$ such that

$$(11.1)\quad Z \coloneqq \sum_{g = \prod g_{\ell_i} \in \prod_{i=1}^s \mathrm{GSpin}(V_D^\diamond)(\mathbb{Q}_{\ell_i}) \backslash \mathrm{GSpin}(V_D)(\mathbb{Q}_{\ell_i}) / K_{\ell_i}} \left(\prod_{i=1}^{s} \varphi'_{\ell_i}(g_{\ell_i}^{-1} \cdot (e_1^T, e_2^T)) \right) \cdot Z(g_0 g, V_T, V_D)_K$$

satisfies $\alpha_* \partial_{\mathrm{AJ},\mathfrak{m}}(Z) \neq 0$. We will check the claimed properties for this choice of g_0. First, if $\varphi'_{\ell_i}(g_{\ell_i}^{-1}(e_1^T, e_2^T)) \neq 0$, then $g_{\ell_i}^{-1} e_1^T \cdot g_{\ell_i}^{-1} e_1^T = e_1^T \cdot e_1^T \in \mathbb{Z}_{(\ell_i)}^\times$, so in particular

$$L_{\ell_i} \cap \mathbb{Q}_{\ell_i} g_{\ell_i}^{-1} e_1^T = \mathbb{Z}_{\ell_i} g_{\ell_i}^{-1} e_1^T.$$

On the other hand, we chose (e_1^T, e_2^T) so that $L_{\ell_i} \cap e_1^T = \mathbb{Z}_{\ell_i} e_1^T$. Hence by Proposition 3.1.8, we have $g_{\ell_i} \in \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i}) \cdot K_{\ell_i}$. In particular, we can rewrite (11.1) as

(11.2)

$$\begin{aligned} Z &= \sum_{g = \prod g_{\ell_i} \in \prod_{i=1}^s \mathrm{GSpin}(V_D^\diamond)(\mathbb{Q}_{\ell_i}) \backslash \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i}) / K_{\ell_i} \cap \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i})} \prod_{i=1}^{s} \varphi'_{\ell_i}(g_{\ell_i}^{-1}(e_1^T, e_2^T)) \cdot Z(g_0 g, V_T, V_D)_K \\ &= \sum_{g = \prod g_{\ell_i} \in \prod_{i=1}^s \mathrm{GSpin}(V_D^\diamond)(\mathbb{Q}_{\ell_i}) \backslash \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i}) / K_{\ell_i} \cap \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i})} \prod_{i=1}^{s} \varphi'_{\ell_i}(g_{\ell_i}^{-1}(e_1^T, e_2^T)) \cdot j_* Z(g, V_T \cap V_D^\circ, V_D^\circ)_{K^\circ}. \end{aligned}$$

Again because $e_1^T \cdot e_1^T \in \mathbb{Z}_{(\ell_i)}^\times$, $L^\circ := L_{\ell_i} \cap (V_D^\circ \otimes \mathbb{Q}_{\ell_i})$ is self-dual, and so its stabilizer $K_{\ell_i}^\circ := K_{\ell_i} \cap \mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i})$ in $\mathrm{GSpin}(V_D^\circ)(\mathbb{Q}_{\ell_i})$ is hyperspecial, which proves (1). To prove (2), by definition we have

$$\begin{aligned}\varphi'_{\ell_i}(e_1^T, g_{\ell_i}^{-1}e_2^T) &= \begin{cases} \ell_i + 1, & g_{\ell_i}^{-1}e_2^T \in L_{\ell_i}, \\ 1, & g_{\ell_i}^{-1}e_2^T \in \ell_i^{-1}L_{\ell_i} - L_{\ell_i}, \\ 0, & g_{\ell_i}^{-1}e_2^T \notin \ell_i^{-1}L_{\ell_i} \end{cases} \\ &= \begin{cases} \ell_i + 1, & g_{\ell_i}^{-1}e_2 \in L^\circ, \\ 1, & g_{\ell_i}^{-1}e_2 \in \ell_i^{-1}L^\circ - L^\circ, \\ 0, & g_{\ell_i}^{-1}e_2 \notin \ell_i^{-1}L^\circ, \end{cases}\end{aligned} \tag{11.3}$$

where

$$e_2 := e_2^T - \frac{e_2^T \cdot e_1^T}{e_1^T \cdot e_1^T} e_1^T \in V_D^\circ.$$

Note that $e_2 \cdot e_2 \in \mathbb{Z}_{(\ell_i)}^\times$ because, for $(e_1^T, g_{\ell_i}^{-1}e_2^T)$ to lie in the support of φ'_{ℓ_i}, we automatically have $e_1^T \cdot e_1^T \in \mathbb{Z}_{(\ell_i)}^\times - (\mathbb{Z}_{\ell_i}^\times)^2$, $e_1^T \cdot e_2^T \in \mathbb{Z}_{(\ell_i)}$, and $e_2^T \cdot e_2^T \in \mathbb{Z}_{(\ell_i)}^\times \cap (\mathbb{Z}_{\ell_i}^\times)^2$.

Let $\varphi_{\ell_i}^\circ \in \mathcal{S}(V_D^\circ \otimes \mathbb{Q}_{\ell_i}, \mathbb{Z})^{K_{\ell_i}^\circ}$ be the indicator function of L°; by our specification $e_2^T \in L_{\ell_i}$ and Proposition 3.1.8, we see that $\varphi_{\ell_i}^\circ(g_{\ell_i}^{-1}e_2)$ is the indicator function of $\mathrm{GSpin}(V_D^\diamond)(\mathbb{Q}_{\ell_i}) \cdot K_{\ell_i}^\circ$. The proposition therefore follows from (11.2), (11.3), and the following:

Claim. When restricted to $\left\{y \in V_D^\circ \otimes \mathbb{Q}_{\ell_i} \,:\, y \cdot y \in \mathbb{Z}_{\ell_i}^\times\right\}$, we have $T_{\ell_i}^\circ \cdot \varphi_{\ell_i}^\circ = (\ell_i + 1)\varphi_{\ell_i}^\circ + \mathbb{1}_{\ell_i^{-1}L^\circ - L^\circ}$.

To prove the claim, observe first that

$$T_{\ell_i}^\circ \cdot \varphi_{\ell_i}^\circ = \sum_{L' \sim L^\circ} \mathbb{1}_{L'},$$

where L' runs over self-dual lattices in $V_D^\circ \otimes \mathbb{Q}_{\ell_i}$ such that

$$\ell_i L^\circ \subset_1 \ell_i L' + \ell_i L^\circ \subset_2 L' \cap L^\circ \subset_1 L^\circ. \tag{11.4}$$

Note that such a chain (11.4) uniquely determines L' because L' and L° correspond to the two isotropic lines in the split two-dimensional $\mathbb{F}_{\ell_i}$-space $L^\circ + L'/L^\circ \cap L'$. We can also write the chain (11.4) as

$$\ell_i L^\circ \subset_1 L_1 \subset_2 L_1^\vee \subset_1 L^\circ, \tag{11.5}$$

i.e. such chains correspond bijectively to isotropic lines $L_1/\ell_i L^\circ$ in the $\mathbb{F}_{\ell_i}$-quadratic space $L^\circ/\ell_i L^\circ$.

Now for any $y \in V_D^\circ \otimes \mathbb{Q}_{\ell_i}$ with $y \cdot y \in \mathbb{Z}_{\ell_i}^\times$, we wish to calculate

$$T_{\ell_i}^\circ \cdot \varphi_{\ell_i}^\circ(y) = \#\left\{L' \sim L^\circ \,:\, y \in L'\right\}.$$

If $y \in L^\circ$, then the choices of L' correspond to isotropic lines orthogonal to y in the four-dimensional $\mathbb{F}_{\ell_i}$-space $L^\circ/\ell_i L^\circ$; there are $\ell_i + 1$ such lines because $y^\perp \subset L^\circ/\ell_i L^\circ$ is a non-degenerate three-dimensional $\mathbb{F}_{\ell_i}$-space. Hence $T_{\ell_i}^\circ \cdot \varphi_{\ell_i}^\circ(y) = \ell_i + 1$ for $y \in L^\circ$.

On the other hand, since any $L' \sim L^\circ$ is contained in $\ell_i^{-1}L^\circ$, if $T_{\ell_i}^\circ \cdot \varphi_{\ell_i}^\circ(y) \neq 0$ we must have $y \in \ell_i^{-1}L^\circ$. So suppose $y \in \ell_i^{-1}L^\circ - L^\circ$. Now, lattices $L' \sim L^\circ$ with $y \in L'$ (equivalently $\ell_i y \in \ell_i L'$) are in bijection with chains (11.5) satisfying $\ell_i y \in L_1$; since $\ell_i y \notin \ell_i L^\circ$, the unique such chain is given by $L_1 = \ell_i L^\circ + \ell_i y$, and this makes sense because $\ell_i y$ is isotropic in $L^\circ/\ell_i L^\circ$. We therefore conclude that $T_{\ell_i}^\circ \cdot \varphi_{\ell_i}^\circ(y) = 1$ for $y \in \ell_i^{-1}L^\circ - L^\circ$, which completes the proof of the claim. □

Theorem 11.2.6. *Suppose π is non-endoscopic, $\kappa^D(1) \neq 0$, and admissible primes exist for ρ_π. Then there exists a constant $C \geq 0$ such that, for all N, there exist infinitely many admissible primes q with $n(q) \geq N$,*

$$\mathrm{ord}_\varpi \mathrm{loc}_q \, \kappa_{n(q)}^D(1) \geq n(q) - C,$$

and

$$\mathrm{ord}_{\varpi}\, \lambda^{D}_{n(q)}(q) \geq n(q) - C.$$

Proof. Since $\kappa^D(1) \neq 0$, we can choose the data φ and K in the beginning of this section so that $\kappa^D(1, \varphi; K) \neq 0$ (where K can be made S-tidy by Lemma 4.4.7). Then fix $g_0 \in \mathrm{GSpin}(V_D)(\mathbb{A}_f^{\{\ell_1,\dots,\ell_s\}})$ as in Proposition 11.2.5, so that

$$\alpha_* \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left(\prod_{i=1}^{s} T^{\circ}_{\ell_i} \left[\mathrm{Sh}_{K^{\diamond}}(V_D^{\diamond}) \right] \right) \neq 0; \tag{11.6}$$

equivalently, by Lemma 4.1.6, (11.6) is non-torsion. Let S° be the set of primes ℓ such that K°_ℓ is not hyperspecial. Without loss of generality, we may replace O by a finite extension such that all the eigenvalues of $\mathbb{T}_O^{\circ S^\circ}$ acting on $H^2_{\text{ét}}(\mathrm{Sh}_{K^\circ}(V_D^\circ)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$ lie in O; let $\boldsymbol{h}_1, \dots, \boldsymbol{h}_m : \mathbb{T}_O^{\circ S^\circ} \to O$ be the finitely many characters that appear in the action on $H^2_{\text{ét},!}(\mathrm{Sh}_{K^\circ}(V_D^\circ)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1))^{G_{\mathbb{Q}}}$, cf. Corollary A.2.2. Then we may choose elements $h_j \in \mathbb{T}_O^{\circ S^\circ}$ such that $\mathbb{T}_O^{\circ S^\circ}$ acts on $h_j H^2_{\text{ét},!}(\mathrm{Sh}_{K^\circ}(V_D^\circ)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1))^{G_{\mathbb{Q}}}$ through $\boldsymbol{h}_j$, and moreover

$$\sum_{j=1}^{m} h_j = \varpi^{C_0} \text{ on } H^2_{\text{ét},!}(\mathrm{Sh}_{K^\circ}(V_D^\circ)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1))^{G_{\mathbb{Q}}}$$

for a constant $C_0 \geq 0$. In particular, by Lemma 10.5.4 and Proposition 11.2.5,

$$c := \alpha \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left(h_j \cdot \left(\prod_{i=1}^{s} T^{\circ}_{\ell_i} \right) \left[\mathrm{Sh}_{K^{\diamond}}(V_D^{\diamond}) \right] \right)$$

is non-torsion for some $1 \leq j \leq m$. Write $\boldsymbol{h} := \boldsymbol{h}_j$ and $h := h_j$. Let us fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. By Corollary A.2.2, we are in one of the following two cases:

Case 1. The character $\iota \circ \boldsymbol{h}$ is given by the action of $\mathbb{T}^{\circ S^\circ}$ on a global newform in an automorphic representation σ of $B_F(\mathbb{A}_F)^\times$ unramified outside primes above S°. Moreover $\mathrm{JL}(\sigma) = \mathrm{BC}_{F/\mathbb{Q}}(\sigma_0) \otimes \chi$ where σ_0 is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ of weight 2 and χ is a finite order character of $F^\times \backslash \mathbb{A}_F^\times$.

Case 2. The character $\iota \circ \boldsymbol{h}$ is given by the action of $\mathbb{T}^{\circ S^\circ}$ on the automorphic representation $\chi_0 \circ \nu$, where $\nu : \mathrm{GSpin}(V_D^\circ) \to \mathbb{G}_m$ is the restriction of the norm character in (A.1.1) and χ_0 is a quadratic character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times / \nu(K^\circ)$.

Claim 1. In Case 1, σ_0 does not have CM by any imaginary quadratic field $K_0 \subset F(\rho_\pi)$.

Proof of claim 1. If so, then $K_0 \neq F$, and K_0 is contained in the compositum of F and one of the quadratic fields $E_i \subset \mathbb{Q}(\rho_\pi)$ from Construction 11.2.1; in particular, since ℓ_i is inert in K_i but split in F by Lemma 11.2.2, ℓ_i is inert in K_0. It follows that $\mathrm{tr}\, \rho_{\sigma,\iota}(\mathrm{Frob}_{\ell_i}) = 0$. On the other hand, it is not difficult to compute using the Satake transform and Theorem 2.2.1(1) that

$$T^{\circ}_{\ell_i} = \mathrm{tr}\, \rho_{\sigma,\iota}(\mathrm{Frob}_{\ell_i}) \cdot \mathrm{tr}\, \rho_{\sigma,\iota}(\tau\, \mathrm{Frob}_{\ell_i}\, \tau^{-1})$$

on a local newform in σ_{ℓ_i}, where $\tau \in G_{\mathbb{Q}}$ projects to the nontrivial element of $\mathrm{Gal}(F/\mathbb{Q})$. Hence $h \cdot T^{\circ}_{\ell_i} = \boldsymbol{h}(T^{\circ}_{\ell_i}) \cdot h = 0$, which contradicts the nontriviality of c. □

Now fix an element $g \in \mathrm{Gal}(F(\rho_\pi)/\mathbb{Q})$ such that:

1. g is admissible for ρ_π and has nontrivial image in $\mathrm{Gal}(F/\mathbb{Q})$.
2. $c(g)$ has nonzero component in the 1-eigenspace for g for any cocycle representative of c.
3. In Case 1 above, $\rho_{\sigma_0,\iota}(g^2)$ has distinct eigenvalues.

This is possible by Proposition C.5.3 because c is nontorsion. Now fix constants $C_1, C_2, C_3 \geq 0$ such that:

(1) The component of $c(g)$ in the 1-eigenspace for g is nonzero modulo ϖ^{C_1}.
(2) In Case 1 above, $\mathrm{tr}(\rho_{\sigma,\iota}(g^2)) - 4\det\rho_{\sigma,\iota}(g^2) \not\equiv 0 \pmod{\varpi^{C_2}}$. In Case 2, $C_2 = 0$.
(3) ϖ^{C_3} annihilates the O-torsion in $H^2_{\text{ét},!}(\mathrm{Sh}_{K^\circ}(V_D^\circ)_{\overline{\mathbb{Q}}}, O)$.

Now fix $N \geq C_1 + C_2 + C_3$ and let $c_N \in H^1(\mathbb{Q}, T_{\pi,N})$ be the reduction of c modulo ϖ^N. All but finitely many primes with Frobenius conjugate to g in $\mathrm{Gal}(F(T_{\pi,N}, c_N)/\mathbb{Q})$ are N-admissible and satisfy Assumption 10.2.1; let q be one such and abbreviate $n := n(q) \geq N$. If $\alpha_n \in \mathrm{Test}_K(V_D, \pi, \mathcal{O}/\varpi^n)$ is the image of α, then:

$$\mathrm{ord}_\varpi \,\mathrm{loc}_q\, \alpha_{n,*} \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left(h \cdot \left(\prod_{i=1}^{s} T^\circ_{\ell_i} \right) [\mathrm{Sh}_{K^\diamond}(V_D^\diamond)] \right) \geq n - C_1. \tag{11.7}$$

We have:

Claim 2. $\mathrm{ord}_\varpi \boldsymbol{h}(T_q^{\circ 2} - 4q^2\langle q\rangle) \leq C_2$.

Proof of Claim 2. In Case 1, we have

$$\boldsymbol{h}(T_q^{\circ 2} - 4q^2\langle q\rangle) = \mathrm{tr}\,\rho_{\sigma,\iota}(\mathrm{Frob}_q^2) - 4\det\rho_{\sigma,\iota}(\mathrm{Frob}_q^2) \equiv \mathrm{tr}\,\rho_{\sigma,\iota}(g^2) - 4\det\rho_{\sigma,\iota}(g^2) \pmod{\varpi^n},$$

so this follows from the choice of C_2. In Case 2, we have $\boldsymbol{h}(T_q^\circ) = (q^2+1)\chi_0(\langle q\rangle)$ and $\boldsymbol{h}(\langle q\rangle) = \chi_0(\langle q\rangle)$, so $\boldsymbol{h}(T_q^{\circ 2} - 4q^2\langle q\rangle) = (q^2-1)^2\chi_0(\langle q\rangle)$, which is a ϖ-adic unit because q is admissible. ☐

Now, by Corollary 10.5.9 we have

$$\begin{aligned}
\mathrm{ord}_\varpi \lambda_n^D(q) &\geq \mathrm{ord}_\varpi \,\mathrm{loc}_q\, \alpha_{n,*} \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left((T_q^{\circ 2} - 4q^2\langle q\rangle) h \left(\prod_{i=1}^{s} T^\circ_{\ell_i} \right) [\mathrm{Sh}_{K^\diamond}(V_D^\diamond)] \right) - C_3 \\
&= \mathrm{ord}_\varpi \,\mathrm{loc}_q\, \alpha_{n,*} \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left(\boldsymbol{h}(T_q^{\circ 2} - 4q^2\langle q\rangle) h \left(\prod_{i=1}^{s} T^\circ_{\ell_i} \right) [\mathrm{Sh}_{K^\diamond}(V_D^\diamond)] \right) - C_3 \\
&\geq \mathrm{ord}_\varpi \,\mathrm{loc}_q\, \alpha_{n,*} \circ \partial_{\mathrm{AJ},\mathfrak{m}} j_* \left(h \left(\prod_{i=1}^{s} T^\circ_{\ell_i} \right) [\mathrm{Sh}_{K^\diamond}(V_D^\diamond)] \right) - C_3 - \mathrm{ord}_\varpi \boldsymbol{h}(T_q^{\circ 2} - 4q^2\langle q\rangle).
\end{aligned}$$

By Claim 2 and (11.7), we conclude that

$$\mathrm{ord}_\varpi \lambda_n^D(q) \geq n - C_1 - C_2 - C_3. \tag{11.8}$$

Since $j_*\left(h \cdot \left(\prod_{i=1}^s T^\circ_{\ell_i}\right)[\mathrm{Sh}_{K^\diamond}(V_D^\diamond)]\right) \in \mathrm{CH}^2(\mathrm{Sh}_K(V_D), O)$ lies in $\mathrm{SC}^2_K(V_D, O)$ by Remark 3.1.6, (11.7) and (11.8) together show the theorem. ☐

In the endoscopic case, we similarly obtain:

Theorem 11.2.7. *Suppose π is endoscopic associated to a pair (π_1, π_2), such that π_1 and π_2 are not both CM for the same imaginary quadratic field. For $j = 1$ or 2, assume $\kappa^D(1)^{(j)} \neq 0$ and admissible primes exist which are BD-admissible for ρ_{π_j}. Then there exists a constant $C \geq 0$ such that, for all N, there exist infinitely many admissible primes q which are N-BD-admissible for ρ_{π_j}, such that*

$$\mathrm{ord}_\varpi \,\mathrm{loc}_q\, \kappa^D_{n(q)}(1)^{(j)} \geq n(q) - C$$

and

$$\mathrm{ord}_\varpi \lambda^D_{n(q)}(q) \geq n(q) - C.$$

Proof. By using Corollary 5.6.7 in place of Corollary 5.6.6, we can refine Lemma 11.2.2 to obtain the same conclusion where $\alpha : H^3_{\text{ét}}(\mathrm{Sh}_K(V_D)_{\overline{\mathbb{Q}}}, O(2))_\mathfrak{m} \to T_\pi$ is required to factor through T_{π_j}. From here, the rest of the proof follows that of Theorem 11.2.6, substituting Proposition C.5.4 for Proposition C.5.3. ☐

12. Main result: rank one case

12.1. **Setup and notation.** Let π, S, and E_0 be as in Notation 4.0.1, and fix a prime $\mathfrak{p}$ of E_0 satisfying Assumption 4.1.1, with residue characteristic p.

Definition 12.1.1. An automorphic representation π of $\mathrm{GL}_4(\mathbb{A})$ is of *general type* if it is neither an automorphic induction nor a symmetric cube lift.

For this section, we shall assume:

Hypothesis (★). If π is non-endoscopic, then $\mathrm{BC}(\pi)$ is either of general type, or a symmetric cube lift of a non-CM automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A})$, or induced from a non-CM automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ with $K/\mathbb{Q}$ real quadratic; in the latter two cases E_0 is also a strong coefficient field of π_0.

12.2. **Choosing Chebotarev primes.**

12.2.1. Let $L_\pi \subset \mathrm{End}_O(T_\pi)$ be the underlying O-module of $\mathrm{ad}^0\,\rho_\pi = \mathrm{ad}^0\,\rho_{\pi,\mathfrak{p}}$, which is free of rank 10 over O. In general, $L_\pi \otimes \mathbb{Q}_p$ is not absolutely irreducible, even if V_π is. In the endoscopic case, we also let L_{π_i} be the underlying O-module of $\mathrm{ad}^0\,\rho_{\pi_i}$, for $i = 1, 2$.

Proposition 12.2.2. *Suppose π is not endoscopic. We have the following cases for L_π:*

1. *If $\mathrm{BC}(\pi)$ is an automorphic induction of π_0 as in Hypothesis (★), then*

$$L_\pi = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \mathrm{ad}^0\,T_{\pi_0} \oplus \left(\otimes - \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} T_{\pi_0}\right)(-1).$$

   Both direct summands are absolutely irreducible after inverting p.

2. *If $\mathrm{BC}(\pi)$ is a symmetric cube lift of π_0 as in Hypothesis (★), then*

$$L_\pi = \mathrm{ad}^0\,T_{\pi_0} \oplus \mathrm{Sym}^6(T_{\pi_0})(-3),$$

   with each summand absolutely irreducible after inverting p.

3. *If $\mathrm{BC}(\pi)$ is of general type, then $L_\pi \otimes \mathbb{Q}_p$ is absolutely irreducible.*

Proof. In the first case, T_{π_0} exists by Lemma 2.2.19, and we have

$$T_\pi = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} T_{\pi_0};$$

so the claimed decomposition follows from, e.g., the discussion in [13, §7.5.16]. For the irreducibility, if π_0^{tw} is the $\mathrm{Gal}(K/\mathbb{Q})$ twist, then one checks using Hodge-Tate weights that $V_{\pi_0} \not\cong V_{\pi_0^{\mathrm{tw}}} \otimes \chi$ for any character χ of G_K. It follows that $\otimes - \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} V_{\pi_0}$ is absolutely irreducible, since if it reduces it must have a one-dimensional constituent. Similarly, if $\mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \mathrm{ad}^0\,V_{\pi_0}$ is not absolutely irreducible, then $\mathrm{ad}^0\,V_{\pi_0} \cong \mathrm{ad}^0\,V_{\pi_0^{\mathrm{tw}}}$, so by Corollary C.3.7, $V_{\pi_0}|_{G_L} = V_{\pi_0^{\mathrm{tw}}}|_{G_L}$ for a finite extension L/K, and this is a contradiction by Lemma C.3.4.

In the second case, T_{π_0} exists by Lemma 2.2.18, and by definition, $T_\pi = \mathrm{Sym}^3\,T_{\pi_0}(-1)$. Then we have

$$\mathrm{End}(T_\pi) = \mathrm{Sym}^6(T_{\pi_0})(-3) \oplus \mathrm{Sym}^4(T_{\pi_0})(-2) \oplus \mathrm{Sym}^2(T_{\pi_0})(-1) \oplus O$$

as an $O[G_\mathbb{Q}]$-module, and by dimension counting, L_π consists of the first and third summands. The irreducibility of each summand of L_π after inverting p is clear because the Zariski closure (over E) of the projective image of ρ_{π_0} is $\mathrm{PGL}_2(E)$ by Theorem C.3.2.

The third case follows from Lemma C.4.3, noting that $p > 3$ by Remark 4.1.2. □

12.2.3. We write $L_{\pi,n} = L_\pi/\varpi^n L_\pi$ for all $n \geq 1$ and likewise for $L_{\pi_i,n}$ when π is endoscopic. We will be applying the results and notations of Appendix B to ρ_π (and ρ_{π_i} when π is endoscopic), using Lemmas 8.2.1 and 8.3.7 to check the relevant assumptions. The notion of admissible primes for Notation B.2.4 is as explained in (8.2.5) and (8.3.10). In particular, if π is not endoscopic and q is an n-admissible prime for ρ_π, let $H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,n}) \subset H^1(\mathbb{Q}_q, L_\pi)$ be the subspace defined in Definition B.2.5.

Lemma 12.2.4. *Suppose π satisfies Hypothesis ($\bigstar$) and is not endoscopic. There exists a $G_\mathbb{Q}$-stable decomposition $L_\pi = L_\pi^\heartsuit \oplus L_\pi^\circ$ such that $L_\pi^\heartsuit \otimes \mathbb{Q}_p$ and $L_\pi^\circ \otimes \mathbb{Q}_p$ are absolutely irreducible and distinct, and moreover, if q is an n-admissible prime:*

(1) We have $H^1(\mathbb{Q}_q, L^\heartsuit_{\pi,n}) + H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,n}) = H^1(\mathbb{Q}_q, L_{\pi,n})$.
(2) We have $H^1_{\mathrm{unr}}(\mathbb{Q}_q, L^\circ_{\pi,n}) = H^1(\mathbb{Q}_q, L^\circ_{\pi,n})$.
(3) If $T_{\pi,n}|_{G_{\mathbb{Q}_q}} = M_{0,n} \oplus M_{1,n}$ is the decomposition of Lemma 4.2.2, then the composite

$$L^\heartsuit_{\pi,n} \hookrightarrow L_{\pi,n} \twoheadrightarrow \mathrm{ad}^0 M_{0,n}$$

is surjective.

Here $L^\heartsuit_{\pi,n} = L^\heartsuit_\pi/\varpi^n L^\heartsuit_\pi$, and likewise for $L^\circ_{\pi,n}$.

Proof. In case (3) of Proposition 12.2.2, the lemma is clear, taking $L^\circ_\pi = 0$. Suppose we are in case (1) of Proposition 12.2.2. Then if q is admissible for ρ_π, we have $\mathrm{tr}\,\rho_\pi(\mathrm{Frob}_q) \not\equiv 0 \pmod{\varpi}$, which implies q splits in K. Let π_0^{tw} be the $\mathrm{Gal}(K/\mathbb{Q})$-twist. Because $\det \rho_{\pi_0} = \chi_p^{\mathrm{cyc}}$, comparing the decomposition $T_{\pi,n} = M_{0,n} \oplus M_{1,n}$ with $T_\pi|_{G_K} = T_{\pi_0} \oplus T_{\pi_0^{\mathrm{tw}}}$ shows that, up to replacing π_0 with π_0^{tw}, we have $M_{0,n} = T_{\pi_0,n}$. Now note that $H^1\left(\mathbb{Q}_q, \left(\otimes - \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} T_{\pi_0,n}\right)(-1)\right) = 0$ because the Frob_q eigenvalues on $\left(\otimes - \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{T}_{\pi_0}\right)(-1)$ are q/α, α/q, α, and $1/\alpha$ for some $\alpha \neq q^2, q^{-1}, \pm 1, \pm q$. Hence the lemma holds in this case with $L^\heartsuit_\pi = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \mathrm{ad}^0 T_{\pi_0}$ and $L^\circ_\pi = \left(\otimes - \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} T_{\pi_0}\right)(-1)$.

Now suppose we are in case (2) of Proposition 12.2.2, and take $L^\heartsuit_\pi = \mathrm{Sym}^6(T_{\pi_0})(-3)$, $L^\circ_\pi = \mathrm{ad}^0 T_{\pi_0}$. If Frob_q acts on $\overline{T}_{\pi_0}$ with generalized eigenvalues $\{\alpha, \beta\}$, then the admissibility of Frob_q implies that (up to reordering) we have $\beta^3 = q$ and $\alpha = \beta^2$. Choose a basis for T_π such that

$$\mathrm{Frob}_q = \begin{pmatrix} \alpha^2/\beta & & & \\ & \alpha & & \\ & & \beta & \\ & & & \beta^2/\alpha \end{pmatrix} = \begin{pmatrix} q & & & \\ & \beta^2 & & \\ & & \beta & \\ & & & 1 \end{pmatrix}.$$

One sees immediately that the eigenvalues of Frob_q on $\mathrm{ad}^0 \overline{T}_{\pi_0}$ are $\left\{1, \beta, \beta^{-1}\right\}$, which are all distinct from q^{-1}, so $H^1_{\mathrm{unr}}(\mathbb{Q}_q, L^\circ_{\pi,n}(1)) = 0$, and this implies (2) by local Poitou-Tate duality.

To prove (1) and (3), we use the following:

Claim. The two decompositions into one-dimensional O/ϖ-vector spaces

$$L_{\pi,1}^{\mathrm{Frob}_q=1} = (L^\circ_{\pi,1})^{\mathrm{Frob}_q=1} \oplus (L^\heartsuit_{\pi,1})^{\mathrm{Frob}_q=1} = (\mathrm{ad}^0 M_{0,1})^{\mathrm{Frob}_q=1} \oplus (\mathrm{ad}^0 M_{1,1})^{\mathrm{Frob}_q=1}$$

are both orthogonal with respect to the Killing form, and not the same.

Before proving the claim, we show it implies (1) and (3). Indeed, by (2), we can rewrite (1) as asserting the surjectivity of

$$H^1_{\mathrm{unr}}(\mathbb{Q}_q, \mathrm{ad}^0 M_{1,n}) = H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,n}) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,n}) \to H^1_{\mathrm{unr}}(\mathbb{Q}_q, L^\circ_{\pi,n}).$$

Since Frob_q acts on L_π with eigenvalues that are all either 1 or not congruent to 1 modulo ϖ, this is equivalent to the surjectivity of

$$(\mathrm{ad}^0 M_{1,1})^{\mathrm{Frob}_q=1} \to (L^\circ_{\pi,1})^{\mathrm{Frob}_q=1},$$

which follows from the claim.

For (3), we can take $n = 1$. Since Frob_q acts with distinct eigenvalues $\{1, q, q^{-1}\}$ on $\mathrm{ad}^0 M_{0,1}$, and the eigenvalues q and q^{-1} do not appear in $L^{\circ}_{\pi,1} = \mathrm{ad}^0 \overline{T}_{\pi_0}$, it again suffices to consider the $\mathrm{Frob}_q = 1$ eigenspaces. In particular, it suffices to show that $(L^{\heartsuit}_{\pi,1})^{\mathrm{Frob}_q=1}$ is not contained in the kernel of the map $L_{\pi,1}^{\mathrm{Frob}_q=1} \to (\mathrm{ad}^0 M_{0,1})^{\mathrm{Frob}_q=1}$, which again follows from the claim.

Now we return to the proof of the claim. In our basis of T_π, $(\mathrm{ad}^0 M_{1,1})^{\mathrm{Frob}_q=1}$ consists of the matrices $\begin{pmatrix} 0 & & & \\ & x & & \\ & & -x & \\ & & & 0 \end{pmatrix}$, and $(\mathrm{ad}^0 M_{0,1})^{\mathrm{Frob}_q=1}$ consists of the matrices $\begin{pmatrix} x & & & \\ & 0 & & \\ & & 0 & \\ & & & x \end{pmatrix}$. This decomposition is plainly orthogonal for the Killing form.

Meanwhile, $(L^{\circ}_{\pi,1})^{\mathrm{Frob}_q=1} = (\mathrm{ad}^0 \overline{T}_{\pi_0})^{\mathrm{Frob}_q=1}$ consists of the matrices of the form

$$\begin{pmatrix} 3y & & & \\ & y & & \\ & & -y & \\ & & & -3y \end{pmatrix}.$$

The decomposition $L^{\circ}_{\pi} \oplus L^{\heartsuit}_{\pi}$ is necessarily orthogonal for the Killing form because the form is Galois-invariant, and the claim follows.

□

Lemma 12.2.5. *Suppose π is not endoscopic and satisfies Hypothesis (★). Then:*

(1) The action of $G_{\mathbb{Q}}$ on T_π contains a scalar of infinite order.
(2) The projective image of ρ_π is a compact p-adic Lie group with semisimple Lie algebra.

Proof. If $\mathrm{BC}(\pi)$ is not an automorphic induction, then the lemma follows from Corollary C.2.6 combined with Lemma C.2.2(4,5). If $\mathrm{BC}(\pi)$ is an automorphic induction of the kind in Hypothesis (★), it follows from Corollary C.3.7. □

Lemma 12.2.6. *Suppose π satisfies Hypothesis (★). Then:*

(1) If π is not endoscopic, there exists a constant $C \geq 0$ such that

$$\varpi^C H^1(\mathbb{Q}(\rho_\pi)/\mathbb{Q}, L_{\pi,n}) = \varpi^C H^1(\mathbb{Q}(\rho_\pi)/\mathbb{Q}, L_{\pi,n}(1)) = 0$$

for all $n \geq 1$.

(2) If π is endoscopic, then for $j = 1$ or 2 such that π_j is non-CM, there exists a constant $C \geq 0$ such that

$$\varpi^C H^1(\mathbb{Q}(\rho_\pi)/\mathbb{Q}, L_{\pi_j,n}) = \varpi^C H^1(\mathbb{Q}(\rho_\pi)/\mathbb{Q}, L_{\pi_j,n}(1)) = 0$$

for all $n \geq 1$.

Proof. We start with the non-endoscopic case. For $i = 0$ or 1, consider the inflation-restriction exact sequence:

$$\begin{aligned} 0 \to H^1(\mathbb{Q}(\mathrm{ad}^0 \rho_\pi)/\mathbb{Q}, L_{\pi,n}(i)^{G_{\mathbb{Q}(\mathrm{ad}^0 \rho_\pi)}}) \to H^1(\mathbb{Q}(\rho_\pi)/\mathbb{Q}, L_{\pi,n}(i)) \to \\ \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\mathrm{ad}^0 \rho_\pi)), L_{\pi,n}(i)). \end{aligned} \tag{12.1}$$

We claim the third term vanishes. Indeed, the $G_{\mathbb{Q}}$ action on $\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\mathrm{ad}^0 \rho_\pi))$ by conjugation is trivial, so the third term is

$$\mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\mathrm{ad}^0 \rho_\pi)), L_{\pi,n}(i)^{G_{\mathbb{Q}}}) = 0$$

because the absolute irreducibility of $\overline{T}_\pi$ implies $\overline{L}_\pi(i)^{G_\mathbb{Q}} = 0$. If $i = 1$, the first term is uniformly bounded in n by Lemma 12.2.5(1), because any $g \in G_\mathbb{Q}$ that acts as a scalar z on T_π lies in $G_{\mathbb{Q}(\mathrm{ad}^0\,\rho_\pi)}$ and acts by z^2 on $L_\pi(1)$, so the lemma is proved in this case.

For the case $i = 0$, we have $L_{\pi,n}^{G_{\mathbb{Q}(\mathrm{ad}^0\,\rho_\pi)}} = L_{\pi,n}$. By Lemma 12.2.5(2) combined with [29, Lemma B.1], the first term in (12.1) is uniformly bounded in n, which proves the lemma when π is non-endoscopic.

Now consider the endoscopic case. Using Theorem C.3.2 to replace Lemma 12.2.5, the same argument as above shows

$$\varpi^C H^1(\mathbb{Q}(\rho_{\pi_j})/\mathbb{Q}, L_{\pi_j,n}) = \varpi^C H^1(\mathbb{Q}(\rho_{\pi_j})/\mathbb{Q}, L_{\pi_j,n}(1)) = 0$$

for some constant $C \geq 0$. By inflation-restriction, it suffices to show

$$\mathrm{Hom}_{G_\mathbb{Q}}(\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\rho_{\pi_j})), L_{\pi_j,n}(i))$$

is uniformly bounded in n for $i = 0, 1$. By the same argument as for the claim in Lemma 9.2.2, any Galois-invariant group homomorphism $\mathrm{Gal}(\mathbb{Q}(\rho_\pi)/\mathbb{Q}(\rho_{\pi_j})) \to L_{\pi_j,n}(i)$ lies in the proper subspace on which complex conjugation acts by -1; since $L_{\pi_j} \otimes \mathbb{Q}_p$ is absolutely irreducible by Theorem C.3.2 again, this suffices by [69, Lemma 2.3.3]. ☐

Lemma 12.2.7. *Let $L = L_1 \oplus L_2$ be a free O-module of finite rank with $G_\mathbb{Q}$ action, where $L_i \otimes \mathbb{Q}_p$ are absolutely irreducible and distinct. Then there is a constant $C \geq 0$ with the following property: for any $G_\mathbb{Q}$-stable O-submodule $H \subset L/\varpi^n L$, we have*

$$H \supset \varpi^C\,\mathrm{pr}_1(H) \oplus \varpi^C\,\mathrm{pr}_2(H).$$

Here $\mathrm{pr}_i : L/\varpi^n L \to L_i/\varpi^n L_i$ is the natural projection.

Proof. Write $L_{i,n} = L_i/\varpi^n$ and note that we have isomorphisms

$$\frac{\mathrm{pr}_1(H)}{H \cap L_{1,n}} \xleftarrow{\sim} \frac{H}{H \cap L_{2,n} \oplus H \cap L_{1,n}} \xrightarrow{\sim} \frac{\mathrm{pr}_2(H)}{H \cap L_{2,n}}.$$

In particular, it is enough to show that any isomorphic $O[G_\mathbb{Q}]$-module subquotients of L_1 and L_2 are ϖ^C-torsion for some universal constant C. Suppose on the contrary that for all integers $m \geq 0$, there exist submodules $B_i^m \subset A_i^m \subset L_i$ with $A_1^m/B_1^m \cong A_2^m/B_2^m$ not ϖ^m-torsion. Rescaling, we may assume without loss of generality that A_i^m has nonzero image in ϖL_i, so by [69, Lemma 2.3.3] we know $\varpi^{C_0} L_i \subset A_i^m$ for some constant C_0 depending on L_1 and L_2.

For each m, let $N_i(m)$ be the maximal integer such that

$$B_i^m \subset \varpi^{N_i(m)} L,$$

which implies

$$\varpi^{N_i(m)+C_0} L \subset B_i^m \subset \varpi^{N_i(m)} L. \tag{12.2}$$

In particular, for A_i^m/B_i^m not to be ϖ^m-torsion, we must have

$$N_i(m) + C_0 > m \tag{12.3}$$

for all m. Now consider the chain of maps:

(12.4)

$$\varpi^{C_0} L_1/\varpi^{N_1(m)+C_0} L_1 \twoheadrightarrow \varpi^{C_0} L_1/(B_1^m \cap \varpi^{C_0} L_1) \hookrightarrow A_1^m/B_1^m \xrightarrow{\sim} A_2^m/B_2^m \hookrightarrow L_2/B_2^m \twoheadrightarrow L_2/\varpi^{N_2(m)} L_2.$$

The two injections in the diagram have cokernel annihilated by ϖ^{C_0}, so in particular the composite has cokernel annihilated by ϖ^{2C_0}. By (12.3), after reindexing, we may assume without loss of generality that $N_i(m)$ is increasing in m. Then by compactness of $\mathrm{Hom}_O(L_1, L_2)$, up to passing to a subsequence, the maps in (12.4) fit together and give in the inverse limit a Galois-equivariant map

$$\varpi^{C_0} L_1 \to L_2$$

with cokernel annihilated by ϖ^{2C_0}. This is absurd because there are no nontrivial maps $L_1 \otimes \mathbb{Q}_p \to L_2 \otimes \mathbb{Q}_p$, so we have a contradiction and the lemma is proved. $\square$

Lemma 12.2.8. *Suppose π is not endoscopic and satisfies Hypothesis ($\bigstar$), and admissible primes exist for ρ_π. There is a constant $C \geq 0$ with the following property: suppose given integers n, m and cocycles $c \in H^1(\mathbb{Q}, T_{\pi,n})$, $\varphi \in H^1(\mathbb{Q}, L_{\pi,m})$, $\psi \in H^1(\mathbb{Q}, L_{\pi,m}(1))$. Let $(\varphi^\heartsuit, \varphi^\circ)$ be the decomposition of φ with respect to*

$$H^1(\mathbb{Q}, L_{\pi,m}) = H^1(\mathbb{Q}, L^\heartsuit_{\pi,m}) \oplus H^1(\mathbb{Q}, L^\circ_{\pi,m}),$$

and likewise for ψ. Then for any $N \geq \max\{n, m\}$, there are infinitely many N-admissible primes q such that all of the cocycles are unramified at q and:

- $\mathrm{ord}_\varpi \operatorname{loc}_q c \geq \mathrm{ord}_\varpi c - C$.
- *We have*
$$\mathrm{ord}_\varpi \left(\operatorname{Res}_q \varphi, \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m})}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m}) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,m})} \right) \geq \mathrm{ord}_\varpi \varphi^\heartsuit - C.$$
- *We have*
$$\mathrm{ord}_\varpi \left(\operatorname{Res}_q \psi, \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m}(1))}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m}(1)) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,m}(1))} \right) \geq \mathrm{ord}_\varpi \psi^\heartsuit - C.$$

Proof. By Lemma 12.2.6(1), Corollary C.2.8, and inflation-restriction, the restrictions of c, φ, and ψ correspond to $G_\mathbb{Q}$-invariant homomorphisms

$$\begin{aligned} &\operatorname{Res}(c) : G_{\mathbb{Q}(T_{\pi,N})} \to T_{\pi,n}, \\ &\operatorname{Res}(\varphi) = \operatorname{Res}(\varphi^\heartsuit) \oplus \operatorname{Res}(\varphi^\circ) : G_{\mathbb{Q}(T_{\pi,N})} \to L_{\pi,m}, \\ &\operatorname{Res}(\psi) = \operatorname{Res}(\psi^\heartsuit) \oplus \operatorname{Res}(\psi^\circ) : G_{\mathbb{Q}(T_{\pi,N})} \to L_{\pi,m}(1) \end{aligned}$$

satisfying

(12.5)

$$\mathrm{ord}_\varpi \operatorname{Res}(c) \geq \mathrm{ord}_\varpi(c) - C_0, \quad \mathrm{ord}_\varpi \operatorname{Res}(\varphi^?) \geq \mathrm{ord}_\varpi(\varphi^?) - C_0, \quad \mathrm{ord}_\varpi \operatorname{Res}(\psi^?) \geq \mathrm{ord}_\varpi(\psi^?) - C_0$$

where $? = \heartsuit$ or $\circ$ and $C_0 \geq 0$ is a constant. We combine these homomorphisms into a map

$$H : G_{\mathbb{Q}(T_{\pi,N})} \to T_{\pi,n} \oplus L_{\pi,m} \oplus L_{\pi,m}(1).$$

Let pr_1, pr_2, pr_3 be the projections onto each of the three factors.

For any $g \in G_\mathbb{Q}$ that acts on T_π as a scalar z, we have

$$\begin{aligned} \operatorname{im}(H) &\supset (g-1)(g-z^2)\operatorname{im}(H) + (g-z)(g-z^2)\operatorname{im}(H) + (g-1)(g-z)\operatorname{im}(H) \\ &= (z-1)(z^2-1)\,\mathrm{pr}_1 \operatorname{im}(H) \oplus (z-1)(z^2-1)\,\mathrm{pr}_2 \operatorname{im}(H) \oplus (z^2-1)(z-1)\,\mathrm{pr}_3 \operatorname{im}(H) \\ &= (z-1)(z^2-1)\,(\operatorname{im}\operatorname{Res}(c) \oplus \operatorname{im}\operatorname{Res}(\varphi) \oplus \operatorname{im}\operatorname{Res}(\psi).) \end{aligned}$$

By Lemma 12.2.7 combined with Lemma 12.2.5(1), this also implies

$$\operatorname{im}(H) \supset \varpi^{C_1} \left(\operatorname{im}\operatorname{Res}(c) \oplus \operatorname{im}\operatorname{Res}(\varphi^\heartsuit) \oplus \operatorname{im}\operatorname{Res}(\varphi^\circ) \oplus \operatorname{im}\operatorname{Res}(\psi^\heartsuit) \oplus \operatorname{im}\operatorname{Res}(\psi^\circ) \right)$$

for some constant $C_1 \geq 0$ indepedent of n amd m. In particular, by (12.5) combined with [69, Lemma 2.3.3], there exists a constant $C \geq 0$ independent of n and m, such that

$$\mathcal{O} \cdot \operatorname{im}(H) \supset \varpi^{n - \mathrm{ord}_\varpi(c) + C} T_{\pi,n} \oplus \varpi^{m - \mathrm{ord}_\varpi(\varphi^\heartsuit) + C} L^\heartsuit_{\pi,m} \oplus \varpi^{m - \mathrm{ord}_\varpi(\psi^\heartsuit) + C} L^\heartsuit_{\pi,m}(1). \tag{12.6}$$

Now fix an admissible element $g \in G_\mathbb{Q}$, which is possible by Lemma 4.2.3. By repeatedly raising g to pth powers and taking the limit, we may assume without loss of generality that $\rho_\pi(g)$ has finite order coprime to

p. Choose a basis $\{e_1, e_2, e_3, e_4\}$ for T_π with respect to which we have

$$\rho_\pi(g) = \begin{pmatrix} \chi_p^{\mathrm{cyc}}(g) & & & \\ & 1 & & \\ & & * & * \\ & & * & * \end{pmatrix},$$

and set $M_0 = \mathrm{Span}_O \{e_1, e_2\}$, $M_1 = \mathrm{Span}_O \{e_3, e_4\}$. Note that, by Lemma 12.2.4(3), $L_\pi^\heartsuit$ surjects onto the direct summand $\mathrm{ad}^0 M_0$ of L_π.

Hence, using (12.6), we may choose $h \in G_{\mathbb{Q}(\rho_\pi)}$ as follows:

(1) The e_2-component of $c(h)$ has order at least $\mathrm{ord}_\varpi(c) - C$.

(2) The component of $\varphi(h)$ in the g-invariant line $\left\{ \begin{pmatrix} x & & & \\ & -x & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \right\} \subset L_\pi$ has order at least $\mathrm{ord}_\varpi(\varphi^\heartsuit) - C$.

(3) The component of $\psi(h)$ in the g-invariant line $\left\{ \begin{pmatrix} 0 & 0 & & \\ * & 0 & & \\ & & & \\ & & & \end{pmatrix} \right\} \subset L_\pi(1)$ has order at least $\mathrm{ord}_\varpi(\psi^\heartsuit) - C$.

In particular, because g has finite order coprime to p, the same is also true for the corresponding components of $c(gh)$, $\varphi(gh)$, and $\psi(gh)$, with respect to any choice of cocycle representatives.

Now suppose $q \notin S \cup \{p\}$ has Frobenius conjugate to g in $\mathrm{Gal}(\mathbb{Q}(T_{\pi,N}, c, \varphi, \psi))$. To show that q satisfies the conclusion of the lemma, it suffices to observe that the O-modules

$$\frac{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m})}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m}) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,m})}, \quad \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m}(1))}{H^1_{\mathrm{unr}}(\mathbb{Q}_q, L_{\pi,m}(1)) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_q, L_{\pi,m}(1))},$$

which are both free of rank one over O/ϖ^m, are generated by the cocyles

$$\mathrm{Frob}_q \mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \mathrm{Frob}_q \mapsto \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & & \\ & & & \end{pmatrix},$$

respectively. □

The Selmer groups in the next lemma (and the rest of this section) are the ones from Definition B.2.5.

Lemma 12.2.9. *Suppose π is non-endoscopic and satisfies Hypothesis (★), and let C be the constant of Lemma 12.2.8. If*

$$\overline{\mathrm{Sel}}_{\mathcal{F}}(\mathbb{Q}, L_{\pi,m}) = 0$$

for some $m \geq \max\{1, C\}$, and q_1 is an N-admissible prime for some $N \geq 5m$, then either:

(1) $\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}) = 0$; or:

(2) For any cocycle $c \in H^1(\mathbb{Q}, T_{\pi,n})$ with $n \leq N$, and any $M \geq N$, there exist infinitely many M-admissible primes q_2 such that

$$\overline{\mathrm{Sel}}_{\mathcal{F}(q_1 q_2)}(\mathbb{Q}, L_{\pi,5m}) = 0$$

and

$$\mathrm{ord}_\varpi \, \mathrm{loc}_{q_2} c \geq \mathrm{ord}_\varpi c - C.$$

Proof. Without loss of generality, assume (1) does not hold, and let $\varphi \in \mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m})$ be an element whose image in $\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m})$ is nonzero; it follows from Lemma B.2.7(1,2) that $\mathrm{ord}_\varpi \varphi = 5m$. Similarly, by Lemma B.2.8, there exists an element $\psi \in \mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}(1))$ whose image in $\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}(1))$ is nonzero, and we have $\mathrm{ord}_\varpi \psi = 5m$. Write

$$\varphi = \varphi^\heartsuit \oplus \varphi^\circ, \quad \psi = \psi^\heartsuit \oplus \psi^\circ$$

as in the statement of Lemma 12.2.8.

Claim 1. We have $\mathrm{ord}_\varpi \varphi^\heartsuit \geq 4m$ and $\mathrm{ord}_\varpi \psi^\heartsuit \geq 4m$.

Proof of claim 1. Suppose that $\mathrm{ord}_\varpi \varphi^\heartsuit < 4m$; then by Lemma B.2.7(1,2), the images of φ and φ° coincide in $H^1(\mathbb{Q}, L_{\pi,m})$. However, by Lemma 12.2.4(2) we have

$$H^1_{\mathrm{unr}}(\mathbb{Q}_{q_1}, L^\circ_{\pi,m}) = H^1(\mathbb{Q}_{q_1}, L^\circ_{\pi,m}),$$

so it follows that the image of φ lies in $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, L_{\pi,m})$; this contradicts the assumption $\overline{\mathrm{Sel}}_{\mathcal{F}}(\mathbb{Q}, L_{\pi,m}) = 0$.

Similarly, if $\mathrm{ord}_\varpi \psi^\heartsuit < 4m$, then ψ and ψ° have the same image in $H^1(\mathbb{Q}, L_{\pi,m}(1))$. However, the local Tate dual of Lemma 12.2.4(1) shows that $H^1(\mathbb{Q}_{q_1}, L^\circ_{\pi,m}(1)) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_{q_1}, L_{\pi,m}(1)) = 0$, so then the image of ψ lies in $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, L_{\pi,m}(1))$, and this contradicts Lemma B.2.8. ☐

By Claim 1 combined with Lemma 12.2.8, there are infinitely many M-admissible primes q_2 such that:

- $\mathrm{ord}_\varpi \mathrm{loc}_{q_2} c \geq \mathrm{ord}_\varpi c - C$.
- We have
$$\mathrm{ord}_\varpi \left(\mathrm{Res}_{q_2} \varphi, \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m})}{H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m}) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m})} \right) \geq \mathrm{ord}_\varpi \varphi^\heartsuit - C \geq 3m.$$
- We have
$$\mathrm{ord}_\varpi \left(\mathrm{Res}_{q_2} \psi, \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1))}{H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1)) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1))} \right) \geq \mathrm{ord}_\varpi \psi^\heartsuit - C \geq 3m.$$

Put

$$\mathrm{Sel}_{\mathcal{F}_{q_2}(q_1)}(\mathbb{Q}, L_{\pi,5m}) = \mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}) \cap \mathrm{Sel}_{\mathcal{F}(q_1 q_2)}(\mathbb{Q}, L_{\pi,5m})$$

and

$$\mathrm{Sel}_{\mathcal{F}_{q_1}}(\mathbb{Q}, L_{\pi,5m}) = \mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, L_{\pi,5m}) \cap \mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}).$$

Our next claim is:

Claim 2. We have

$$\varpi^{3m-1} \mathrm{Sel}_{\mathcal{F}_{q_2}(q_1)}(\mathbb{Q}, L_{\pi,5m}) = 0.$$

Proof of claim 2. First, we have the exact sequence

(12.7)
$$0 \to \mathrm{Sel}_{\mathcal{F}_{q_1}}(\mathbb{Q}, L_{\pi,5m}) \to \mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}) \to \frac{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_1}, L_{\pi,5m})}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_1}, L_{\pi,5m}) \cap H^1_{\mathrm{unr}}(\mathbb{Q}_{q_1}, L_{\pi,5m})} \simeq O/\varpi^{5m},$$

where the final isomorphism is by Lemma 8.2.11. Because $\mathrm{Sel}_{\mathcal{F}_{q_1}}(\mathbb{Q}, L_{\pi,5m})$ is ϖ^{m-1}-torsion by Lemma B.2.7(3), but $\varpi^{5m-1} \mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}) \neq 0$ by assumption, from (12.7) we have an isomorphism of O-modules

$$\mathrm{Sel}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m}) \simeq O/\varpi^{5m} \oplus T \tag{12.8}$$

where T is ϖ^{m-1}-torsion. In particular, for any $x \in \mathrm{Sel}_{\mathcal{F}_{q_2}(q_1)}(\mathbb{Q}, L_{\pi,5m})$, $\varpi^{m-1}x = \varpi^a\varphi$ for some $a \geq 0$. We conclude

$$0 = \mathrm{ord}_{\varpi}\left(\mathrm{Res}_{q_2}(\varpi^{m-1}x), \frac{H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m})}{H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m}) \cap H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m})}\right) \geq 3m - a, \tag{12.9}$$

so $a \geq 3m$, hence $\varpi^{3m-1}x = 0$. ☐

Now we are ready to complete the proof of the lemma. We have the exact sequence

$$0 \to \mathrm{Sel}_{\mathcal{F}_{q_2}(q_1)}(\mathbb{Q}, L_{\pi,5m}) \to \mathrm{Sel}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, L_{\pi,5m}) \to \frac{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m})}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}) \cap H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m})} \simeq O/\varpi^{5m}. \tag{12.10}$$

On the other hand, for any $\varphi' \in \mathrm{Sel}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, L_{\pi,5m})$, we can compute the global Tate pairing

$$0 = \sum_v \langle \varphi', \psi \rangle_v = \langle \varphi', \psi \rangle_{q_2}$$

by the definition of the local conditions for $\mathcal{F}(q_1q_2)$ and $\mathcal{F}(q_1)$. Now, the induced local Tate pairing

$$O/\varpi^{5m} \times O/\varpi^{5m} \simeq$$
$$\frac{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m})}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}) \cap H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m})} \times \frac{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1)) + H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1))}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1))} \to O/\varpi^{5m}$$

is perfect, so we conclude

$$\begin{aligned}\mathrm{ord}_{\varpi}&\left(\mathrm{Res}_{q_2}\varphi', \frac{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m})}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}) \cap H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m})} \simeq O/\varpi^{5m}\right) \\ &\leq 5m - \mathrm{ord}_{\varpi}\left(\mathrm{Res}_{q_2}\psi, \frac{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1)) + H^1_{\mathrm{unr}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1))}{H^1_{\mathrm{ord}}(\mathbb{Q}_{q_2}, L_{\pi,5m}(1))}\right) \leq 2m.\end{aligned}$$

In particular, the image of the final map in (12.10) is ϖ^{2m}-torsion, so we have

$$\varpi^{5m-1}\,\mathrm{Sel}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, L_{\pi,5m}) = 0$$

by Claim 2. Hence by Lemma B.2.7(1,2), $\overline{\mathrm{Sel}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, L_{\pi,5m})} = 0$, as desired. ☐

We also have an endoscopic analogue:

Lemma 12.2.10. *Suppose π is endoscopic associated to a pair (π_1, π_2), and let $j = 1$ or 2. Suppose π_j is non-CM. Then there exists a constant C with the following property: if*

$$\overline{\mathrm{Sel}}_{\mathcal{F}}(\mathbb{Q}, L_{\pi_j,m}) = 0$$

for some $m \geq \max\{1, C\}$, and q_1 is an N-admissible prime for some $N \geq 5m$ which is BD-admissible for π_j, then either:

(1) $\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi_j,5m}) = 0$; or:

(2) For any cocycle $c \in H^1(\mathbb{Q}, T_{\pi_j,n})$ with $n \leq N$, and for any $M \geq N$, there exist infinitely many M-admissible primes q_2, BD-admissible for π_j, such that

$$\overline{\mathrm{Sel}}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, L_{\pi_j,5m}) = 0$$

and

$$\mathrm{ord}_{\varpi}\,\mathrm{loc}_{q_2} c \geq \mathrm{ord}_{\varpi} c - C.$$

Proof. The same argument used to prove Lemma 12.2.9 applies formally, taking $L^{\circ}_{\pi} := 0$ and $L^{\heartsuit}_{\pi} := L_{\pi_j}$. When proving the appropriate analogue of Lemma 12.2.8, one uses the claim in the proof of Lemma 9.2.2 in place of Corollary C.2.8, Lemma 12.2.6(2) in place of Lemma 12.2.6(1), and Theorem C.3.2 in place of Lemma 12.2.5(1). ☐

12.3. **Proof of the main results.**

Theorem 12.3.1. *Suppose π is non-endoscopic and satisfies Hypothesis ($\bigstar$), and let $\mathfrak{p}$ be a prime of E_0 satisfying Assumption 4.1.1, such that admissible primes exist for $\rho_{\pi,\mathfrak{p}}$. Then for all $D \geq 1$ squarefree with $\sigma(D)$ even,*

$$\kappa^D(1)_{\mathfrak{p}} \neq 0 \implies \dim H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = 1.$$

Proof. Let N be a large integer to be specified later, and let $M \geq N$ be the number from Lemma 1.6.3(3) applied to T_π and $n = N$. By Theorem 11.2.6, there exists a constant $C_0 \geq 0$ independent of N and an M-admissible prime q_1 such that

$$\operatorname{ord}_\varpi \operatorname{loc}_{q_1} \kappa^D_{n(q_1)}(1) \geq n(q_1) - C_0$$

and

$$\operatorname{ord}_\varpi \lambda^D_{n(q_1)}(q_1) \geq n(q_1) - C_0.$$

Because $\kappa^D(1) \subset H^1(\mathbb{Q}, T_\pi)$ is a free O-module by Lemma 4.1.6(1), we may fix a class $\kappa^D(1)_0 \in \kappa^D(1)$, with image $\kappa^D_m(1)_0$ in $H^1(\mathbb{Q}, T_{\pi,m})$ for all $m \geq 1$, such that

$$\operatorname{ord}_\varpi \operatorname{loc}_{q_1} \kappa^D_m(1)_0 \geq m - C_0 \tag{12.11}$$

for all $m \leq n(q_1)$.

Claim. Suppose $\dim H^1_f(\mathbb{Q}, V_\pi) > 1$. Then there exists a class $c \in H^1_f(\mathbb{Q}, T_\pi)$, with images $c_m \in H^1_f(\mathbb{Q}, T_{\pi,m})$ for all $m \geq 1$, such that:

(1) $\operatorname{ord}_\varpi c_m = m$ for all m.
(2) $\operatorname{ord}_\varpi \operatorname{loc}_{q_1} c_m \leq C_0$ for all $m \leq n(q_1)$.

Proof of claim. By the assumption $\dim H^1_f(\mathbb{Q}, V_\pi) > 1$, we may choose $c \in H^1_f(\mathbb{Q}, T_\pi)$ such that

$$c + \alpha\kappa^D(1)_0 \notin \varpi H^1_f(\mathbb{Q}, T_\pi), \quad \forall \alpha \in O. \tag{12.12}$$

Adjusting c by an O-multiple of $\kappa^D(1)_0$ and using (12.11), we can ensure that (2) holds. By definition, we have an injection

$$\frac{H^1(\mathbb{Q}, T_\pi)}{H^1_f(\mathbb{Q}, T_\pi)} \hookrightarrow \prod_v \frac{H^1(\mathbb{Q}_v, V_\pi)}{H^1_f(\mathbb{Q}_v, V_\pi)},$$

hence the quotient $H^1(\mathbb{Q}, T_\pi)/H^1_f(\mathbb{Q}, T_\pi)$ is O-torsion-free, and in particular $c \notin \varpi H^1(\mathbb{Q}, T_\pi)$ by (12.12). Then (1) holds as well by Lemma 4.1.6(2).

□

By Theorem 9.1.3 and Lemma B.3.6, there exists a constant $m_0 \geq 1$ such that $\overline{\operatorname{Sel}}_{\mathcal{F}}(\mathbb{Q}, \operatorname{ad}^0 \rho_{m_0}) = 0$. Without loss of generality, we assume $N \geq 10m_0$ and $m_0 \geq \max\{1, C\}$, for the constant C of Lemma 12.2.8. Now consider the following two cases:

(1) If $\overline{\operatorname{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,10m_0-1}) = 0$, then we choose (by Lemma 9.1.2 and (1) of the claim) an $n(q_1)$-admissible prime q_2 such that $\operatorname{ord}_\varpi \operatorname{loc}_{q_2} c_{n(q_1)} \geq n(q_1) - C_1$ for a constant $C_1 \geq 0$ independent of N, q_1, and q_2.
(2) If $\overline{\operatorname{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,10m_0-1}) \neq 0$, then *a fortiori* we have $\overline{\operatorname{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, L_{\pi,5m_0}) \neq 0$. We choose (by Lemma 12.2.9) an $(n(q_1) + 5m_0)$-admissible prime $q_2 \neq q_1$ such that $\overline{\operatorname{Sel}}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, L_{\pi,5m_0}) = 0$ and $\operatorname{ord}_\varpi \operatorname{loc}_{q_2} c_{n(q_1)} \geq \operatorname{ord}_\varpi n(q_1) - C_1$ for a constant $C_1 \geq 0$ independent of N, q_1, and q_2.

By Theorem 8.5.1 combined with Corollary B.4.3, in either case we can conclude – as long as N is sufficiently large in a manner depending only on π, $\mathfrak{p}$, and m_0 – that

$$\partial_{q_2}\kappa^D_{n(q_1)}(q_1q_2) \supset \lambda^D_{n(q_1)}(q_1) \cdot (\varpi^{C_2})$$

for a constant C_2 that is independent of N, q_1, and q_2. Hence we may choose an element $\kappa^D_{n(q_1)}(q_1q_2)_0 \in \kappa^D_{n(q_1)}(q_1q_2)$, with images $\kappa^D_m(q_1q_2)_0 \in H^1(\mathbb{Q}, T_{\pi,m})$ for all $m \leq n(q_1)$, such that

$$\mathrm{ord}_\varpi \partial_{q_2}\kappa^D_m(q_1q_2)_0 \geq m - C_0 - C_2. \tag{12.13}$$

We now compute the Tate pairing

$$\langle c_N, \kappa^D_N(q_1q_2)_0\rangle = \sum_v \langle c_N, \kappa^D_N(q_1q_2)_0\rangle_v = 0 \in O/\varpi^N. \tag{12.14}$$

Arguing as in the proof of Theorem 9.1.4 and using that $M \leq n(q_1)$, there is a constant $C_3 \geq 0$ so that

$$\mathrm{ord}_\varpi \langle c_N, \kappa^D_N(q_1q_2)_0\rangle_v \leq C_3, \quad \forall v \notin \{q_1, q_2\}.$$

By (2) of the claim, we also have

$$\mathrm{ord}_\varpi \langle c_N, \kappa^D_N(q_1q_2)_0\rangle_{q_1} \leq C_0.$$

In particular, (12.14) implies

$$\mathrm{ord}_\varpi \langle c_N, \kappa^D_N(q_1q_2)_0\rangle_{q_2} \leq \max\{C_0, C_3\}.$$

However, by Proposition 4.2.8, (12.13) combined with the choice of q_2 implies

$$\mathrm{ord}_\varpi \langle c_N, \kappa^D_N(q_1q_2)_0\rangle_{q_2} \geq N - C_0 - C_1 - C_2.$$

This is a contradiction if we choose $N > C_0 + C_1 + C_2 + \max\{C_0, C_3\}$, and the proof of the theorem is complete. ☐

Theorem 12.3.2. *Suppose π is endoscopic, associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ (in any order), and $\mathfrak{p}$ is a prime of E_0 satisfying Assumption 4.1.1, such that $H^1_f(\mathbb{Q}, V_{\pi_1,\mathfrak{p}} \otimes V_{\pi_2,\mathfrak{p}}(-1)) = 0$. Assume as well that $\kappa^D(1)^{(1)}_\mathfrak{p} \neq 0$ for some squarefree D with $\sigma(D)$ even. Then the following hold:*

1. *If π_1 is non-CM and there exist admissible primes which are BD-admissible for $\rho_{\pi_1,\mathfrak{p}}$, then*
$$\dim H^1_f(\mathbb{Q}, V_{\pi_1,\mathfrak{p}}) = 1.$$
2. *If for each $j = 1, 2$, there exist admissible primes which are BD-admissible for $\rho_{\pi_j,\mathfrak{p}}$, then*
$$\dim H^1_f(\mathbb{Q}, V_{\pi_2,\mathfrak{p}}) = 0.$$

In particular, if for each $j = 1, 2$, π_j is non-CM and there exist admissible primes which are BD-admissible for $\rho_{\pi_j,\mathfrak{p}}$, then

$$\kappa^D(1)_\mathfrak{p} \neq 0 \implies \dim H^1_f(\mathbb{Q}, V_{\pi,\mathfrak{p}}) = 1.$$

Remark 12.3.3. The existence of admissible primes which are BD-admissible for each $\rho_{\pi_j,\mathfrak{p}}$ is considered in Proposition C.4.12.

Proof. Let N be a large integer to be specified later, and let $M \geq N$ be the number from Lemma 1.6.3(3) for T_π and $n = N$. By Theorem 11.2.7, there exists a constant $C_0 \geq 0$ independent of N and an M-admissible prime q_1, BD-admissible for ρ_{π_1}, such that

$$\mathrm{ord}_\varpi \, \mathrm{loc}_{q_1} \kappa^D_{n(q_1)}(1)^{(1)} \geq n(q_1) - C_0$$

and

$$\mathrm{ord}_\varpi \lambda^D_{n(q_1)}(q_1) \geq n(q_1) - C_0.$$

As in Theorem 12.3.1, we fix a class $\kappa^D(1)_0 \in \kappa^D(1)^{(1)}$, with image $\kappa^D_m(1)_0 \in H^1(\mathbb{Q}, T_{\pi_1,m})$ for all $m \geq 1$, such that

$$\mathrm{ord}_\varpi \, \mathrm{loc}_{q_1} \kappa^D_m(1)_0 \geq m - C_0 \tag{12.15}$$

for all $m \leq n(q_1)$.

Now we suppose we are in case (1) of the theorem; assume for contradiction $\dim H^1_f(\mathbb{Q}, V_{\pi_1}) > 1$. By the same argument as for the claim in Theorem 12.3.1, we have a class $c \in H^1_f(\mathbb{Q}, T_{\pi_1})$, with images $c_m \in H^1_f(\mathbb{Q}, T_{\pi_1,m})$, such that:

1. $\mathrm{ord}_\varpi c_m = m$ for all $m \geq 1$.
2. $\mathrm{ord}\, \mathrm{loc}_{q_1} c_m \leq C_0$ for all $1 \leq m \leq n(q_1)$.

By Proposition 9.2.3 and Lemma B.3.6, there exists a constant $m_0 \geq 1$ such that

$$\overline{\mathrm{Sel}}_{\mathcal{F}}(\mathbb{Q}, \mathrm{ad}^0\, \rho_{\pi_1,m_0}) = 0.$$

Increasing N if necessary, we assume $N > 10m_0$. Now consider the following cases:

1. If $\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, \mathrm{ad}^0\, T_{\pi_1,10m_0-1}) = 0$, then choose (by Lemma 9.2.2) q_2 to be $n(q_1)$-admissible and BD-admissible for ρ_{π_1} such that
$$\mathrm{ord}_\varpi\, \mathrm{loc}_{q_2} c_{n(q_1)} \geq n(q_1) - C_1$$
for a constant $C_1 \geq 0$ independent of N, q_1, and q_2.
2. If $\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, \mathrm{ad}^0\, T_{\pi_1,10m_0-1}) \neq 0$, then *a fortiori* we have
$$\overline{\mathrm{Sel}}_{\mathcal{F}(q_1)}(\mathbb{Q}, \mathrm{ad}^0\, T_{\pi_1,5m_0}) \neq 0.$$
We choose, by Lemma 12.2.10, an $(n(q_1) + 5m_0)$-admissible prime $q_2 \neq q_1$, BD-admissible for π_1, such that
$$\overline{\mathrm{Sel}}_{\mathcal{F}(q_1q_2)}(\mathbb{Q}, \mathrm{ad}^0\, T_{\pi_1,5m_0}) = 0$$
and $\mathrm{ord}_\varpi\, \mathrm{loc}_{q_2} c_{n(q_1)} \geq n(q_1) - C_1$ for a constant $C_1 \geq 0$ independent of N, q_1, and q_2.

By Theorem 8.5.2 and Corollary B.4.3, in either case we can conclude

$$\partial_{q_2} \kappa^D_{n(q_1)}(q_1q_2) \supset \lambda^D_{n(q_1)}(q_1) \cdot (\varpi^{C_2})$$

for all N sufficiently large in a manner depending only on π, $\mathfrak{p}$, and m_0, and for a constant $C_2 \geq 0$ that is independent of N, q_1, and q_2. The remainder of the proof of (1) is now identical to Theorem 12.3.1.

Now we suppose we are in case (2), and assume for contradiction that there exists a non-torsion class $c \in H^1_f(\mathbb{Q}, T_{\pi_2})$ with images $c_m \in H^1(\mathbb{Q}, T_{\pi_2,m})$ for all $m \geq 1$. By the proof of the claim in Theorem 12.3.1, we may assume $\mathrm{ord}_\varpi c_m = m$ for all m. Then we choose (by Lemma 9.2.2) q_2 to be $n(q_1)$-admissible and BD-admissible for ρ_{π_2}, such that $\mathrm{ord}_\varpi\, \mathrm{loc}_{q_2} c_{n(q_1)} \geq n(q_1) - C_1$ for a constant $C_1 \geq 0$ independent of N, q_1, and q_2. By Proposition 9.2.3 and Lemma B.3.6, there exists a constant $m_0 \geq 1$ such that

$$\overline{\mathrm{Sel}}_{\mathcal{F}}(\mathbb{Q}, \mathrm{ad}^0\, \rho_{\pi_2,m_0}) = 0.$$

By Theorem 8.5.2(1) and Corollary B.4.3, we can conclude that

$$\partial_{q_2} \kappa^D_N(q_1q_2) \supset \lambda^D_N(q_1) \cdot (\varpi^{C_2})$$

for all N sufficiently large and for a constant $C_2 \geq 0$ that is independent of N, q_1, and q_2. The remainder of the proof of (2) now follows the proof of Theorem 12.3.1, using that $\mathrm{loc}_{q_1} c_N = 0$ because $H^1_f(\mathbb{Q}_{q_1}, T_{\pi_2,N}) = 0$. □

APPENDIX A. COHOMOLOGY OF GSpin_4 SHIMURA VARIETIES

A.1. **The auxiliary quaternionic Shimura variety.**

A.1.1. For this section, let V be a quadratic space over $\mathbb{Q}$ of signature $(2,2)$ and nontrivial discriminant character χ. Then by [50, Appendix A], $C^+(V) = B \otimes_{\mathbb{Q}} F =: B_F$, where B is an indefinite quaternion algebra over $\mathbb{Q}$ and $F/\mathbb{Q}$ is the real quadratic field associated to χ. We abbreviate $G \coloneqq \mathrm{GSpin}(V)$, $\widetilde{G} \coloneqq \mathrm{Res}_{F/\mathbb{Q}} B_F^\times$. Then we have an exact sequence of algebraic groups over $\mathbb{Q}$:

$$1 \to G \to \widetilde{G} \to (\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m)/\mathbb{G}_m \to 1. \tag{A.1}$$

Let $K = \prod K_\ell \subset G(\mathbb{A}_f)$ be a neat compact open subgroup, and let S be a nonempty set of primes such that K_ℓ is hyperspecial for $\ell \notin S$. Fix a neat compact open subgroup $\widetilde{K} = \prod \widetilde{K}_\ell \subset \widetilde{G}(\mathbb{A}_f)$ satisfying the following conditions:

1. For all $\ell \notin S$, $\widetilde{K}_\ell$ is hyperspecial.
2. $\widetilde{K} \cap \mathrm{GSpin}(V)(\mathbb{A}_f) = K$.
3. We have $\nu(\widetilde{K}) \cap O_{F,+}^\times = (O_F^\times \cap \widetilde{K})^2$, where $\nu : \widetilde{G} \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ is the norm character and $O_{F,+}^\times$ is the group of totally positive units of F.

Such a $\widetilde{K}$ exists because S is nonempty. Let $\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)$ be the Shimura variety for $\widetilde{G}$ at level $\widetilde{K}$.

Proposition A.1.2. *Under conditions (1) - (3) above, the natural map*

$$\mathrm{Sh}_K(V) \to \widetilde{\mathrm{Sh}}_{\widetilde{K}}(V) \tag{A.2}$$

is an open and closed embedding.

Proof. This follows from [68, Proposition 2.10, Remark 2.11]. ☐

A.1.3. *Hecke algebras.* For a prime $\ell \notin S$ and a ring R, let $\mathbb{T}_{\ell,R}$ (resp. $\widetilde{\mathbb{T}}_{\ell,R}$) denote the spherical Hecke algebra of K_ℓ-biinvariant (resp. $\widetilde{K}_\ell$-biinvariant) R-valued functions on $G(\mathbb{Q}_\ell)$ (resp. $\widetilde{G}(\mathbb{Q}_\ell)$). If $S' \supset S$ is a finite set of primes, then we set

$$\mathbb{T}_R^{S'} \coloneqq \otimes'_{\ell \notin S'} \mathbb{T}_{\ell,R}, \quad \widetilde{\mathbb{T}}_R^{S'} \coloneqq \otimes'_{\ell \notin S'} \widetilde{\mathbb{T}}_{\ell,R}.$$

When $R = \mathbb{Z}$ we drop it from the notation.

Proposition A.1.4. *Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. There is a decomposition of $\widetilde{\mathbb{T}}^S$-modules:*

$$H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) = \bigoplus_{\pi_f} H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)_{\pi_f} \otimes \iota^{-1}\pi_f^{\widetilde{K}} \oplus \bigoplus_{\chi} H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)_{\chi \circ \det} \otimes \iota^{-1}(\chi \circ \det)^{\widetilde{K}},$$

where π_f runs over finite parts of cuspidal, infinite-dimensional automorphic representations π of $B_F(\mathbb{A}_F)^\times$ with discrete series archimedean components of parallel weight 2, and χ runs over finite order characters of $F^\times \backslash \mathbb{A}_F^\times$. As $G_{\mathbb{Q}}$-representations, we have

$$H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)^{ss}_{\pi_f} = \otimes - \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_{\mathrm{JL}(\pi),\iota}$$

and

$$H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)^{ss}_{\chi \circ \det} = \iota^{-1} \operatorname{rec}(\chi|_{\mathbb{A}_{\mathbb{Q}}^\times})(-1) \oplus \iota^{-1} \operatorname{rec}(\chi|_{\mathbb{A}_{\mathbb{Q}}^\times}) \cdot \omega_{F/\mathbb{Q}}(-1),$$

where $\omega_{F/\mathbb{Q}}$ is the quadratic character of $G_{\mathbb{Q}}$ associated to F.

Proof. When B_F is split, this follows from the discussion in [112, §XI.2]. In the nonsplit case, the Hecke module decomposition is clear from Matsushima's formula, and the Galois actions follow from [58]. ☐

A.1.5. Let $\ell \notin S$ be a prime; then we may identify $B \otimes \mathbb{Q}_\ell \simeq M_2(\mathbb{Q}_\ell)$ in such a way that $\widetilde{K}_\ell = \mathrm{GL}_2(O_F \otimes \mathbb{Z}_\ell)$ and $G(\mathbb{Q}_\ell) \subset \widetilde{G}(\mathbb{Q}_\ell) \simeq \mathrm{GL}_2(F \otimes \mathbb{Q}_\ell)$ consists of those matrices having determinant in $\mathbb{Q}_\ell^\times$. Let $T(\mathbb{Q}_\ell)$ and $\widetilde{T}(\mathbb{Q}_\ell)$ be the standard diagonal tori in $G(\mathbb{Q}_\ell)$ and $\widetilde{G}(\mathbb{Q}_\ell)$, respectively, and let $B(\mathbb{Q}_\ell)$ and $\widetilde{B}(\mathbb{Q}_\ell)$ be the upper triangular Borel subgroups.

Proposition A.1.6. *Let* $\chi_0 : \widetilde{T}(\mathbb{Q}_\ell) \to \mathbb{C}^\times$ *be an unramified character. Then* $\left(\mathrm{Ind}_{\widetilde{B}(\mathbb{Q}_\ell)}^{\widetilde{G}(\mathbb{Q}_\ell)} \chi_0\right)|_{G(\mathbb{Q}_\ell)}$ *has a unique* K_ℓ*-spherical constituent, which is isomorphic to the unique spherical constituent of* $\mathrm{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}(\chi_0|_{T(\mathbb{Q}_\ell)})$.

Proof. This is clear from the observation that $\widetilde{B}(\mathbb{Q}_\ell) \cdot K_\ell = \widetilde{B}(\mathbb{Q}_\ell) \cdot \widetilde{K}_\ell = \widetilde{G}(\mathbb{Q}_\ell)$; indeed, $K_\ell \cdot (\widetilde{K}_\ell \cap \widetilde{T}(\mathbb{Q}_\ell)) = \widetilde{K}_\ell$ because $\nu(\widetilde{T}(\mathbb{Q}_\ell) \cap \widetilde{K}_\ell) = \nu(\widetilde{K}_\ell)$. □

It follows immediately that:

Corollary A.1.7. *Let* $\widetilde{\pi}$ *be an irreducible admissible representation of* $\widetilde{G}(\mathbb{Q}_\ell)$ *which is* $\widetilde{K}_\ell$*-spherical, and let* $\chi_{\widetilde{\pi}} : \widetilde{\mathbb{T}}_\ell \to \mathbb{C}$ *be the character giving the Hecke action on* $\widetilde{\pi}^{\widetilde{K}_\ell}$. *Then, viewing* $\widetilde{\pi}$ *as an admissible representation of* $G(\mathbb{Q}_\ell)$, $\mathbb{T}_\ell$ *stabilizes the one-dimensional space* $\widetilde{\pi}^{\widetilde{K}_\ell}$ *and acts on it via the composite of* $\chi_{\widetilde{\pi}}$ *with the homomorphism* $\mathbb{T}_\ell \to \widetilde{\mathbb{T}}_\ell$ *determined by the Satake transform and the map of dual groups* ${}^L\mathrm{Res}_{F/\mathbb{Q}} B_F^\times \twoheadrightarrow {}^L\mathrm{GSpin}(V)$.

A.2. **Hecke action on Tate classes for** GSpin_4**.** For an automorphic represention π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, let $\mathrm{BC}_{F/\mathbb{Q}}(\pi)$ denote the base change to $\mathrm{GL}_2(\mathbb{A}_F)$.

Lemma A.2.1. *Continue the notation of Proposition A.1.4.*

(1) If $H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1))^{G_\mathbb{Q}}_{\pi_f} \neq 0$, *then there exists an automorphic representation* π_0 *of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, *with* $\pi_{0,\infty}$ *discrete series of weight 2, and a finite-order character* χ *of* $F^\times\backslash\mathbb{A}_F^\times$, *such that* $\mathrm{JL}(\pi) = \mathrm{BC}_{F/\mathbb{Q}}(\pi_0) \otimes \chi$.
(2) If $H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1))^{G_\mathbb{Q}}_{\chi\circ\det} \neq 0$, *then* $\chi|_{\mathbb{A}_\mathbb{Q}^\times} = \omega_{F/\mathbb{Q}}$ *or* $\mathbb{1}$.

Proof. Part (2) is obvious from Proposition A.1.4. Part (1) follows from the proof of [112, Theorem XI.4.6(i)], except for the assertion about $\pi_{0,\infty}$; but this is clear by [3, Chapter 3, Theorem 5.1] and the archimedean condition on π in Proposition A.1.4. □

Corollary A.2.2. *The* $\mathbb{T}^S$*-module* $H^2_{\text{ét},!}(\mathrm{Sh}_K(V), \overline{\mathbb{Q}}_p(1))^{G_\mathbb{Q}}$ *decomposes completely into a direct sum of characters* $\boldsymbol{h} : \mathbb{T}^S \to \overline{\mathbb{Q}}_p$, *each of which arises from the action of* $\mathbb{T}^S$ *on either:*

- *A newform in an automorphic representation* π *of* $B_F(\mathbb{A}_F)^\times$, *unramified outside* S, *such that* π *satisfies the conclusion of Lemma A.2.1(1).*
- *The automorphic character* $\chi_0 \circ \nu$ *of* $\mathrm{GSpin}(V)(\mathbb{A}_\mathbb{Q})$, *where* χ_0 *is either trivial or the Hecke character associated to* $F/\mathbb{Q}$.

Proof. Because (A.2) is an open and closed embedding, we have a split inclusion of $\mathbb{T}^S$-modules

$$H^2_{\text{ét},!}(\mathrm{Sh}_K(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1)) \subset H^2_{\text{ét},!}(\widetilde{\mathrm{Sh}}_{\widetilde{K}}(V)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p(1))$$

and so the corollary is immediate from Lemma A.2.1 and Corollary A.1.7. □

APPENDIX B. RELATIVE DEFORMATION THEORY AND LEVEL-RAISING

In this appendix, we recall the relative deformation theory of Fakhruddin, Khare, and Patrikis [29] in a format useful for characteristic zero level-raising. In the hope that this discussion will be helpful in future work, we work with representations valued in more general groups than are needed for the main text.

B.1. **Notation.**

B.1.1. *The group.* Let G be a smooth, split group scheme over the ring of integers O of a finite extension E of $\mathbb{Q}_p$, such that the neutral component of G is a connected reductive group. Let G^{der} be the derived subgroup of G. Write ϖ for the uniformizer of O. We suppose $p > 2$ and G satisfy [29, Assumption 2.1]. When $G = \mathrm{GSp}_4$ or GL_2, which are the cases relevant for the main text, *loc. cit.* is satisfied for all odd p. Let d_G be the dimension of $\operatorname{Lie} G^{\mathrm{der}}$.

B.1.2. *The Galois representation.* Let k be a number field, and fix a Galois representation $\rho : G_k \to G(O)$. For an integer $n \geq 1$, let $\rho_n : G_k \to G(O/\varpi^n)$ be the reduction of ρ, and let $\overline{\rho} := \rho_1$. Also write $\mathrm{ad}^0\rho$, $\mathrm{ad}^0\rho_n$, and $\mathrm{ad}^0\overline{\rho}$ for the natural G_k-representations on $\operatorname{Lie} G^{\mathrm{der}}$, $\operatorname{Lie} G^{\mathrm{der}} \otimes_O O/\varpi^n$, and $\operatorname{Lie} G^{\mathrm{der}} \otimes_O O/\varpi$, respectively. Let Σ_p and Σ_∞ be the set of places of k lying above p and ∞, respectively. We will always suppose fixed a finite set S of nonarchimedean places of k such that $\Sigma_p \cap S = \emptyset$ and $\rho|_{G_{k_v}}$ is unramified for $v \notin S \cup \Sigma_p$. We make the following assumptions on ρ:

Assumption B.1.3.

(1) $H^0(k, \mathrm{ad}^0\overline{\rho}) = H^0(k, \mathrm{ad}^0\overline{\rho}(1)) = 0$.
(2) ρ is odd in the sense of [29, Definition 1.2].
(3) For all primes $v \in \Sigma_p$, $\rho|_{G_{k_v}}$ is potentially semistable with regular Hodge-Tate cocharacter $\mu_v : \mathbb{G}_m \to G$.

Notation B.1.4. Recall the category CNL_O from (1.1.3). Let $\mu : G \to H = G/G^{\mathrm{der}}$ be the maximal abelian quotient of G, and let

$$\chi : G_k \xrightarrow{\rho} G(O) \xrightarrow{\mu} H(O)$$

be the multiplier character of ρ. For all primes v of k, let $\mathcal{D}_v$ be the functor on CNL_O defined by

$$\mathcal{D}_v(A) = \{\rho_A : G_{k_v} \to G(A) \,:\, \rho_A \otimes_A (O/\varpi) = \overline{\rho},\ \mu \circ \rho_A = \chi\}\,. \tag{B.1}$$

The functor $\mathcal{D}_v$ is represented by a universal deformation ring $\widetilde{R}_v$. For $v \in \Sigma_p$, let R_v be the quotient of $\widetilde{R}_v$ corresponding to potentially semistable deformations with fixed Hodge type μ_v [4, Proposition 3.0.12]; for $v \notin \Sigma_p$, set $R_v := \widetilde{R}_v$.

Our final assumption on ρ is:

Assumption B.1.5. For all primes $v \in S \cup \Sigma_p$, the point y_v of $\operatorname{Spec} R_v[1/\varpi]$ defined by $\rho|_{G_{k_v}} : G_{k_v} \to G(O) \to G(E)$ is formally smooth.

In particular, Assumption B.1.5 implies that y_v lies on a unique irreducible component of $\operatorname{Spec} R_v[1/\varpi]$. Let $R_v \twoheadrightarrow \overline{R}_v$ be the quotient corresponding to the Zariski closure of this irreducible component. We have the following simple criterion for Assumption B.1.5 to hold:

Lemma B.1.6. *For all v, y_v is a formally smooth point of* $\operatorname{Spec} R_v[1/\varpi]$ *if and only if*

$$H^0(\mathrm{WD}(\mathrm{ad}^0\,\rho|_{G_{k_v}})(1)) = 0.$$

Proof. This is immediate from [6, Corollary 3.3.4]; note that, in the notation of *loc. cit.*, $\operatorname{ad}\mathrm{WD}(\rho|_{G_{k_v}})$ is by definition the Weil-Deligne representation associated to $\operatorname{ad}\rho|_{G_{k_v}}$, cf. [6, §2]. ☐

B.2. **Selmer groups and relative deformation theory.** Recall that a *Selmer structure* $\mathcal{F}$ for an $O[G_k]$-module M is a collection of O-submodules (the "local conditions")

$$H^1_{\mathcal{F}}(k_v, M) \subset H^1(k_v, M)$$

for all nonarchimedean[13] places v of k, such that $H^1_{\mathcal{F}}(k_v, M) = H^1_{\mathrm{unr}}(k_v, M)$ for all but finitely many v. The associated *Selmer group* is

$$H^1_{\mathcal{F}}(k, M) = \ker\left(H^1(k, M) \to \prod_v \frac{H^1(k_v, M)}{H^1_{\mathcal{F}}(k_v, M)}\right).$$

If M is finite and $M' = \mathrm{Hom}(M, (E/O)(1))$ is the Cartier dual, then $H^1(k_v, M)$ and $H^1(k_v, M')$ are dual under the local Tate pairing. The *dual Selmer structure* $\mathcal{F}^*$ to $\mathcal{F}$ is the Selmer structure for M' defined by the orthogonal complement local conditions.

Proposition B.2.1. *Let v be a nonarchimedean place of k. There exists a nonempty open set $Y_v \subset \mathrm{Spec}\,\overline{R}_v(O)$ containing the point corresponding to ρ_v, and a collection of submodules $Z_{r,v} \subset Z^1(G_{k_v}, \mathrm{ad}^0\rho_r)$ with the following properties.*

1. *$Z_{r,v}$ is free over O/ϖ^r of rank $\dim \mathrm{Spec}\, R_v[1/\varpi]$ ($= d_G$ if $v \notin \Sigma_p$).*
2. *Let Y_n^v be the image of Y_v in $\mathrm{Spec}\, R_v(O/\varpi^n)$ and denote by $\varphi^{Y_v}_{n,r} : Y^v_{n+r} \to Y^v_n$ the reduction maps for $n, r \geq 1$. Then given $r_0 \geq 1$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$ and all $0 \leq r \leq r_0$, the fibers of $\varphi^{Y_v}_{n,r}$ are nonempty principal homogeneous spaces for $Z_{r,v}$.*
3. *The natural O-module maps $\mathrm{ad}^0\rho_r \twoheadrightarrow \mathrm{ad}^0\rho_{r-1}$ and $\mathrm{ad}^0\rho_{r-1} \hookrightarrow \mathrm{ad}^0\rho_r$ induce surjections $Z_{r,v} \twoheadrightarrow Z_{r-1,v}$ and inclusions $Z_{r-1,v} \hookrightarrow Z_{r,v}$.*
4. *$Z_{r,v}$ contains all coboundaries in $Z^1(G_{k_v}, \mathrm{ad}^0\rho_r)$.*

Proof. See [29, Proposition 4.7]. ☐

Remark B.2.2. Although $Y_v \subset \mathrm{Spec}\,\overline{R}_v(O)$ is not uniquely determined by the properties in Proposition B.2.1, the property (2) shows that $Z_{r,v}$ depends only on $\rho|_{G_{k_v}}$ (by considering the fiber over $\rho_n|_{G_{k_v}}$).

Definition B.2.3. For all $n \geq 1$, we define a Selmer structure $\mathcal{F}$ for $\mathrm{ad}^0\,\rho_n$ by

$$H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0\,\rho_n) = \begin{cases} \mathrm{im}\left(Z_{n,v} \to H^1(k_v, \mathrm{ad}^0\,\rho_n)\right), & v \in S \cup \Sigma_p, \\ H^1_{\mathrm{unr}}(k_v, \mathrm{ad}^0\,\rho_n), & v \notin S \cup \Sigma_p \cup \Sigma_\infty. \end{cases}$$

Notation B.2.4. Now suppose given a set $\mathfrak{Q}$ of finite primes $\boldsymbol{q}$ of k called *admissible*, and, for each $\boldsymbol{q} \in \mathfrak{Q}$, a quotient $R^{\mathrm{ord}}_{\boldsymbol{q}}$ of $R_{\boldsymbol{q}}$ with the following properties:

1. $R^{\mathrm{ord}}_{\boldsymbol{q}}$ is formally smooth of dimension d_G.
2. $R^{\mathrm{ord}}_{\boldsymbol{q}}$ is stable under the conjugation action by
$$\ker\left(G(O) \to G(O/\varpi)\right).$$

We also suppose $\mathfrak{Q} \cap (S \cup \Sigma_p) = \emptyset$.

Definition B.2.5.

1. A lift $\tau_{\boldsymbol{q}} : G_{k_{\boldsymbol{q}}} \to G(A)$ of $\overline{\rho}|_{G_{k_{\boldsymbol{q}}}}$, for a complete local Noetherian O-algebra A, is called *ordinary* if the corresponding map $R_{\boldsymbol{q}} \to A$ factors through $R^{\mathrm{ord}}_{\boldsymbol{q}}$.
2. For an admissible prime $\boldsymbol{q} \in \mathfrak{Q}$, a global lift $\tau : G_k \to G(A)$ of $\overline{\rho}$ is called $\boldsymbol{q}$-ordinary if $\tau|_{G_{k_{\boldsymbol{q}}}}$ is ordinary.
3. For $\boldsymbol{q} \in \mathfrak{Q}$, we say $\boldsymbol{q}$ is *n-admissible* if ρ_n is $\boldsymbol{q}$-ordinary.

---

[13] Since $p \neq 2$, for all $v|\infty$ we have $H^1(k_v, M) = 0$.

(4) If $\boldsymbol{q}$ is n-admissible, then we define $Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\rho_n) \subset Z^1(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_n)$ as the relative tangent space to $\operatorname{Spec} R^{\mathrm{ord}}_{\boldsymbol{q}} \otimes_O O/\varpi^n$ at the point corresponding to $\rho_n|_{G_{k_{\boldsymbol{q}}}}$. Let

$$H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0\,\rho_n) = \operatorname{Im}\left(Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_n) \to H^1(k_{\boldsymbol{q}}, \mathrm{ad}^0\,\rho_n)\right),$$

and let $H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0\,\rho_n(1)) \subset H^1(k_{\boldsymbol{q}}, \mathrm{ad}^0\,\rho_n(1))$ be the orthogonal complement of $H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0\,\rho_n)$.

(5) If $\mathfrak{Q}$ is a finite set of n-admissible primes, then we define a Selmer structure $\mathcal{F}(\mathfrak{Q})$ for $\mathrm{ad}^0\rho_n$ by

$$H^1_{\mathcal{F}(\mathfrak{Q})}(k_v, \mathrm{ad}^0\rho_n) = \begin{cases} H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0\rho_n), & v \notin \mathfrak{Q}, \\ H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0\rho_n), & v = \boldsymbol{q} \in \mathfrak{Q}. \end{cases}$$

(6) If $\mathfrak{Q}$ is a finite set of n-admissible primes, then we define the *relative Selmer groups* by

$$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_n) = \operatorname{Im}\left(\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_n) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\overline{\rho})\right)$$

and, dually,

$$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_n(1)) = \operatorname{Im}\left(\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_n(1)) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\overline{\rho}(1))\right).$$

Proposition B.2.6. *Suppose $\boldsymbol{q}$ is n-admissible.*

(1) $Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\rho_n)$ is free of rank d_G over O/ϖ^n and contains all coboundaries.

(2) For all $1 < r \leq n$, the natural maps $\mathrm{ad}^0\,\rho_r \twoheadrightarrow \mathrm{ad}^0\,\rho_{r-1}$ and $\mathrm{ad}^0\,\rho_{r-1} \hookrightarrow \mathrm{ad}^0\,\rho_r$ induce surjections $Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\rho_r) \twoheadrightarrow Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_{r-1})$ and injections $Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\rho_{r-1}) \hookrightarrow Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_r)$.

(3) Let $Y_{\boldsymbol{q},n,\mathrm{ord}} \subset \operatorname{Spec} R^{\mathrm{ord}}_{\boldsymbol{q}}(O)$ be the set of points reducing to $\rho_n|_{G_{k_{\boldsymbol{q}}}}$ modulo ϖ^n, and let $Y^{\boldsymbol{q}}_{m,n,\mathrm{ord}}$ be the image in $\operatorname{Spec} R^{\mathrm{ord}}_{\boldsymbol{q}}(O/\varpi^m)$ for all $m \geq n$. Then for any $1 \leq r \leq n$, the fibers of $Y^{\boldsymbol{q}}_{m+r,n,\mathrm{ord}} \to Y^{\boldsymbol{q}}_{m,n,\mathrm{ord}}$ are nonempty principal homogeneous spaces over $Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_r)$.

Proof. Parts (1) and (3) are immediate from the conditions on $R^{\mathrm{ord}}_{\boldsymbol{q}}$ in Notation B.2.4. For (2), it is clear from the definition that

$$Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_{r-1}) \supseteq \operatorname{Im}\left(Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_r) \to Z^1_{\mathrm{ord}}(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_{r-1})\right),$$

and equality holds by (1) and counting. A similar argument shows the compatibility with

$$Z^1(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_{r-1}) \hookrightarrow Z^1(G_{k_{\boldsymbol{q}}}, \mathrm{ad}^0\,\rho_r).$$

□

Lemma B.2.7. *Let $\mathfrak{Q}$ be a finite set of n-admissible primes. Then:*

(1) For all $a, b \geq 0$ with $a + b \leq n$, there are natural exact sequences

$$0 \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_a) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_{a+b}) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_b)$$

and

$$0 \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_a(1)) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_{a+b}(1)) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_b(1)).$$

(2) The exact sequences in (1) identify

$$\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\,\rho_a) = \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\,\rho_n)[\varpi^a]$$

and

$$\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\,\rho_a(1)) = \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\,\rho_n(1))[\varpi^a]$$

for all $a \leq n$.

(3) If $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_m) = 0$ for some integer $m \leq n$, then for all m' with $m - 1 \leq m' \leq n$, the natural map induces an isomorphism

$$\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_{m-1}) \xrightarrow{\sim} \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_{m'}).$$

Proof. Using Proposition B.2.6(1,2), the first part is [29, Lemma 6.1], except for the injectivity on the left, which follows from Assumption B.1.3(1). Part (2) is a corollary of (1), because the kernel of ϖ^a : $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\, \rho_n) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\, \rho_n)$ coincides with the kernel of $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\, \rho_n) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\, \rho_{n-a})$, and likewise for dual Selmer groups. So we show (3). For any m' with $m - 1 < m' \leq n$, the map

$$\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_{m'}) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \overline{\mathrm{ad}}^0\rho)$$

factors through $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_m)$, hence vanishes; in particular, we have an isomorphism

$$\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_{m'-1}) \xrightarrow{\sim} \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_{m'})$$

by (1). The claim follows by downwards induction on m'.

□

Lemma B.2.8. *Suppose $\mathfrak{Q}$ is a finite set of n-admissible primes. Then for all $m \leq n$,*

$$\dim_{O/\varpi} \overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_m) = \dim_{O/\varpi} \overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_m).$$

Proof. This is [29, Lemma 6.3]. Note that the local conditions are balanced in the sense of *loc. cit.*: for $v \notin \mathfrak{Q}$, this is [29, Proposition 4.7(3)], and for $\mathfrak{q} \in \mathfrak{Q}$ the same calculation applies because by Proposition B.2.6(1). □

Remark B.2.9. The proof of [29, Lemma 6.3] uses Assumption B.1.3(1).

Definition B.2.10. For any finite set of primes $\mathfrak{Q}$ disjoint from $S \cup \Sigma_p$, we define the Shafarevich-Tate groups:

$$\text{(B.2)} \qquad Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\rho_n) := \ker\left(H^2(k^{S\cup\Sigma_p\cup\mathfrak{Q}}/k, \mathrm{ad}^0\rho_n) \to \prod_{v \in S\cup\Sigma_p\cup\mathfrak{Q}} H^2(k_v, \mathrm{ad}^0\rho_n)\right)$$

and

$$\text{(B.3)} \qquad Ш^1_{\mathfrak{Q}}(\mathrm{ad}^0\rho_n(1)) := \ker\left(H^1(k^{S\cup\Sigma_p\cup\mathfrak{Q}}/k, \mathrm{ad}^0\rho_n(1)) \to \prod_{v \in S\cup\Sigma_p\cup\mathfrak{Q}} H^1(k_v, \mathrm{ad}^0\rho_n(1))\right),$$

for all $n \geq 1$.

Lemma B.2.11. *Suppose given a finite set $\mathfrak{Q}$ of n-admissible primes such that*

$$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_n) = 0.$$

Then the natural map

$$Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\overline{\rho}) \to Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\rho_n)$$

is identically zero.

Proof. In the commutative diagram

$$\begin{tikzcd}
Ш^1_{\mathfrak{Q}}(\mathrm{ad}^0\rho_n(1)) \arrow[r] \arrow[d, hook] & Ш^1_{\mathfrak{Q}}(\mathrm{ad}^0\overline{\rho}(1)) \arrow[d, hook] \\
\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_n(1)) \arrow[r] & \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\overline{\rho}(1)),
\end{tikzcd}$$

the bottom map is identically zero because $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0\rho_n(1)) = 0$ by Lemma B.2.8. Hence the top map is identically zero as well. But by global Poitou-Tate duality, the top map is canonically dual to the map

$$Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\overline{\rho}) \to Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\overline{\rho}_n),$$

so this shows the lemma. $\square$

Theorem B.2.12 (Fakhruddin-Khare-Patrikis, [29]). *Fix $m \geq 1$. Then there exists a constant $n_0 = n_0(m, \rho) \geq 1$ with the following property. For any $N \geq n_0$, if $\mathfrak{Q}$ is a finite set of N-admissible primes and*

$$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_m) = 0,$$

then there exists a representation

$$\rho^{\mathfrak{Q}} : G_k \to G(O)$$

satisfying:

1. *$\rho \equiv \rho^{\mathfrak{Q}} \pmod{\varpi^{N-m+1}}$.*
2. *$\rho^{\mathfrak{Q}}$ is unramified outside $S \cup \Sigma_p \cup \mathfrak{Q}$.*
3. *For all $v \in S \cup \Sigma_p$, the points of* $\operatorname{Spec} R_v[1/\varpi]$ *corresponding to $\rho^{\mathfrak{Q}}|_{G_{k_v}}$ and $\rho|_{G_{k_v}}$ lie on the same irreducible component.*
4. *For all $\boldsymbol{q} \in \mathfrak{Q}$, $\rho^{\mathfrak{Q}}$ is $\boldsymbol{q}$-ordinary.*

Proof. For all primes $v \in S \cup \Sigma_p$, apply Proposition B.2.1 with $r_0 = m$ to obtain an integer $n_{0,v}$ and a subset $Y_v \subset \overline{R}_v(O)$. Then let

$$n_0 = \max\left\{\max_{v \in S \cup \Sigma_p}\{n_{0,v}\} + m, 2m\right\}.$$

We construct $\rho^{\mathfrak{Q}}$ as the inverse limit of representations $\rho^{\mathfrak{Q}}_n : G_k \to G(O)$, compatible under reduction maps, with the following properties for all $n \geq 1$:

(i) For all $v \notin S \cup \Sigma_p \cup \mathfrak{Q}$, $\rho^{\mathfrak{Q}}_n|_{G_{k_v}}$ is unramified.
(ii) For $\boldsymbol{q} \in \mathfrak{Q}$, $\rho^{\mathfrak{Q}}_n|_{G_{k_{\boldsymbol{q}}}}$ is ordinary.
(iii) For $v \in S \cup \Sigma_p$, $\rho^{\mathfrak{Q}}_n|_{G_{k_v}}$ lies in the set Y^v_n (cf. Proposition B.2.1(2)).

This suffices because $Y_v \subset \operatorname{Spec} \overline{R}_v(O) \subset \operatorname{Spec} R_v(O)$ and $\operatorname{Spec} R^{\mathrm{ord}}_{\boldsymbol{q}}(O) \subset \operatorname{Spec} R_{\boldsymbol{q}}(O)$ are both closed in the ϖ-adic topology, for all $v \in S \cup \Sigma_p$ and $\boldsymbol{q} \in \mathfrak{Q}$. The representations $\rho^{\mathfrak{Q}}_n$ are constructed inductively, *but*, when constructing $\rho^{\mathfrak{Q}}_{n+1}$, we will allow ourselves to modify the representations

$$\rho^{\mathfrak{Q}}_{n-m+2}, \cdots, \rho^{\mathfrak{Q}}_{n-1}, \rho^{\mathfrak{Q}}_n.$$

The base case of the induction is $\rho^{\mathfrak{Q}}_n \equiv \rho_n$ for $n \leq N$.

For the inductive step, first fix local lifts $\rho^{\mathfrak{Q}}_{n+1,v}$ of $\rho^{\mathfrak{Q}}_{n-m+1}|_{G_{k_v}}$ for $v \in S \cup \Sigma_p \cup \mathfrak{Q}$, with the following property: if $v \in S \cup \Sigma_p$, then $\rho^{\mathfrak{Q}}_{n+1,v}$ lies in Y^v_{n+1}, and if $v = \boldsymbol{q} \in \mathfrak{Q}$, then $\rho^{\mathfrak{Q}}_{n+1,\boldsymbol{q}}$ lies in $\operatorname{Spec} R^{\mathrm{ord}}_{\boldsymbol{q}}(O/\varpi^{n+1})$. Such choices are possible by Proposition B.2.1(2) and the formal smoothness of $\operatorname{Spec} R^{\mathrm{ord}}_{\boldsymbol{q}}$. Now let $c \in H^2(k^{S \cup \Sigma_p \cup \mathfrak{Q}}/k, \mathrm{ad}^0\overline{\rho})$ denote the obstruction class defined by choosing a set-theoretic lift

$$\widetilde{\rho}^{\mathfrak{Q}}_{n+1} : G_{k, S \cup \Sigma_p \cup \mathfrak{Q}} \to G(O/\varpi^{n+1})$$

of $\rho^{\mathfrak{Q}}_n$; we have

$$c \in Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\overline{\rho}) \tag{B.4}$$

since the local lifts $\rho^{\mathfrak{Q}}_{n+1,v}$ exist for all $v \in S \cup \Sigma_p \cup \mathfrak{Q}$.

Then by Lemma B.2.11, c has trivial image c_m in $Ш^2_{\mathfrak{Q}}(\mathrm{ad}^0\rho_m)$. But, because $n-m+1 \geq N-m+1 \geq m$, c_m is precisely the obstruction to lifting $\rho^{\mathfrak{Q}}_{n-m+1}$ modulo ϖ^{n+1}. Thus we may choose a lift

$$\widetilde{\rho}^{\mathfrak{Q}}_{n+1} : G_{k, S \cup \Sigma_p \cup \cup \mathfrak{Q}} \to G(O/\varpi^{n+1})$$

with

$$\widetilde{\rho}^{\mathfrak{Q}}_{n+1} \equiv \rho^{\mathfrak{Q}}_{n-m+1} \pmod{\varpi^{n-m+1}}.$$

For each place $v \in S \cup \mathfrak{Q} \cup \Sigma_p$, comparing $\widetilde{\rho}^{\mathfrak{Q}}_{n+1}|_{G_{k_v}}$ to $\rho^{\mathfrak{Q}}_{n+1,v}$ as lifts of $\rho^{\mathfrak{Q}}_{n-m+1}|_{G_{k_v}}$ produces a collection of local classes

$$(f_v) \in \prod_{v \in S \cup \mathfrak{Q} \cup \Sigma_p} \frac{H^1(k_v, \mathrm{ad}^0 \rho_m)}{H^1_{\mathcal{F}(\mathfrak{Q})}(k_v, \mathrm{ad}^0 \rho_m)}.$$

Now consider the commutative diagram with exact rows coming from the Poitou-Tate long exact sequence: $(*)$

$$\begin{tikzcd}
H^1(k^{S \cup \mathfrak{Q} \cup \Sigma_p}/k, \mathrm{ad}^0 \rho_m) \arrow[r, "\mathrm{loc}_m"] \arrow[d] & \prod_{v \in S \cup \mathfrak{Q} \cup \Sigma_p} \frac{H^1(k_v, \mathrm{ad}^0 \rho_m)}{H^1_{\mathcal{F}(\mathfrak{Q})}(k_v, \mathrm{ad}^0 \rho_m)} \arrow[r] \arrow[d] & \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0 \rho_m(1))^\vee \arrow[d, hook] \\
H^1(k^{S \cup \mathfrak{Q} \cup \Sigma_p}/k, \mathrm{ad}^0 \rho_{m-1}) \arrow[r, "\mathrm{loc}_{m-1}"] & \prod_{v \in S \cup \mathfrak{Q} \cup \Sigma_p} \frac{H^1(k_v, \mathrm{ad}^0 \rho_{m-1})}{H^1_{\mathcal{F}(\mathfrak{Q})}(k_v, \mathrm{ad}^0 \rho_{m-1})} \arrow[r] & \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0 \rho_{m-1}(1))^\vee,
\end{tikzcd}$$

where the superscript $\vee$ denotes Pontryagin duality. The injectivity of the rightmost map, or equivalently the surjectivity of its dual $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0 \rho_{m-1}(1)) \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0 \rho_m(1))$, follows from Lemma B.2.7(1), Lemma B.2.8, and the vanishing of $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})^*}(k, \mathrm{ad}^0 \rho_m(1))$.

Our next claim is that the image of (f_v) under the central vertical map of $(*)$ lies in the image of loc_{m-1}. Indeed, because $\rho^{\mathfrak{Q}}_n$ satisfies (i)-(iii) above, it follows from Propositions B.2.1(2) and B.2.6(3) that the image of (f_v) coincides with the image of the global cocycle formed by comparing $\widetilde{\rho}^{\mathfrak{Q}}_{n+1} \pmod{\varpi^n}$ and $\rho^{\mathfrak{Q}}_n$ as lifts of $\rho^{\mathfrak{Q}}_{n-m+1}$. Then because the rows of $(*)$ are exact, we conclude that (f_v) lies in the image of loc_m. Picking a preimage, we may then modify $\widetilde{\rho}^{\mathfrak{Q}}_{n+1}$ to a representation $\rho'_{n+1} : G_k \to G(O/\varpi^{n+1})$ with

$$\rho'_{n+1} \equiv \rho^{\mathfrak{Q}}_{n-m+1} \pmod{\varpi^{n-m+1}}.$$

Properties (i)-(iii) hold for ρ'_{n+1} by Propositions B.2.1(2), B.2.6(3) again. To complete the inductive step, we set $\rho^{\mathfrak{Q}}_{n+1} := \rho'_{n+1}$, and *relabel* $\rho^{\mathfrak{Q}}_{n-m+2}, \cdots, \rho^{\mathfrak{Q}}_n$ to be the reductions of ρ'_{n+1}. ☐

B.3. **Relation to Bloch-Kato Selmer groups.**

Notation B.3.1. Let $\Sigma_p^{\mathrm{cris}} \subset \Sigma_p$ be the set of places $v|p$ of k such that $\rho|_{G_{k_v}}$ is crystalline. For $v \in \Sigma_p^{\mathrm{cris}}$, we write R_v^{cris} for the crystalline quotient of R_v (constructed in [4]).

Remark B.3.2. Recall that $\mathrm{Spec}\, R_v^{\mathrm{cris}}[1/\varpi]$ is a union of irreducible components of $\mathrm{Spec}\, R_v[1/\varpi]$; in particular, $\overline{R}_v$ is a quotient of R_v^{cris}, and $\mathrm{Spec}\, R_v^{\mathrm{cris}}[1/\varpi]$ is equidimensional of the same dimension as $\mathrm{Spec}\, R_v[1/\varpi]$.

Definition B.3.3. For all finite places v of k, and all $r \geq 0$, define

$$Z^{\mathrm{rel}}_{r,v} \subset Z^1(k_v, \mathrm{ad}^0\, \rho_r)$$

to be the subspace of cocycles c corresponding to lifts $\rho_c : G_{k_v} \to G(O[\epsilon]/(\epsilon^2, \varpi^r \epsilon))$ such that the corresponding map $f_c : \widetilde{R}_v \to O[\epsilon]/(\epsilon^2, \varpi^r \epsilon)$ factors through R_v (resp. R_v^{cris}) if $v \notin \Sigma_p^{\mathrm{cris}}$ (resp. $v \in \Sigma_p^{\mathrm{cris}}$). In particular, $Z^{\mathrm{rel}}_{r,v} = Z^1(k_v, \mathrm{ad}^0\, \rho_r)$ if $v \notin \Sigma_p$.

Proposition B.3.4. *Fix a place v of k.*

(1) For all $r \geq 1$, we have $Z_{r,v} \subset Z^{\mathrm{rel}}_{r,v}$.

(2) The cardinality of $Z^{\mathrm{rel}}_{r,v}/Z_{r,v}$ is uniformly bounded in r.

Proof. Let $I \subset \widetilde{R}_v$ be the kernel of the map to R_v (resp. R_v^{cris}) if $v \notin \Sigma_p^{\mathrm{cris}}$ (resp. $v \in \Sigma_p^{\mathrm{cris}}$). For (1), suppose given a cocycle $c \in Z_{r,v}$ which corresponds to a lift

$$f_c : \widetilde{R}_v \to O[\epsilon]/(\epsilon^2, \varpi^r \epsilon)$$

of the map $f : \widetilde{R}_v \to O$ determined by $\rho|_{G_{k_v}}$. Because $c \in Z_{r,v}$, for all n sufficiently large the map

$$\widetilde{R}_v \xrightarrow{f_c} O[\epsilon]/(\epsilon^2, \varpi^r \epsilon) \xrightarrow{\epsilon \mapsto \varpi^n} O/\varpi^{n+r}$$

factors through $\widetilde{R}_v/I$; hence

$$f_c(I) \subset (\varpi^n - \epsilon) \cap (\epsilon) \text{ in } O[\epsilon]/(\epsilon^2, \varpi^r \epsilon)$$

for all n sufficiently large. We conclude $f_c(I) = 0$, hence c lies in $Z_{r,v}^{\mathrm{rel}}$, which proves (1).

For (2), let $\mathfrak{p} \subset \widetilde{R}_v$ be the kernel of f, and note that $Z_{r,v}^{\mathrm{rel}}$ is canonically identified with

$$\mathrm{Hom}_O(\mathfrak{p}/(\mathfrak{p}^2, I), O/\varpi^r).$$

Since $\widetilde{R}_v$ is Noetherian, $\mathfrak{p}/(\mathfrak{p}^2, I)$ is a finitely-generated O-module, hence

$$\lg_O Z_{r,v}^{\mathrm{rel}} = r \cdot \mathrm{rank}_O(\mathfrak{p}/(\mathfrak{p}^2, I)) + \mathrm{O}(1)$$

as r varies. But because $\rho|_{G_{k_v}}$ defines a formally smooth point of $\mathrm{Spec}\, R_v[1/\varpi]$, the O-rank of $\mathfrak{p}/(\mathfrak{p}^2, I)$ is also $\dim \mathrm{Spec}\, R_v[1/\varpi]$ (which equals $\dim \mathrm{Spec}\, R_v^{\mathrm{cris}}[1/\varpi]$ for $v \in \Sigma_p^{\mathrm{cris}}$). Moreover

$$\lg_O Z_{r,v} = r \dim \mathrm{Spec}\, R_v[1/\varpi]$$

by Proposition B.2.1(1), so (2) follows.

□

Proposition B.3.5. *For all places v of k, let*

$$H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) = \varprojlim_n H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho_n) \subset H^1(k_v, \mathrm{ad}^0 \rho).$$

Then we have

$$H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) \otimes_O E = H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E).$$

Proof. Suppose first that $v \notin \Sigma_p$. Then

$$\dim H^1(k_v, \mathrm{ad}^0 \rho \otimes_O E) - \dim H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E) = \dim H^0(k_v, \mathrm{ad}^0 \rho(1) \otimes_O E) = 0$$

by the local Euler characteristic formula, local duality, and Lemma B.1.6 (under Assumption B.1.5). By Proposition B.2.1(1,3), we also have

$$\begin{aligned}\dim H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) \otimes_O E &= d_G - \dim H^0(k_v, \mathrm{ad}^0 \rho \otimes_O E)\\ &= \dim_E \mathrm{Lie}\, G^{\mathrm{der}} \otimes_O E - \dim H^0(k_v, \mathrm{ad}^0 \rho \otimes_O E)\\ &= \dim H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E),\end{aligned}$$

and the proposition follows.

Now we consider the case $v \in \Sigma_p$. By Proposition B.2.1(1), we have

$$\begin{aligned}\dim H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) \otimes_O E &= \dim \mathrm{Spec}\, R_v[1/\varpi] + \dim \mathrm{ad}^0 \rho \otimes_O E - \dim H^0(k_v, \mathrm{ad}^0 \rho \otimes_O E)\\ &= \dim D_{\mathrm{dR}}(\mathrm{ad}^0 \rho \otimes_O E)/\,\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathrm{ad}^0 \rho \otimes_O E) - \dim H^0(k_v, \mathrm{ad}^0 \rho \otimes_O E),\end{aligned}$$

where the latter equality is by the proof of [6, Theorem 3.3.2] and by Assumption B.1.5. In particular, by [9, Corollary 3.8.4], we have

$$\dim H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) \otimes_O E = \dim H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E).$$

It therefore suffices to show that $H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E) \subset H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) \otimes_O E$.

By Proposition B.3.4, we have

$$(\varprojlim_r Z_{r,v}) \otimes_O E = (\varprojlim_r Z_{r,v}^{\mathrm{rel}}) \otimes_O E \subset Z^1(k_v, \mathrm{ad}^0 \rho \otimes_O E).$$

In particular, by the definition of $Z^{\mathrm{rel}}_{r,v}$, $H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0 \rho) \otimes_O E$ consists of cocycles c such that the corresponding G-valued deformation ρ_c of ρ to $E[\epsilon]/(\epsilon^2)$ is crystalline (resp. potentially semistable) with Hodge type μ_v if $v \in \Sigma_p^{\mathrm{cris}}$ (resp. $v \in \Sigma_p - \Sigma_p^{\mathrm{cris}}$). By [4, Proposition 2.3.2], it suffices to check this condition for the representation $\sigma \circ \rho_c : G_{k_v} \to \mathrm{GL}_n(E[\epsilon]/(\epsilon^2))$ obtained by composing ρ_c with any faithful algebraic representation $\sigma : G \to \mathrm{GL}_{n,E}$.

Consider the case when $v \in \Sigma_p - \Sigma_p^{\mathrm{cris}}$. Suppose given a cocycle $c \in H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E)$; *a fortiori*, c lies in the kernel of the map

$$H^1(k_v, \mathrm{ad}^0 \rho \otimes_O E) \to H^1(k_v, \mathrm{ad}^0 \rho \otimes_O B_{\mathrm{dR}}),$$

hence the cocycle corresponding to the deformation $\sigma \circ \rho_c$ of $\sigma \circ \rho$ lies in the kernel of the map

$$H^1(k_v, \mathrm{ad}^0(\sigma \circ \rho)) \to H^1(k_v, \mathrm{ad}^0(\sigma \circ \rho) \otimes B_{\mathrm{dR}}).$$

In particular, $\sigma \circ \rho_c$ is potentially semistable by the argument of [2, Lemma 1.2.5], and this completes the proof. When $v \in \Sigma_p^{\mathrm{cris}}$, an analogous argument applies, using that c lies in the kernel of the map

$$H^1(k_v, \mathrm{ad}^0 \rho \otimes_O E) \to H^1(k_v, \mathrm{ad}^0 \rho \otimes_O B_{\mathrm{cris}})$$

by definition of $H^1_f(k_v, \mathrm{ad}^0 \rho \otimes_O E)$. ☐

Lemma B.3.6. *Suppose $H^1_f(k, \mathrm{ad}^0 \rho \otimes_O E) = 0$. Then for all n sufficiently large,*

$$\overline{\mathrm{Sel}}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho_n) = 0.$$

Proof. We first claim:

Claim. We have $\mathrm{Sel}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho) = \varprojlim_n \mathrm{Sel}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho_n)$.

Proof of claim. Set $Z_{n,v} = Z^1_{\mathrm{unr}}(G_{k_v}, \mathrm{ad}^0 \rho_n)$ for all finite $v \notin S \cup \Sigma_p$, and set $Z_v := \varprojlim_n Z_{n,v}$ for all finite v. It follows from Proposition B.2.1(1,3) and a direct calculation in the unramified case that $Z^1(G_{k_v}, \mathrm{ad}^0 \rho)/Z_v$ is torsion-free and $Z_{n,v}$ is the image of the map $Z_v \to Z^1(G_{k_v}, \mathrm{ad}^0 \rho_n)$ for all $n \geq 1$. Then the claim follows from [75, Lemma 3.7.1]. ☐

Now we return to the proof of the lemma. By Proposition B.3.5, the assumption $H^1_f(k, \mathrm{ad}^0 \rho \otimes_O E) = 0$ implies that $\mathrm{Sel}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho)$ is torsion, hence trivial because $H^1(k, \mathrm{ad}^0 \rho)$ is ϖ-torsion-free by Assumption B.1.3(1). So by the claim, we have

$$\varprojlim_n \mathrm{Sel}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho_n) = 0,$$

which implies

$$\varprojlim_n \overline{\mathrm{Sel}}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho_n) = \cap_n \overline{\mathrm{Sel}}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho_n) = 0.$$

Hence $\overline{\mathrm{Sel}}_{\mathcal{F}}(k, \mathrm{ad}^0 \rho_n) = 0$ for n sufficiently large. ☐

B.4. **Controlling congruences for level-raised representations.**

Definition B.4.1. (1) For each place v of k, we define

$$C_v := \sup_r \# \left(Z^{\mathrm{rel}}_{r,v} / Z_{r,v} \right),$$

which is finite by Proposition B.3.4.

(2) If $\mathfrak{Q}$ is a finite set of n-admissible primes for some $n \geq 1$, define a Selmer structure $\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}$ for $\mathrm{ad}^0 \rho_n$ by

$$H^1_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k_v, \mathrm{ad}^0 \rho_n) = \begin{cases} \mathrm{im} \left(Z^{\mathrm{rel}}_{n,v} \to H^1(k_v, \mathrm{ad}^0 \rho_n) \right), & v \in S \cup \Sigma_p, \\ H^1_{\mathcal{F}(\mathfrak{Q})}(k_v, \mathrm{ad}^0 \rho_n), & \text{otherwise}. \end{cases}$$

Proposition B.4.2. *Let $C_1 = \sum_{v \in S \cup \Sigma_p} C_v$, where C_v is as in Definition B.4.1. Then*

$$\lg_O \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k, \mathrm{ad}^0 \rho_n) \leq \lg_O \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_n) + C_1$$

for all $n \geq 1$ and all n-admissible $\mathfrak{Q}$.

Proof. This follows from the exactness of the sequence

$$0 \to \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_n) \to \mathrm{Sel}_{\mathcal{F}^{\mathrm{rel}}(\mathfrak{Q})}(k, \mathrm{ad}^0\rho_n) \to \prod_{v \in S \cup \Sigma_p} \frac{H^1_{\mathcal{F}^{\mathrm{rel}}}(k_v, \mathrm{ad}^0\rho_n)}{H^1_{\mathcal{F}}(k_v, \mathrm{ad}^0\rho_n)}. \tag{B.5}$$

☐

Corollary B.4.3. *Fix $c \geq 0$. There exists a constant $C_2 \geq 0$, depending only on c and ρ, with the following property: for all $n \geq m - 1 \geq 0$ and all n-admissible $\mathfrak{Q}$ with $|\mathfrak{Q}| = c$,*

$$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_m) = 0 \implies \lg_O \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k, \mathrm{ad}^0 \rho_n) \leq C_2(m-1) + C_1,$$

where C_1 is the constant in Proposition B.4.2.

Proof. By Lemma B.2.7(3), if $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_m) = 0$ then

$$\varpi^{m-1} \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_n) = 0.$$

Hence

$$\begin{aligned} \lg_O \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_n) &\leq \left(\dim_{O/\varpi} \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_n)[\varpi]\right)(m-1) \\ &= \left(\dim_{O/\varpi} \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \overline{\rho})\right)(m-1) \\ &\leq \left(\dim_{O/\varpi} \mathrm{Sel}_{\mathcal{F}}(k, \mathrm{ad}^0 \overline{\rho}) + \sum_{\mathfrak{q} \in \mathfrak{Q}} \dim_{O/\varpi} \frac{H^1(k_{\mathfrak{q}}, \mathrm{ad}^0 \overline{\rho})}{H^1_{\mathcal{F}}(k_{\mathfrak{q}}, \mathrm{ad}^0 \overline{\rho})}\right)(m-1); \end{aligned} \tag{B.6}$$

in the second line we have used Lemma B.2.7(2). By the local Euler characteristic formula, $\dim_{O/\varpi} H^1(k_{\mathfrak{q}}, \mathrm{ad}^0 \overline{\rho})$ is uniformly bounded in $\mathfrak{q}$, so (B.6) becomes

$$\lg_O \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_n) \leq C_2(m-1) \tag{B.7}$$

for a constant C_2 depending only on $c = |\mathfrak{Q}|$ and ρ. Combined with Proposition B.4.2, this proves the corollary. ☐

Notation B.4.4. Let Σ be a finite set of places of k.

(1) Let $A \in \mathrm{CNL}_O$. A lift $\rho_A : G_k \to G(A)$ of $\overline{\rho}$ is called Σ-*good* if:
  (i) $\mu \circ \rho_A = \chi$ (notation as in Notation B.1.4);
  (ii) ρ_A is unramified outside $S \cup \Sigma_p \cup \Sigma_\infty \cup \Sigma$;
  (iii) For all $v \in \Sigma_p^{\mathrm{cris}}$ (resp. $v \in \Sigma_p - \Sigma_p^{\mathrm{cris}}$), the map $\widetilde{R}_v \to A$ defined by $\rho_A|_{G_{k_v}}$ factors through R_v^{cris} (resp. R_v).

(2) Let $\mathcal{D}_\Sigma^{\mathrm{global}}$ be the functor on CNL_O defined by

$$\mathcal{D}_\Sigma^{\mathrm{global}}(A) = \{\rho_A : G_k \to G(A) \,:\, \rho_A \otimes_A (O/\varpi) = \overline{\rho} \text{ and } \rho_A \text{ is } \Sigma\text{-good}\} \,/\sim,$$

where the equivalence relation is $\ker(G(A) \to G(O/\varpi))$-conjugacy. By (the same argument of) [25, Proposition 2.2.9], $\mathcal{D}_\Sigma^{\mathrm{global}}$ is represented by a global deformation ring which we denote R^Σ.

(3) Now suppose $\Sigma = \mathfrak{Q}$ for some finite subset $\mathfrak{Q} \subset \mathfrak{Q}$. Let $\mathcal{D}_{\mathfrak{Q}\text{-ord}}^{\mathrm{global}} \subset \mathcal{D}_{\mathfrak{Q}}^{\mathrm{global}}$ be the subfunctor consisting of deformations which are $\mathfrak{q}$-ordinary for all $\mathfrak{q} \in \mathfrak{Q}$. By (the same argument of) [25, Proposition 2.2.9], $\mathcal{D}_{\mathfrak{Q}\text{-ord}}^{\mathrm{global}}$ is represented by the $\mathfrak{Q}$-*ordinary quotient* of $R^{\mathfrak{Q}}$, which we denote $R_{\mathfrak{Q}}$.

(4) Given a homomorphism $f_{\mathfrak{Q}} : R_{\mathfrak{Q}} \to O$, define the *congruence ideal*

$$\eta_{f_{\mathfrak{Q}}} \subset O := f_{\mathfrak{Q}}(\mathrm{Ann}_{R_{\mathfrak{Q}}}(\ker f_{\mathfrak{Q}})) \subset O. \tag{B.8}$$

Lemma B.4.5. *Let $m \geq 1$ be an integer. Then we may choose the integer $n_0(m, \rho) \geq 1$ in Theorem B.2.12 such that the following holds: suppose $n \geq n_0(m, \rho)$, and $\mathfrak{Q}$ is an n-admissible set such that*

$$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(\mathrm{ad}^0 \rho_m) = 0.$$

Let $f_{\mathfrak{Q}} : R_{\mathfrak{Q}} \to O$ be the homomorphism corresponding to the representation $\rho^{\mathfrak{Q}}$ from Theorem B.2.12. Then

$$\mathrm{ord}_{\varpi}\, \eta_{f_{\mathfrak{Q}}} \leq \lg_O \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(\mathrm{ad}^0 \rho_{n-m+1}).$$

Proof. Let $n_1(m, \rho) \geq 1$ satisfy the conclusion of Theorem B.2.12, and let C_1 be the constant from Proposition B.4.2. We set $n_0(m, \rho) := \max\{n_1(m, \rho), C_1 + 2m - 1\}$, and check the claimed property. Write $I := \ker f_{\mathfrak{Q}}$. We have

$$\mathrm{Fitt}_{R_{\mathfrak{Q}}}(I) \subset \mathrm{Ann}_{R_{\mathfrak{Q}}}(I),$$

so by base change for Fitting ideals,

$$\mathrm{Fitt}_{R_{\mathfrak{Q}}/I}(I/I^2) \subset \eta_{f_{\mathfrak{Q}}}.$$

Because $R_{\mathfrak{Q}}/I = O$, it therefore suffices to bound $\lg_O I/I^2$.

Now note that O-module maps $I/I^2 \to O/\varpi^s$, for any integer $s \geq 1$, are canonically in bijection with lifts $R_{\mathfrak{Q}} \to O[\epsilon]/(\epsilon^2, \varpi^s \epsilon)$ of $f_{\mathfrak{Q}}$. Taking $s = n - m + 1$ and using that $\rho^{\mathfrak{Q}} \equiv \rho \pmod{\varpi^{n-m+1}}$, such lifts are in bijection with classes in $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k, \mathrm{ad}^0 \rho_{n-m+1})$. Hence

$$\mathrm{Hom}(I/I^2, O/\varpi^{n-m+1}) \cong \mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k, \mathrm{ad}^0\, \rho_{n-m+1}). \tag{B.9}$$

Now, by Lemma B.2.7(3), $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_{n-m+1})$ is ϖ^{m-1}-torsion. In particular, (B.5) shows that $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k, \mathrm{ad}^0 \rho_{n-m+1})$ is ϖ^{m+C_1-1}-torsion, hence *a fortiori* ϖ^{n-m}-torsion. Since I is finitely generated over $R_{\mathfrak{Q}}$, we conclude that I/I^2 is ϖ^{n-m}-torsion.

Thus

$$\lg_O I/I^2 = \lg_O \mathrm{Hom}_O(I/I^2, O/\varpi^{n-m+1}),$$

and the lemma follows from (B.9). ☐

We remark that essentially the same argument shows:

Remark B.4.6. The map $f_{\mathfrak{Q}} : R_{\mathfrak{Q}} \to O$ is an isomorphism if and only if $\mathrm{Sel}_{\mathcal{F}(\mathfrak{Q})^{\mathrm{rel}}}(k, \mathrm{ad}^0\, \overline{\rho}) = 0$.

Definition B.4.7. Let $\boldsymbol{q}$ be n-admissible. We say that $\boldsymbol{q}$ is *standard* if:

1. There exists a representation $\tau_{\boldsymbol{q}} : G_{k_{\boldsymbol{q}}} \to G(O)$ which is both ordinary and unramified.
2. For all $m \leq n$, both
$$\frac{H^1_{\mathrm{unr}}(k_{\boldsymbol{q}}, \mathrm{ad}^0 \rho_m) + H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0 \rho_m)}{H^1_{\mathrm{unr}}(k_{\boldsymbol{q}}, \mathrm{ad}^0 \rho_m)}$$
and
$$\frac{H^1_{\mathrm{unr}}(k_{\boldsymbol{q}}, \mathrm{ad}^0 \rho_m) + H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0 \rho_m)}{H^1_{\mathrm{ord}}(k_{\boldsymbol{q}}, \mathrm{ad}^0 \rho_m)}$$
are free of rank one over O/ϖ^m.

Lemma B.4.8. *Fix an integer $m \geq 1$ and let $n_0 = n_0(m, \rho)$ be the integer of Theorem B.2.12. Let $n \geq \max\{n_0, 3m\}$ be an integer and suppose given a finite set $\mathfrak{Q}$ of $(n+m)$-admissible primes and two additional n-admissible primes $\boldsymbol{p}, \boldsymbol{q} \notin \mathfrak{Q}$, such that:*

1. *$\boldsymbol{q}$ is standard and* not *$(n + 1)$-admissible.*
2. *$\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q})}(k, \mathrm{ad}^0 \rho_m) = \overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q}\boldsymbol{q}\boldsymbol{p})}(k, \mathrm{ad}^0 \rho_m) = 0$ but $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathfrak{Q}\boldsymbol{q})}(k, \mathrm{ad}^0 \rho_{2m-1}) \neq 0$.*

Then the representation $\rho^{\mathcal{Q}qp}$ constructed in Theorem B.2.12 is ramified at p modulo ϖ^{n+m}.

Proof. Let $\rho^{\mathcal{Q}}$ and $\rho^{\mathcal{Q}qp}$ be the representations afforded by Theorem B.2.12, so that $\rho^{\mathcal{Q}} \equiv \rho \pmod{\varpi^{n+1}}$ and $\rho^{\mathcal{Q}qp} \equiv \rho \pmod{\varpi^{n-m+1}}$. Modulo ϖ^{n+m}, $\rho_{\mathcal{Q}}$ and $\rho_{\mathcal{Q}qp}$ differ by a cocycle $c \in H^1(k, \mathrm{ad}^0\rho_{2m-1})$. (This make sense because $2m-1 \leq n-m+1$.) Also let $d \in \mathrm{Sel}_{\mathcal{F}(\mathcal{Q}q)}(k, \mathrm{ad}^0\rho_{2m-1}(1))$ be an element whose image in $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathcal{Q}q)^*}(k, \mathrm{ad}^0\rho_{2m-1}(1))$ is nonzero, which exists Lemma B.2.8. By global Poitou-Tate duality, we have

$$\sum_v \langle c, d\rangle_v = 0, \tag{B.10}$$

where $\langle c, d\rangle_v$ is the local Tate pairing. By Proposition B.2.1(2) and by the choice of n_0, $\mathrm{loc}_v\, c$ lies in $H^1_{\mathcal{F}(\mathcal{Q}q)}(k_v, \mathrm{ad}^0\rho_{2m-1})$ for all $v \neq p, q$. In particular,

$$\langle c, d\rangle_p \neq 0 \iff \langle c, d\rangle_q \neq 0. \tag{B.11}$$

Our next claim is that:

$$\langle c, d\rangle_q \neq 0. \tag{B.12}$$

Indeed, $\mathrm{Res}_q\, c$ lies in $H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\rho_{2m-1}) + H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1})$; one can see this by comparing both $\rho^{\mathcal{Q}}$ and $\rho^{\mathcal{Q}qp}$ to the representation τ_q in Definition B.4.7(1). Because q is not $(n+1)$-admissible, $\rho^{\mathcal{Q}}$ is not ordinary at q modulo ϖ^{n+1}, so

$$\mathrm{Res}_q\, c \in \frac{H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\, \rho_{2m-1}) + H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\, \rho_{2m-1})}{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\, \rho_{2m-1})} \approx O/\varpi^{2m-1}$$

is nonzero modulo ϖ^m. On the other hand,

$$\begin{aligned}\mathrm{Res}_q\, d \in{} & \frac{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1))}{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1)) \cap H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1))} \\ ={} & \frac{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1)) + H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1))}{H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1))} \approx O/\varpi^{2m-1}\end{aligned}$$

is also nonzero modulo ϖ^m. Otherwise, the image of d modulo ϖ^m would lie in $\mathrm{Sel}_{\mathcal{F}(\mathcal{Q})^*}(k, \mathrm{ad}^0\rho_m(1))$, which contradicts the assumption that $\overline{\mathrm{Sel}}_{\mathcal{F}(\mathcal{Q})^*}(k, \mathrm{ad}^0\rho_m(1)) = 0$. Since local Poitou-Tate duality gives a perfect pairing

$$\frac{H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\rho_{2m-1}) + H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1})}{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1})} \times \frac{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1))}{H^1_{\mathrm{ord}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1)) \cap H^1_{\mathrm{unr}}(k_q, \mathrm{ad}^0\rho_{2m-1}(1))} \to O/\varpi^{2m-1},$$

we indeed have (B.12). Then by (B.11), we conclude

$$\langle c, d\rangle_p \neq 0.$$

Since $\mathrm{loc}_p\, d$ is unramified, we must have

$$\mathrm{Res}_p\, c \notin H^1_{\mathrm{unr}}(k_p, \mathrm{ad}^0\rho_{2m-1}).$$

Since $\rho^{\mathcal{Q}}|_{G_{k_p}}$ is unramified, and $\mathrm{Res}_p\, c$ measures the difference between $\rho^{\mathcal{Q}}|_{G_{k_p}}$ and $\rho^{\mathcal{Q}qp}|_{G_{k_p}}$ modulo ϖ^{n+m}, this proves the lemma. ☐

APPENDIX C. LARGE IMAGE RESULTS

Throughout this appendix, let E be a finite extension of $\mathbb{Q}_p$, with ring of integers $O_E \subset E$.

C.1. Generalities on p-adic Lie groups.

Lemma C.1.1. *Let $\mathfrak{h}$ be a simple Lie algebra over E.*

1. *If $\mathfrak{g} \subset \mathfrak{h}^{\oplus n}$ is a Lie subalgebra that surjects onto each factor, then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}^{\oplus m}$ for some integer $m \leq n$. Up to an automorphism of $\mathfrak{h}^{\oplus n}$, the map $\mathfrak{g} \cong \mathfrak{h}^{\oplus m} \to \mathfrak{h}^{\oplus n}$ is given by*
$$(h_1, \dots, h_m) \mapsto (\underbrace{h_1, \dots, h_1}_{n_1 \text{ times}}, \underbrace{h_2, \dots, h_2}_{n_2 \text{ times}}, \dots, \underbrace{h_m, \cdots, h_m}_{n_m \text{ times}})$$
with $n_1 + \cdots + n_m = n$.
2. *The only ideal $I \subset \mathfrak{h}^{\oplus n}$ that surjects onto each factor is $I = \mathfrak{h}^{\oplus n}$.*

Proof. We prove (1) by induction on n, with the case $n = 1$ being trivial. Supposing we know (1) for $n - 1$, let $\mathfrak{g} \subset \mathfrak{h}^{\oplus n}$ be a subalgebra surjective onto each factor, and let $\mathfrak{g}'$ be the image of $\mathfrak{g}$ under the projection $\mathfrak{h}^{\oplus n} = \mathfrak{h}^{\oplus(n-1)} \oplus \mathfrak{h} \to \mathfrak{h}^{\oplus(n-1)}$. Then by the inductive hypothesis, $\mathfrak{g}' \cong \mathfrak{h}^{\oplus m}$ for some integer $m \leq n - 1$. Now, $\mathfrak{g} \subset \mathfrak{g}' \oplus \mathfrak{h}$ is a subalgebra surjective onto each factor, so by Goursat's Lemma for Lie algebras, $\mathfrak{g}$ is either $\mathfrak{g}' \oplus \mathfrak{h}$ or the graph of isomorphism between $\mathfrak{h}$ and a simple factor of $\mathfrak{g}'$. In particular, $\mathfrak{g}$ is isomorphic to either $\mathfrak{g}' \cong \mathfrak{h}^{\oplus m}$ or $\mathfrak{g}' \oplus \mathfrak{h} \cong \mathfrak{h}^{\oplus(m+1)}$, and it is easy to check that the embedding $\mathfrak{g} \to \mathfrak{h}^{\oplus n}$ is of the desired form using that $\mathfrak{g}' \to \mathfrak{h}^{\oplus(n-1)}$ is. For (2), it suffices to check that the subalgebras in (1) are never ideals unless $n = m$ (and hence $n_1 = n_2 = \cdots = n_m = 1$). Indeed, it suffices to check that the diagonal subalgebra $\mathfrak{h} \subset \mathfrak{h} \oplus \mathfrak{h}$ is not an ideal, but this is clear: since $\mathfrak{h}$ is simple, it is not abelian, so for some $h_1, h_2 \in \mathfrak{h}$ we have $[h_1, h_2] \neq 0$. In particular, the bracket $[(h_1, h_1), (h_2, 0)]$ is not contained in the diagonal subalgebra, which witnesses that the latter is not an ideal. ☐

Corollary C.1.2. *Let $\mathfrak{h}$ be an absolutely simple Lie algebra over $\mathbb{Q}_p$. Then for any finite extension $E/\mathbb{Q}_p$:*

1. *The base change $\mathfrak{h}_E := \mathfrak{h} \otimes_{\mathbb{Q}_p} E$ is simple as a Lie algebra over $\mathbb{Q}_p$.*
2. *For any $\mathbb{Q}_p$-Lie subalgebra $\mathfrak{g} \subset \mathfrak{h}_E$ such that $E \cdot \mathfrak{g} = \mathfrak{h}_E$, $\mathfrak{g}$ is simple.*

Proof. For any subalgebra $\mathfrak{g} \subset \mathfrak{h}_E$, consider the extension of scalars
$$\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \subset \mathfrak{h}_E \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}.$$
For (1), suppose $\mathfrak{g}$ is an ideal; then the image of $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ in each factor of $\mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}$ is a $\overline{\mathbb{Q}}_p$-stable ideal, hence either 0 or $\mathfrak{h}_{\overline{\mathbb{Q}}_p}$. Now, $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \subset \mathfrak{h}_E \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ is stable under the action of $G_{\mathbb{Q}_p}$, which transitively permutes the factors of $\mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}$. Hence if $\mathfrak{g} \neq 0$, then $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ surjects onto each factor of $\mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}$. Then by Lemma C.1.1(2), $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = \mathfrak{h}_E \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$, so $\mathfrak{g} = \mathfrak{h}_E$. This proves (1). For (2), if $\mathfrak{g} \cdot E = \mathfrak{h}_E$, then $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ generates $\mathfrak{h}_E \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}$ under the action of $E \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \overline{\mathbb{Q}}_p^{[E:\mathbb{Q}_p]}$, so $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ surjects onto each factor. By Lemma C.1.1(1), we conclude $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \mathfrak{h}_{\overline{\mathbb{Q}}_p}^{\oplus m}$ for some $m \leq [E : \mathbb{Q}_p]$.

If $I \subset \mathfrak{g}$ is a nonzero ideal, then $I \cdot E$ is a nonzero ideal of $\mathfrak{g} \cdot E = \mathfrak{h}_E$, so $I \cdot E = \mathfrak{h}_E$. But then $I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ is an ideal of $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ that surjects onto each factor of $\mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}$, and, inspecting the possible embeddings
$$\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \mathfrak{h}_{\overline{\mathbb{Q}}_p}^{\oplus m} \hookrightarrow \mathfrak{h}_{\overline{\mathbb{Q}}_p}^{[E:\mathbb{Q}_p]}$$
from Lemma C.1.1(1), we conclude that $I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ surjects onto each factor of $\mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \mathfrak{h}_{\overline{\mathbb{Q}}_p}^{\oplus m}$. But by Lemma C.1.1(2), $I \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = \mathfrak{g} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$, so then $I = \mathfrak{g}$. This proves (2). ☐

C.2. Strongly irreducible representations.

For the following definition only, we allow E to be an arbitrary algebraic extension of $\mathbb{Q}_p$.

Definition C.2.1. Suppose V is a finite-dimensional E-vector space, G is a group, and $\rho : G \to \mathrm{GL}_E(V)$ is a representation. Then V (or ρ) is said to be strongly irreducible if, for any finite-index subgroup $H \subset G$, (ρ, V) is absolutely irreducible as a representation of H.

Lemma C.2.2. *Let V be an E-vector space of finite dimension, and let $G \subset \mathrm{GL}_E(V)$ be a compact p-adic Lie subgroup. If V is strongly irreducible as a representation of G, then:*

1. *No nontrivial element of G/Z_G is fixed under conjugation by an open subgroup $U \subset G$; in particular G/Z_G has trivial center.*
2. *The group G/Z_G contains no finite normal subgroup.*
3. *If $g \in G$ acts unipotently on $\mathrm{Lie}\, G$, then g has only one eigenvalue on V.*
4. *The Lie algebra $\mathrm{Lie}\, G/Z_G$ is semisimple.*
5. *The natural maps $G \cap \mathrm{SL}_E(V) \to G/Z_G$ and $Z_G \to \det G$ induce isomorphisms on Lie algebras.*
6. *V is absolutely irreducible as a representation of $\mathrm{Lie}(G \cap \mathrm{SL}_E(V))$.*

Proof. (1) Let $h \in G$ be an element whose image in G/Z_G is invariant under conjugation by U. Then for all $g \in U$, $hgh^{-1}g^{-1}$ lies in Z_G, so by Schur's Lemma

$$hgh^{-1} = g\lambda_h(g) \text{ for a scalar } \lambda_h(g) \in E^\times. \tag{C.1}$$

It is easy to check that $g \mapsto \lambda_h(g)$ is a group homomorphism $U \to E^\times$. On the other hand, if $\dim_E V = n$, then (C.1) implies that $\lambda_h(g)$ lies in $\mu_n(E)$ for all $g \in U$. In particular, the homomorphism $g \mapsto \lambda_h(g)$ has open kernel, so h commutes with an open subgroup of U. By strong irreducibility and Schur's Lemma again, h is scalar, so has trivial image in G/Z_G.

(2) Let $H \subset G/Z_G$ be a finite normal subgroup. Then the map $G \to \mathrm{Aut}(H)$ has open kernel, so (1) implies that H is trivial.

(3) Let $g = g^{ss}g^u$ be the Jordan decomposition in $\mathrm{GL}_E(V)$. Then

$$\mathrm{ad}(g) - 1 = (\mathrm{ad}(g^{ss}) - 1)(\mathrm{ad}(g^u) - 1) + (\mathrm{ad}(g^u) - 1) + (\mathrm{ad}(g^{ss}) - 1)$$

as operators on $\mathfrak{gl}_E(V)$. In particular, if $\mathrm{ad}(g)-1$ is nilpotent on $\mathfrak{g} \subset \mathfrak{gl}_E(V)$, then for N sufficiently large, $\mathfrak{g}$ lies in the kernel of $(\mathrm{ad}(g^{ss}) - 1)^N$.

But since $\mathrm{ad} g^{ss} - 1$ is diagonalizable over $\overline{\mathbb{Q}}_p$ as an operator on $\mathfrak{gl}_E(V)$, we conclude $\mathfrak{g}$ lies in the kernel of $\mathrm{ad}(g^{ss}) - 1$; hence g^{ss} commutes with an open subgroup of G, so by Schur's Lemma g^{ss} is a scalar in $\mathrm{GL}_E(V)$. In particular, $g = g^{ss}g^u$ has a single eigenvalue on V.

(4) By [11, §6, Proposition 5], $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$, with $\mathfrak{h}$ semisimple and $\mathfrak{s}$ abelian. Since $\mathrm{Lie}(G/Z_G)$ has trivial center by (1), it follows that the natural maps induce isomorphisms $\mathfrak{h} \xrightarrow{\sim} \mathrm{Lie}(G/Z_G)$ and $\mathrm{Lie}\, Z_G \xrightarrow{\sim} \mathfrak{s}$. In particular, $\mathrm{Lie}(G/Z_G)$ is semisimple.

(5) The map $\mathrm{Lie}(G \cap \mathrm{SL}_E(V)) \to \mathrm{Lie}(G/Z_G) = \mathfrak{h}$ is injective with abelian cokernel; hence it is an isomorphism. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$, it follows that the determinant identifies $\mathrm{Lie}\, Z_G = \mathfrak{s} \xrightarrow{\sim} \mathrm{Lie}(\det(G))$.

(6) By (5), $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathrm{Lie}(Z_G) \oplus \mathrm{Lie}(G \cap \mathrm{SL}_E(V))$. So if $\mathrm{Lie}(G \cap \mathrm{SL}_E(V))$ stabilized any subspace of V after extending scalars, $\mathfrak{g}$ would as well, which contradicts strong irreducibility.

□

Lemma C.2.3. *Let V be a symplectic E-vector space of dimension 2 or 4, and let $G \subset \mathrm{GSp}_E(V)$ be a compact p-adic Lie subgroup. If V is strongly irreducible as a representation of G, then every nontrivial closed normal subgroup of G/Z_G has finite index.*

Proof. Abbreviate $\overline{G} = G/Z_G$ and $\overline{\mathfrak{g}} = \mathrm{Lie}\, \overline{G}$. Then $\overline{\mathfrak{g}}$ is a Lie subalgebra (over $\mathbb{Q}_p$) of $\mathfrak{sp}_{n,E}$, with $n = 2$ or 4. After replacing E with a finite extension, we may assume $E \cdot \overline{\mathfrak{g}} \subset \mathfrak{sp}_{n,E}$ is split. It is also semisimple (by Lemma C.2.2(4)) of rank at most 2, so $E \cdot \overline{\mathfrak{g}}$ is isomorphic to $\mathfrak{sl}_{2,E}$, $\mathfrak{sl}_{2,E} \times \mathfrak{sl}_{2,E}$, or $\mathfrak{sp}_{4,E}$. The second case is impossible by Lemma C.2.2(5, 6), since $\mathfrak{sl}_{2,E} \times \mathfrak{sl}_{2,E}$ admits no faithful irreducible two- or four-dimensional symplectic representation. Hence $E \cdot \overline{\mathfrak{g}}$ is simple, so $\overline{\mathfrak{g}}$ is simple by Corollary C.1.2(2). Now if $H \subset \overline{G}$ is any closed normal subgroup, it has the structure of a compact p-adic Lie subgroup by [59, Ch. III, Théorème 3.2.3]. In particular, $\mathfrak{h} := \operatorname{Lie} H$ is an ideal of $\overline{\mathfrak{g}}$, hence $\mathfrak{h} = 0$ or $\mathfrak{h} = \overline{\mathfrak{g}}$. By arguing with the exponential map, we see that H is either finite or has finite index; but if finite it is trivial by Lemma C.2.2(2), so the lemma is proved. ☐

In fact, we extract the following more precise statement in the four-dimensional case.

Proposition C.2.4. *Let $G \subset \mathrm{GSp}_4(E)$ be a compact p-adic Lie subgroup, such that the defining representation is strongly irreducible. Write $\mathfrak{h} := \mathrm{Lie}(G/Z_G) \subset \mathfrak{sp}_{4,E}$. Then after a finite extension of E:*

(1) $E \cdot \mathfrak{h}$ is isomorphic to either $\mathfrak{sp}_{4,E}$ or $\mathfrak{sl}_{2,E}$.

(2) In the latter case G is contained in the image of the symmetric cube representation $\mathrm{Sym}^3 : \mathrm{GL}_2(E) \to \mathrm{GL}_4(E)$ up to $\mathrm{GL}_4(E)$-conjugacy.

Proof. We have seen (1) in the proof of Lemma C.2.3, so we prove (2). Let V be the four-dimensional defining representation of G; as a representation of $E \cdot \mathfrak{h} \cong \mathfrak{sl}_{2,E}$, V is isomorphic to the symmetric cube. In particular, after extending E if necessary, the embedding $E \cdot \mathfrak{h} \cong \mathfrak{sl}_{2,E} \hookrightarrow \mathfrak{sp}_{4,E}$ is conjugate to $\operatorname{Lie} \mathrm{Sym}^3$ by some $g \in \mathrm{GL}_4(E)$. We may assume without loss of generality that $g = 1$. Let $S \subset \mathrm{GSp}_4(E)$ be the image of the symmetric cube embedding over $\overline{\mathbb{Q}}_p$; we first claim that G is contained in $S \cdot E^\times \subset \mathrm{GL}_4(E)$. Indeed, for any $g \in G$, $\mathrm{Ad}(g)$ preserves $\mathfrak{sl}_{2,E} = E \cdot \mathfrak{h}$. Since the automorphism group of $\mathfrak{sl}_{2,E}$ is $\mathrm{PGL}_2(E)$, for each $g \in G$ there exists $h \in S$ such that $\mathrm{Ad}(h) = \mathrm{Ad}(g)$ on $\mathfrak{sl}_{2,E}$. In particular $h^{-1}g \in \mathrm{GL}_4(E)$ commutes with an open subgroup of G (by arguing with the exponential map), so by Schur's Lemma and strong irreducibility $h^{-1}g$ is scalar. So $G \subset S \cdot E^\times$, as desired.

If E' denotes the compositum of the finitely many cubic extensions of E, then $S \cdot E^\times$ is contained in the image of the symmetric cube map $\mathrm{GL}_2(E') \to \mathrm{GL}_4(E')$, and this completes the proof. ☐

The following lemma is a corollary of [16, Lemma 4.3].

Lemma C.2.5. *Fix a number field F, and let $\rho : G_F \to \mathrm{GL}_E(V)$ be a continuous, absolutely irreducible representation of G_F. Assume there exists a place $\mathfrak{p}|p$ of F such that $V|_{G_{F_\mathfrak{p}}}$ is Hodge-Tate with distinct weights. Then, after possibly replacing E by a finite extension, there exists a number field $K \supset F$ and a strongly irreducible, continuous representation $\rho_0 : G_K \to \mathrm{GL}_E(V_0)$ such that $\rho \cong \mathrm{Ind}_{G_K}^{G_F} \rho_0$.*

Proof. After taking a finite extension of E, there exists a Galois extension K of F such that each constituent of $\rho|_{G_K}$ is strongly irreducible. Write

$$\rho|_{G_K}^{ss} = \bigoplus_{i=0}^{j} \rho_i$$

for some $0 \leq j < n$. Then the ρ_i are all distinct because ρ has distinct Hodge-Tate weights, so there is a well-defined action of $\mathrm{Gal}(K/F)$ on the set of ρ_i's. This action must be transitive or else ρ would be reducible; hence $\rho|_{G_K}$ is semisimple and each ρ_i has the same dimension m. Replacing K by the fixed field of the stabilizer of ρ_0, it follows that $\rho = \mathrm{Ind}_{G_K}^{G_F} \rho_0$. ☐

Corollary C.2.6. *Let π be a relevant, non-endoscopic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ such that $\mathrm{BC}(\pi)$ (Lemma 2.2.17) is not an automorphic induction. Then for each isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ with $p > 3$, $V_{\pi,\iota}$ is strongly irreducible.*

Proof. By Lemma 2.2.12, $V_{\pi,\iota}$ is absolutely irreducible. Suppose for contradiction that it is not strongly irreducible. By Lemma C.2.5, we may assume that $V_{\pi,\iota} = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \rho_0$, where $K/\mathbb{Q}$ is either quartic or quadratic and ρ_0 is strongly irreducible and Hodge-Tate. If K is quartic, ρ_0 corresponds to Hecke character of K with algebraic infinity type via the usual recipe, and $\mathrm{BC}(\pi)$ is the automorphic induction of $\chi|\cdot|^{-1/2}$, a contradiction.

If K is quadratic, then $V_{\pi,\iota} \cong V_{\pi,\iota} \otimes \omega_{K/\mathbb{Q}}$ where $\omega_{K/\mathbb{Q}}$ is the quadratic character of $G_{\mathbb{Q}}$ corresponding to K. Hence $\mathrm{BC}(\pi) \cong \mathrm{BC}(\pi) \otimes \omega_{K/\mathbb{Q}}$ by strong multiplicity one for GL_4, so by [3, Theorem 4.2(b)] $\mathrm{BC}(\pi)$ is an automorphic induction. This is a contradiction, so indeed $V_{\pi,\iota}$ is strongly irreducible, as desired. ☐

C.2.7. For the next corollary, we use the following notation. Let $\varpi \in O_E$ be a uniformizer; then for an O_E-lattice T in an E-vector space V, we write $T_n \coloneqq T/\varpi^n T$ for all $n \geq 1$.

Corollary C.2.8. *Let F be a number field, and let (ρ, V) be as in Lemma C.2.5 above. Assume, if $\dim V = 1$, that there exists a place $\mathfrak{p}|p$ of F such that the character $\rho|_{G_{F_\mathfrak{p}}}$ has nonzero Hodge-Tate weight. Then for any Galois-stable O_E-lattice $T \subset V$, there exists a constant $C \geq 0$ such that*

$$\varpi^C H^1(F(\rho)/F, T_n) = 0$$

for all $n \geq 1$.

Proof. Without loss of generality, we can extend E so that the conclusion of Lemma C.2.5 holds, for some finite extension K/F and some $\rho_0 : G_K \to \mathrm{GL}_E(V_0)$. Let K^c be the Galois closure of K; then by inflation-restriction, it suffices to show

$$H^1(K^c(\rho)/K^c, T_n) \text{ is uniformly bounded in } n. \tag{C.2}$$

We label the $\mathrm{Gal}(K^c/F)$-conjugates of (ρ_0, V_0) as (ρ_i, V_i), for $0 \leq i < \dim V/\dim V_0$, and let

$$G \subset \prod_i \mathrm{GL}_E(V_i)$$

be the image of G_{K^c} under ρ. We assume without loss of generality that $T = \oplus T_i$ for Galois-stable O_E-lattices $T_i \subset V_i$. Hence to show (C.2), it suffices to show $H^1(G, T_{0,n})$ is uniformly bounded in n.

Case 1. Z_G contains an element z that acts nontrivially on V_0.

Then by inflation-restriction, we have an exact sequence

$$0 \to H^1(G/\langle z\rangle, T_{0,n}^z) \to H^1(G, T_{0,n}) \to H^1(\langle z\rangle, T_{0,n})^{G/\langle z\rangle}.$$

The outer terms are clearly uniformly bounded, so we are done in this case.

Case 2. Z_G acts trivially on V_0.

For this case, note that G/Z_G is a compact p-adic Lie group with semisimple Lie algebra; indeed, if $G_i = \rho_i(G_{K^c})$ with center $Z_{G_i} \subset G_i$, then we have an injection

$$\mathrm{Lie}(G/Z_G) \hookrightarrow \prod_i \mathrm{Lie}(G_i/Z_{G_i}), \tag{C.3}$$

and the semisimplicity of $\mathrm{Lie}(G/Z_G)$ follows by Goursat's Lemma and Lemma C.2.2(4). In particular, by [29, Lemma B.1], $H^1(G/Z_G, T_{0,n})$ is uniformly bounded in n. By inflation-restriction again, it then suffices to show

$$H^1(Z_G, T_{0,n})^{G/Z_G} = \mathrm{Hom}_G(Z_G, T_{0,n})$$

is uniformly bounded. Since G acts trivially on Z_G and V_0 is strongly irreducible, it suffices to ensure V_0 is not the trivial representation of G. However, if this occurs then $\rho_0 : G_K \to \mathrm{GL}_E(V_0)$ is a finite-order character, so all its Galois conjugates ρ_i also have finite order. This would mean that $\rho|_{G_{F_v}}$ has trivial Hodge-Tate weights, which is ruled out by our assumptions on ρ.

☐

Lemma C.2.9. *Let F be a number field, and let $\rho : G_F \to \mathrm{GL}_E(V)$ be as in Lemma C.2.5 above, so after an extension of scalars we can write*

$$\rho \cong \mathrm{Ind}_{G_K}^{G_F} \rho_0$$

for a number field $K \supset F$ and a strongly irreducible representation $\rho_0 : G_K \to \mathrm{GL}_E(V_0)$. Let L be an abelian Galois extension of F (possibly infinite) which is disjoint from K; then $\rho|_{G_L}$ is absolutely irreducible.

Proof. Let K^c be the Galois closure of K; then

$$\rho|_{G_{K^c}} = \bigoplus \rho_0^\sigma|_{G_{K^c}},$$

where ρ_0^σ runs over the distinct $\mathrm{Gal}(K^c/K)$-conjugates of ρ_0. In particular, if v is a prime of K^c lying over the $\mathfrak{p}$ from Lemma C.2.5, then

$$\rho_0^\sigma|_{G_{K^c_v}} \text{ and } \rho_0|_{G_{K^c_v}} \text{ have distinct Hodge-Tate weights if } \rho_0^\sigma \not\cong \rho_0. \tag{C.4}$$

If $\rho|_{G_L}$ is reducible (after replacing E by any finite extension), then

$$\begin{aligned} 1 &< \dim_E \mathrm{Hom}_{E[G_L]} \left((\mathrm{Ind}_{G_K}^{G_F} \rho_0)|_{G_L}, (\mathrm{Ind}_{G_K}^{G_F} \rho_0)|_{G_L} \right) \\ &= \dim_E \mathrm{Hom}_{E[G_L]} \left(\mathrm{Ind}_{G_{KL}}^{G_L} \rho_0|_{G_{KL}}, \mathrm{Ind}_{G_{KL}}^{G_L} \rho_0|_{G_{KL}} \right) \\ &= \dim_E \mathrm{Hom}_{E[G_{KL}]} \left(\rho_0|_{G_{KL}}, \mathrm{Res}_{G_{KL}}^{G_L} \mathrm{Ind}_{G_{KL}}^{G_L} \rho_0|_{G_{KL}} \right) \\ &\leq \dim_E \mathrm{Hom}_{E[G_{K^cL}]} \left(\rho_0|_{G_{K^cL}}, \bigoplus \rho_0^\sigma|_{G_{K^cL}} \right). \end{aligned}$$

In particular, we may fix $\sigma \in \mathrm{Gal}(K^c/K)$ such that $\rho_0^\sigma \not\cong \rho_0$ but

$$\mathrm{Hom}_{E[G_{K^cL}]}(\rho_0|_{G_{K^cL}}, \rho_0^\sigma|_{G_{K^cL}}) \neq 0.$$

We claim $\rho_0|_{G_{K^cL}}$ is absolutely irreducible; indeed, if $G_0 = \rho_0(G_{K^c})$, then $H = \rho_0(G_{K^cL})$ is a normal subgroup of G_0 with abelian cokernel. Then $\mathrm{Lie}(H \cap \mathrm{SL}_E(V_0)) \subset \mathrm{Lie}(G_0 \cap \mathrm{SL}_E(V_0))$ has abelian cokernel, which implies $\mathrm{Lie}(H \cap \mathrm{SL}_E(V_0)) = \mathrm{Lie}(G_0 \cap \mathrm{SL}_E(V_0))$ by Lemma C.2.2(4, 5). Then $H \cap \mathrm{SL}_E(V_0)$ acts strongly irreducibly on V_0 by Lemma C.2.2(6), so *a fortiori* $\rho_0|_{G_{K^cL}}$ is absolutely irreducible, as desired.

This implies that $\mathrm{Hom}_{E[G_{K^cL}]}(\rho_0, \rho_0^\sigma)$, which is nonzero by assumption, is in fact one-dimensional. It is also preserved by the natural action of G_{K^c} on $\mathrm{Hom}_E(\rho_0, \rho_0^\sigma)$, because G_{K^cL} is normal in G_{K^c}. Hence $\mathrm{Gal}(K^cL/K^c)$ acts on $\mathrm{Hom}_{E[G_{K^cL}]}(\rho_0, \rho_0^\sigma)$ by scalars, and in particular we conclude that

$$\rho_0|_{G_{K^c}} \cong \rho_0^\sigma|_{G_{K^c}} \otimes \chi \tag{C.5}$$

for a character χ of $\mathrm{Gal}(K^cL/K^c) \subset \mathrm{Gal}(L/F)$. Because $\mathrm{Gal}(K^c/K)$ acts trivially by conjugation on $\mathrm{Gal}(L/F)$ and $\sigma \in \mathrm{Gal}(K^c/K)$ has finite order, (C.5) implies

$$\rho_0 \cong \rho_0 \otimes \chi^n \text{ for some } n \geq 1,$$

hence χ has finite order. But then (C.5) contradicts (C.4), so the lemma is proved. □

C.3. **Galois representations associated to Hilbert modular forms.**

C.3.1. Fix a totally real field F. The following result is due to Nekovar:

Theorem C.3.2. *Let π be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to a non-CM Hilbert modular form of weight $(2k_v)_{v|\infty}$, with each $k_v \geq 1$. If E_0 is a strong coefficient field for π, then there exists a subfield $E_1 \subset E_0$ and a quaternion algebra D over E_1, along with a finite abelian extension K of F, such that for all primes $\mathfrak{p}$ of E_0, with residue characteristic p:*

(1) The image of $\rho_{\pi,\mathfrak{p}}$ contains an open subgroup of

$$H_{\mathfrak{p}} := \left\{ x \in (D \otimes_{E_1} E_{1,\mathfrak{p}})^\times \ : \ \mathrm{Nm}(x) \in \mathbb{Q}_p^\times \right\},$$

where the embedding $H_{\mathfrak{p}} \hookrightarrow \mathrm{GL}_2(E_{0,\mathfrak{p}})$ is induced by the natural embedding $D \otimes_{E_1} E_{1,\mathfrak{p}} \hookrightarrow D \otimes_{E_1} E_{0,\mathfrak{p}}$ and an isomorphism $D \otimes_{E_1} E_{0,\mathfrak{p}} \simeq M_2(E_{0,\mathfrak{p}})$.

(2) The image $\rho_{\pi,\mathfrak{p}}(G_K)$ is contained in $H_{\mathfrak{p}}$.

Moreover, for any finite abelian extension K'/K and all but finitely many $\mathfrak{p}$, the image of $\rho_{\pi,\mathfrak{p}}(G_{K'})$ is a conjugate of

$$\left\{ g \in \mathrm{GL}_2(O_{E_{1,\mathfrak{p}}}) \ : \ \det g \in \mathbb{Z}_p^\times \right\}.$$

Proof. Let K be the field written F_Γ in [80, Theorem B.5.2]. Then the first two claims, and the last part when $K' = K$, follow from *loc. cit.*

For the general case, note that $\det \rho_{\pi,\mathfrak{p}}|_{G_K} = \chi_{p,\mathrm{cyc}}$. If K'/K is a finite abelian extension, restrict to those primes $\mathfrak{p}$ such that $\rho_{\pi,\mathfrak{p}}(G_K)$ contains $\mathrm{SL}_2(O_{E_1,\mathfrak{p}})$ and $K' \cap \mathbb{Q}(\mu_{p^\infty}) = K \cap \mathbb{Q}(\mu_{p^\infty})$. Then $\det \rho_{\pi,\mathfrak{p}}(G_K) = \det \rho_{\pi,\mathfrak{p}}(G_{K'})$. On the other hand, $\rho_{\pi,\mathfrak{p}}(G_{K'})$ is a normal subgroup of $\rho_{\pi,\mathfrak{p}}(G_K)$ with abelian cokernel, which necessarily contains $\mathrm{SL}_2(O_{E_1,\mathfrak{p}})$; it follows that $\rho_{\pi,\mathfrak{p}}(G_K) = \rho_{\pi,\mathfrak{p}}(G_{K'})$, which completes the proof. ☐

C.3.3. Now let π_1 and π_2 be two automorphic representations as in Theorem C.3.2, with E_0 a common strong coefficient field. For the rest of this section we will always write p for the residue characteristic of a prime $\mathfrak{p}$ of E_0. Also, let E_1, D_1, K_1, $H_{1,\mathfrak{p}}$, E_2, D_2, K_2, and $H_{2,\mathfrak{p}}$ be as in the conclusion of Theorem C.3.2 applied to π_1 and π_2, respectively; we can and do fix a finite abelian extension $K \supset K_1 \cdot K_2$ such that

$$\det \rho_{\pi_1,\mathfrak{p}}|_{G_K} = \det \rho_{\pi_2,\mathfrak{p}}|_{G_K} = \chi_{p,\mathrm{cyc}} \tag{C.6}$$

for all $\mathfrak{p}$, and the conclusions of Theorem C.3.2 hold for this choice of K.

We will also consider the joint representation

$$\rho_{\pi_1,\pi_2,\mathfrak{p}} : G_F \xrightarrow{\rho_{\pi_1,\mathfrak{p}} \times \rho_{\pi_2,\mathfrak{p}}} \mathrm{GL}_2(E_{0,\mathfrak{p}}) \times \mathrm{GL}_2(E_{0,\mathfrak{p}}). \tag{C.7}$$

Lemma C.3.4. *Suppose there exists a prime $\mathfrak{p}$ of E_0 and a finite extension L/F such that*

$$\rho_{\pi_1,\mathfrak{p}}|_{G_L} \cong \rho_{\pi_2,\mathfrak{p}}|_{G_L}.$$

Then π_1 is the twist of π_2 by a finite-order automorphic character of $\mathbb{A}_F^\times$.

Proof. By [90, Theorem 2], we have

$$\rho_{\pi_1,\mathfrak{p}} = \rho_{\pi_2,\mathfrak{p}} \otimes \chi$$

for some character χ of G_F, which is of finite order because it vanishes on G_L. Viewing χ as a finite-order character of $F^\times \backslash \mathbb{A}_F^\times$ via class field theory, we conclude that $\rho_{\pi_1,\mathfrak{p}} = \rho_{\pi_2 \otimes \chi,\mathfrak{p}}$, and hence $\pi_1 = \pi_2 \otimes \chi$. ☐

C.3.5. If π is as in Theorem C.3.2, corresponding to a holomorphic Hilbert modular form f, then for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we write π^σ for the automorphic representation corresponding to f^σ. The following result generalizes [70] to the setting of Theorem C.3.2.

Theorem C.3.6. *In the setting of (C.3.3), suppose $\pi_1 \neq \pi_2^\sigma \otimes \chi$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any finite-order automorphic character χ of $\mathbb{A}_F^\times$. Then:*

(1) For all primes $\mathfrak{p}$ of E_0, $\rho_{\pi_1,\pi_2,\mathfrak{p}}(G_K)$ contains an open subgroup of $H_{1,\mathfrak{p}} \times_{\mathbb{Q}_p^\times} H_{2,\mathfrak{p}}$.

(2) For all but finitely many $\mathfrak{p}$, the image of $\rho_{\pi_1,\pi_2,\mathfrak{p}}(G_K)$ contains a conjugate of

$$\left\{ (g_1, g_2) \in \mathrm{GL}_2(O_{E_1,\mathfrak{p}}) \times \mathrm{GL}_2(O_{E_2,\mathfrak{p}}) : \ \det g = \det h \in \mathbb{Z}_p^\times \right\}.$$

Proof. The proof is analogous to [70, Theorem 3.2.2, Proposition 3.3.2], where we replace Lemma 3.1.1 of *op. cit.* with Lemma C.3.4 above. For completeness, we recall the argument. Let $G_{\mathfrak{p}} = \rho_{\pi_1,\pi_2,\mathfrak{p}}(G_K)$, so that we have a natural embedding

$$G_{\mathfrak{p}} \hookrightarrow H_{1,\mathfrak{p}} \times_{\mathbb{Q}_p^\times} H_{2,\mathfrak{p}}.$$

It follows from Goursat's Lemma for Lie algebras that $\operatorname{Lie} G_{\mathfrak{p}}$ is either $\operatorname{Lie}\left(H_{1,\mathfrak{p}} \times_{\mathbb{Q}_p^\times} H_{2\mathfrak{p}}\right)$ or $\operatorname{Lie} G_{1,\mathfrak{p}} = \operatorname{Lie} H_{1,\mathfrak{p}}$, diagonally embedded by an isomorphism $\operatorname{Lie} H_{1,\mathfrak{p}} \xrightarrow{\sim} \operatorname{Lie} H_{2,\mathfrak{p}}$ that preserves the linearized determinant maps to $\mathbb{Q}_p$. To prove (1), we assume that we are in the latter case, and aim to show π_1 is a conjugate twist of π_2.

By [70, Lemma 1.1.4], any isomorphism $\operatorname{Lie} H_{1,\mathfrak{p}} \xrightarrow{\sim} \operatorname{Lie} H_{2,\mathfrak{p}}$ is induced by an isomorphism $i : E_{1,\mathfrak{p}} \xrightarrow{\sim} E_{2,\mathfrak{p}}$ and an i-linear isomorphism $D_1 \otimes_{E_1} E_{1,\mathfrak{p}} \xrightarrow{\sim} D_2 \otimes_{E_2} E_{2,\mathfrak{p}}$. In particular, assuming without loss of generality that E_0 is Galois, there exists an automorphism $\sigma \in \operatorname{Gal}(E_0/\mathbb{Q})$ that preserves $\mathfrak{p}$ and induces $i : E_{1,\mathfrak{p}} \xrightarrow{\sim} E_{2,\mathfrak{p}}$. Since all automorphisms of $M_2(E_{0,\mathfrak{p}})$ are inner, it follows from the description of the embedding in Theorem C.3.2(1) that, after conjugating $\rho_{\pi_2,\mathfrak{p}}$, $\operatorname{Lie} G_{\mathfrak{p}} \subset \mathfrak{gl}_2(E_{0,\mathfrak{p}}) \times \mathfrak{gl}_2(E_{0,\mathfrak{p}})$ is contained in a subalgebra of the form

$$\{(X, \sigma X) \, : X \in \mathfrak{gl}_2(E_{0,\mathfrak{p}})\}.$$

Exponentiating, for some finite extension L/K we have

$$\rho_{\pi_1,\mathfrak{p}}|_{G_L} = \sigma \circ \rho_{\pi_2,\mathfrak{p}}|_{G_L}.$$

Since $\sigma \circ \rho_{\pi_2,\mathfrak{p}} = \rho_{\pi_2^\sigma,\mathfrak{p}}$, Lemma C.3.4 concludes the proof of (1).

For (2), we restrict our attention to those $\mathfrak{p}$ such that $\rho_{\pi_i,\mathfrak{p}}(G_K) = \mathrm{GL}_2(O_{E_i,\mathfrak{p}})$ (after conjugating) for $i = 1, 2$, which eliminates only finitely many primes of E_0 by Theorem C.3.2. Let S be the set of primes $\mathfrak{p}$ as above such that the conclusion of (2) does *not* hold; we assume for contradiction that S is infinite. By [70, Proposition 3.2.1], for all $\mathfrak{p} \in S$, we have an element $\sigma \in \operatorname{Gal}(E_0/\mathbb{Q})$ preserving $\mathfrak{p}$ such that, after conjugating $\rho_{\pi_2,\mathfrak{p}}$:

$$\rho_{\pi_1,\mathfrak{p}}(g) = \pm\sigma \circ \rho_{\pi_2,\mathfrak{p}}(g) \pmod{\mathfrak{p}}, \ \forall g \in G_K. \tag{C.8}$$

If S is infinite, then there exists a single $\sigma \in \operatorname{Gal}(E_0/\mathbb{Q})$ such that (C.8) holds for infinitely many of the $\mathfrak{p} \in S$ fixed by σ.

Let Σ be the set of primes v of F such that either $\pi_{1,v}$ or $\pi_{2,v}$ is ramified, and for $v \notin \Sigma$, let $a_{1,v}$ and $a_{2,v}$ be the eigenvalues of the standard Hecke operator at v on the spherical vectors of $\pi_{1,v}$ and $\pi_{2,v}$, respectively. By (C.8), for all $v \notin \Sigma$ that split completely in K/F, $a_{1,v}^2 - \sigma(a_{2,v})^2 \in O_{E_0}$ is divisible by infinitely many primes $\mathfrak{p} \in S$, hence vanishes. Now take a single $\mathfrak{p} \in S$ fixed by σ such that (C.8) holds, and assume without loss of generality that $p \neq 2$. Let K' be the compositum of the fixed fields of $\overline{\rho}_{\pi_1,\mathfrak{p}}|_{G_K}$ and $\overline{\rho}_{\pi_2,\mathfrak{p}}|_{G_K}$. Then for all $v \notin \Sigma$ that split completely in K',

$$a_{1,v} \equiv \sigma(a_{2,v}) \not\equiv 0 \pmod{\mathfrak{p}},$$

so the identity $a_{1,v}^2 - \sigma(a_{2,v})^2 = 0$ implies $a_{1,v} = \sigma(a_{2,v})$. In particular, the traces of $\rho_{\pi_1,\mathfrak{p}}|_{G_{K'}}$ and $\sigma \circ \rho_{\pi_2,\mathfrak{p}}|_{G_{K'}}$ coincide, so we have $\rho_{\pi_1,\mathfrak{p}}|_{G_{K'}} \cong \sigma \circ \rho_{\pi_2,\mathfrak{p}}|_{G_{K'}}$. Now we can conclude by Lemma C.3.4. □

We also have the following complementary result.

Corollary C.3.7. *In the setting of (C.3.3), there exists a finite abelian extension* L *of* K *with the following property:*

(1) *For all primes* $\mathfrak{p}$ *of* E_0*, either* $\rho_{\pi_1,\pi_2,\mathfrak{p}}(G_L)$ *is an open subgroup of* $H_{1,\mathfrak{p}} \times_{\mathbb{Q}_p^\times} H_{2,\mathfrak{p}}$*, or there exists an isomorphism* $\sigma_{\mathfrak{p}} : E_{1,\mathfrak{p}} \xrightarrow{\sim} E_{2,\mathfrak{p}}$ *and a* $\sigma_{\mathfrak{p}}$*-linear isomorphism* $i_{\mathfrak{p}} : D_1 \otimes_{E_1} E_{1,\mathfrak{p}} \xrightarrow{\sim} D_2 \otimes_{E_2} E_{2,\mathfrak{p}}$ *such that* $\rho_{\pi_1,\pi_2,\mathfrak{p}}(G_L)$ *is an open subgroup of*

$$H_{1,\mathfrak{p}} \xhookrightarrow{\mathrm{id}, i_{\mathfrak{p}}} H_{1,\mathfrak{p}} \times_{\mathbb{Q}_p^\times} H_{2,\mathfrak{p}}.$$

(2) *For all but finitely many* $\mathfrak{p}$*, either the image of* $\rho_{\pi_1,\pi_2,\mathfrak{p}}(G_L)$ *is a conjugate of*

$$\left\{(g_1, g_2) \in \mathrm{GL}_2(O_{E_1,\mathfrak{p}}) \times \mathrm{GL}_2(O_{E_2,\mathfrak{p}}) : \det g = \det h \in \mathbb{Z}_p^\times\right\},$$

or there exists an isomorphism $\sigma_\mathfrak{p} : E_{1,\mathfrak{p}} \xrightarrow{\sim} E_{2,\mathfrak{p}}$ *such that* $\rho_{\pi_1,\pi_2,\mathfrak{p}}(G_L)$ *is a conjugate of*

$$\left\{(g, \sigma_\mathfrak{p}(g)) \in \mathrm{GL}_2(O_{E_{1,\mathfrak{p}}}) \times \mathrm{GL}_2(O_{E_{2,\mathfrak{p}}}) \,:\, g \in \mathrm{GL}_2(O_{E_{1,\mathfrak{p}}}),\, \det g \in \mathbb{Z}_p^\times\right\}.$$

Proof. By Theorem C.3.6, we may assume without loss of generality that $\pi_1 \cong \pi_2^\sigma \otimes \chi$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and some finite-order automorphic character χ of $\mathbb{A}_F^\times$, which we also view as a character of G_F via class field theory. Then for all primes $\mathfrak{p}$ of E_0,

$$\rho_{\pi_2,\mathfrak{p}} \otimes_{E_{0,\mathfrak{p}},\sigma} E_{0,\sigma(\mathfrak{p})} \cong \rho_{\pi_2^\sigma,\sigma(\mathfrak{p})} \cong \rho_{\pi_1,\sigma(\mathfrak{p})} \otimes \chi.$$

By [80, Theorem B.4.10], for all but finitely many $\mathfrak{p}$ the image of $\rho_{\pi_2^\sigma,\pi_2,\mathfrak{p}}(G_{K_0})$ is one of the groups listed in part (2), for a certain abelian extension K_0 of F such that $\det_{\rho_{\pi_2,\mathfrak{p}}}|_{G_{K_0}} = \chi_{p,\mathrm{cyc}}$. If L is the compositum of K_0 with K and with the fixed field of the kernel of χ, it follows from the same argument as in Theorem C.3.2 that the image of $\rho_{\pi_2^\sigma,\pi_2,\mathfrak{p}}(G_L) = \rho_{\pi_1,\pi_2,\mathfrak{p}}(G_L)$ coincides with that of $\rho_{\pi_2^\sigma,\pi_2,\mathfrak{p}}(G_{K_0})$ for all but finitely many $\mathfrak{p}$, and this proves (2). ☐

C.4. **Large image for relevant representations.**

C.4.1. Fix a relevant automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, with trivial central character and with strong coefficient field E_0. In this subsection, we prove some results on the image of the Galois representation $\rho_{\pi,\mathfrak{p}}$ associated to π, with an eye towards studying the existence of admissible elements (Definition 4.2.1) and assumption (R1) from (9.3.1). Throughout this section we write p for the residue characteristic of a prime $\mathfrak{p}$ of E_0.

Lemma C.4.2. *Suppose* π *is not endoscopic, and* $\mathrm{BC}(\pi)$ *is the symmetric cube lift of a non-CM automorphic representation* π_0 *of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$*. Consider the map of algebraic groups*

$$f = \mathrm{Sym}^3 \otimes \det{}^{-1} : \mathrm{GL}_2 \to \mathrm{GSp}_4\,.$$

For all but finitely many primes $\mathfrak{p}$ *of* E_0*, the image of* $\rho_{\pi,\mathfrak{p}}$ *contains a conjugate of* $f(\mathrm{GL}_2(\mathbb{Z}_p))$*. In particular, for all but finitely many* $\mathfrak{p}$*, admissible elements exist for* $\rho_{\pi,\mathfrak{p}}$*.*

Proof. By Lemma 2.2.18, $\pi_{0,\infty}$ is discrete series of weight 2. Without loss of generality, extend E_0 so that it is also a strong coefficient field for π_0. Then for all primes $\mathfrak{p}$ of E_0, $\rho_{\pi,\mathfrak{p}} = \mathrm{Sym}^3 \rho_{\pi_0,\mathfrak{p}}(-1)$. Comparing similitude characters, we see that the central character of π_0 is cubic; by twisting, we may assume without loss of generality that it is trivial. Then the claim about the image of $\rho_{\pi,\mathfrak{p}}$ follows from [94, Theorem 3.1]. Restricting to these $\mathfrak{p}$, if p is sufficiently large we may fix $z \in \mathbb{Z}_p^\times$ with

$$z \not\equiv \pm 1, \pm z^3, z^6, z^{-3} \pmod{p}, \quad z^{12} \not\equiv 1 \pmod{p}. \tag{C.9}$$

Then applying f to the diagonal matrix $\begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}$, it follows that the image of $\rho_{\pi,\mathfrak{p}}$ contains a matrix with eigenvalues $\{1, z, z^2, z^3\}$; in particular, admissible elements exist for $\rho_{\pi,\mathfrak{p}}$. ☐

Lemma C.4.3. *Suppose* π *is not endoscopic, and* $\mathrm{BC}(\pi)$ *is neither a (weak) symmetric cube lift nor a (weak) automorphic induction. For all but finitely many primes* $\mathfrak{p}$ *of* E_0*:*

(1) *The image of* $\overline{\rho}_{\pi,\mathfrak{p}}$ *contains a conjugate of* $\mathrm{Sp}_4(\mathbb{F}_p)$*.*
(2) *If* $E_{0,\mathfrak{p}} = \mathbb{Q}_p$*, the image of* $\overline{\rho}_{\pi,\mathfrak{p}}$ *is a conjugate of* $\mathrm{GSp}_4(\mathbb{F}_p)$*.*

Moreover, for all primes $\mathfrak{p}$ *with* $p > 3$*, the Zariski closure (over* $E_{0,\mathfrak{p}}$*) of the image of* $\rho_{\pi,\mathfrak{p}}$ *is equal to* $\mathrm{GSp}_4(E_{0,\mathfrak{p}})$*.*

Proof. Part (1) is [123, Theorem 1.2(ii)]. Part (2) follows immediately. For the final claim, by Proposition C.2.4(2) and Corollary C.2.6, it suffices to rule out the case that $\rho_{\pi,\mathfrak{p}}$ factors through the image of the symmetric cube representation. In this case, we can write $\rho_{\pi,\mathfrak{p}}(1) = \mathrm{Sym}^3 \rho_0$, for some $\rho_0 : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)/\mu_3$ with Zariski-dense image (over E). By Lemma C.4.4 below, ρ_0 lifts to

$$\rho_0 : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p).$$

Comparing similitude factors, we see that $\det \rho_0/\chi_{p,\mathrm{cyc}}$ is cubic; so after twisting, we may assume without loss of generality that $\det \rho_0 = \chi_{p,\mathrm{cyc}}$. For all but finitely many primes ℓ, we have $\rho_0(I_\ell) \subset \mu_3$, so the determinant condition implies ρ_0 is unramified almost everywhere. By [87, Corollary 3.2.13], $\rho_0|_{G_{\mathbb{Q}_p}}$ is geometric. Since ρ_0 is also clearly odd, [86, Theorem 1.0.4] implies ρ_0 arises from a modular form, hence $\mathrm{BC}(\pi)$ is a symmetric cube lift, and this concludes the proof of the final claim. ☐

Lemma C.4.4. *For all n, we have $H^2(\mathbb{Q}, \mathbb{Z}/n\mathbb{Z}) = 0$.*

Proof. This lemma is well-known, but we were unable to find a reference. Without loss of generality, $n = p$ is prime. By a theorem of Tate [87, Theorem 2.1.1], $H^2(\mathbb{Q}, \mathbb{Z}_p/\mathbb{Q}_p) = 0$. On the other hand, by the Kronecker-Weber theorem, $H^1(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(\widehat{\mathbb{Z}}, \mathbb{Q}_p/\mathbb{Z}_p)$ is p-divisible. So the lemma follows from the long exact sequence

$$\cdots \to H^1(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\times p} H^1(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^2(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) \to H^2(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p) \to \cdots .$$

☐

C.4.5. For an automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ as in Theorem C.3.2 with $K/\mathbb{Q}$ real quadratic, let π_0^{tw} denote the $\mathrm{Gal}(K/\mathbb{Q})$-twist. We say π_0 is *exceptional* if there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and a finite-order automorphic character χ of $\mathbb{A}_K^\times$ such that

$$\pi_0^{\mathrm{tw}} \cong \pi_0^\sigma \otimes \chi.$$

Lemma C.4.6. *Suppose π is not endoscopic, and $\mathrm{BC}(\pi)$ is the automorphic induction of a non-CM automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ with $K/\mathbb{Q}$ real quadratic. Then for all but finitely many primes $\mathfrak{p}$ of E_0, the following hold.*

1. *The image of $\rho_{\pi,\mathfrak{p}}$ contains a conjugate of $\mathrm{GL}_2(\mathbb{Z}_p)$, embedded diagonally via*
$$\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4 .$$
2. *If p splits in K or π_0 is not exceptional, the image of $\rho_{\pi,\mathfrak{p}}$ contains a conjugate of $\mathrm{GL}_2(\mathbb{Z}_p) \times_{\mathbb{Z}_p^\times} \mathrm{GL}_2(\mathbb{Z}_p)$.*

Proof. Recall from Lemma 2.2.19 that π_0 is the automorphic representation associated to a Hilbert modular form of weights $(2, 4)$. Assume without loss of generality that E_0 is Galois and is also a strong coefficient field for π_0^{tw}; then

$$\rho_{\pi,\mathfrak{p}}|_{G_K} = \rho_{\pi_0,\mathfrak{p}} \oplus \rho_{\pi_0^{\mathrm{tw}},\mathfrak{p}}.$$

Hence part (1) follows from Corollary C.3.7(2). For part (2), the non-exceptional case is immediate from Theorem C.3.6(2), so suppose without loss of generality that p splits in K. Then for any $\sigma \in \mathrm{Gal}(E_{0,\mathfrak{p}}/\mathbb{Q}_p)$, and any fixed embedding $j : K \hookrightarrow \mathbb{Q}_p$, the Hodge-Tate weights of $\rho_{\pi_0,\mathfrak{p}}$ and $\sigma \circ \rho_{\pi_0,\mathfrak{p}}$ with respect to j coincide. This rules out that $\rho_{\pi_0^{\mathrm{tw}},\mathfrak{p}}|_{G_L} \cong \sigma \circ \rho_{\pi_0,\mathfrak{p}}|_{G_L}$ for any finite extension L/K, so by Corollary C.3.7(2), we obtain (2). ☐

Lemma C.4.7. *Suppose $\mathrm{BC}(\pi)$ is the automorphic induction of an automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ with $K/\mathbb{Q}$ imaginary quadratic, and π_0 is not an automorphic induction. Then for all but finitely many primes $\mathfrak{p}$ of E_0:*

(1) The image of $\rho_{\pi,\mathfrak{p}}$ contains a conjugate of $\mathrm{SL}_2(\mathbb{Z}_p)$, where $\mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_4$ is embedded into the Levi factor of a Siegel parabolic.

(2) If p splits in K, then admissible elements exist for $\rho_{\pi,\mathfrak{p}}$.

Proof. By Lemma 2.2.20, after possibly extending E_0 we can write

$$\rho_{\pi,\mathfrak{p}}|_{G_K} = \rho_{f,\mathfrak{p}} \otimes \chi_{\mathfrak{p}} \oplus \rho_{f,\mathfrak{p}} \otimes \chi_{\mathfrak{p}}^{\mathrm{tw}} \tag{C.10}$$

for all $\mathfrak{p}$, where:

- $\rho_{f,\mathfrak{p}}$ is the Galois representation attached to a classical modular form f of weight $k = 2$ or 3 with coefficients in E_0; here the normalization is as usual, i.e. $\det \rho_{f,\mathfrak{p}} = \omega_f \cdot \chi_{p,\mathrm{cyc}}^{k-1}$ where ω_f is the nebentype character of f, viewed as a character of $G_{\mathbb{Q}}$ via class field theory.
- $\chi_{\mathfrak{p}}$ is the G_K-representation attached to an algebraic Hecke character χ' of infinity type $(-1, 3-k)$.
- $\chi_{\mathfrak{p}}^{\mathrm{tw}}$ is the $\mathrm{Gal}(K/\mathbb{Q})$-twist of $\chi_{\mathfrak{p}}$, which is also associated to the twist $(\chi')^{\mathrm{tw}}$.

The symplectic form in (C.10) is given by the natural pairing

$$\rho_{f,\mathfrak{p}} \otimes \rho_{f,\mathfrak{p}} \otimes \chi_{\mathfrak{p}} \otimes \chi_{\mathfrak{p}}^{\mathrm{tw}} \to \det \rho_{f,\mathfrak{p}} \otimes \chi_{\mathfrak{p}}\chi_{\mathfrak{p}}^{\mathrm{tw}} = \chi_{p,\mathrm{cyc}}.$$

Let L be the fixed field of ω_f, which is independent of $\mathfrak{p}$. After discarding finitely many primes $\mathfrak{p}$ and changing basis, we may assume by [94, Theorem 3.1] that

$$\rho_{f,\mathfrak{p}}(G_L) = \left\{ g \in \mathrm{GL}_2(O) \ : \ \det g \in (\mathbb{Z}_p^\times)^{k-1} \right\},$$

where O is the ring of integers of a subfield of $E_{0,\mathfrak{p}}$. Then the Galois group

$$\mathrm{Gal}(L(\rho_{f,\mathfrak{p}}) \cap LK(\chi_{p,\mathrm{cyc}}, \chi_{\mathfrak{p}}, \chi_{\mathfrak{p}}^{\mathrm{tw}}, \omega_f)/L(\det(\rho_{f,\mathfrak{p}})))$$

is a solvable quotient of $\mathrm{SL}_2(O)$, hence trivial if p is sufficiently large; so we have

$$L(\rho_{f,\mathfrak{p}}) \cap LK(\chi_{p,\mathrm{cyc}}, \chi_{\mathfrak{p}}, \chi_{\mathfrak{p}}^{\mathrm{tw}}, \omega_f) = L(\det(\rho_{f,\mathfrak{p}})). \tag{C.11}$$

In particular, this immediately implies (1).

For (2), we further restrict to those $\mathfrak{p}$ such that $\chi_{\mathfrak{p}}$ is crystalline at all primes above p. Fix a prime $v|p$ of K, and let $\overline{v}$ be its complex conjugate, with inertia subgroups $I_v, I_{\overline{v}} \subset G_K^{\mathrm{ab}}$; these are disjoint and each naturally identified with $\mathbb{Z}_p^\times$ since we are assuming p is split (and unramified) in K. When restricted to inertia, the characters χ_p^{cyc}, $\chi_{\mathfrak{p}}$, and $\chi_{\mathfrak{p}}^{\mathrm{tw}}$ have the form:

$$\begin{aligned} \chi_p^{\mathrm{cyc}}|_{I_v \times I_{\overline{v}}} \ : \ \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times &\to \mathbb{Z}_p^\times \\ (z_1, z_2) &\mapsto z_1 z_2, \\ \chi_{\mathfrak{p}}|_{I_v \times I_{\overline{v}}} \ : \ \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times &\to \mathbb{Z}_p^\times \subset O_{E,\mathfrak{p}}^\times \\ (z_1, z_2) &\mapsto z_1^{-1} z_2^{3-k} \\ \chi_{\mathfrak{p}}^{\mathrm{tw}}|_{I_v \times I_{\overline{v}}} \ : \ \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times &\to \mathbb{Z}_p^\times \subset O_{E,\mathfrak{p}}^\times \\ (z_1, z_2) &\mapsto z_1^{3-k} z_2^{-1}. \end{aligned}$$

In particular, one can calculate that, for p unramified in L, the image of

$$(\chi_{\mathfrak{p}}, \chi_{\mathfrak{p}}^{\mathrm{tw}}, \chi_{p,\mathrm{cyc}}) : G_L \to O_{E_0,\mathfrak{p}}^\times \times O_{E_0,\mathfrak{p}}^\times \times \mathbb{Z}_p^\times$$

contains a subgroup of

$$\left\{ (a, b, c) \in (\mathbb{Z}_p^\times)^3 \ : \ ab = c^{2-k} \right\}$$

with index at most 2. Comparing with (C.11), we see that the image of

$$(\rho_{f,\mathfrak{p}}, \chi_{\mathfrak{p}}, \chi_{\mathfrak{p}}^{\mathrm{tw}}) : G_L \to \mathrm{GL}_2(O) \times O_{E_0,\mathfrak{p}}^\times \times O_{E_0,\mathfrak{p}}^\times \tag{C.12}$$

contains a subgroup of

$$\left\{(g,x,y)\in \mathrm{GL}_2(O)\times \mathbb{Z}_p^\times\times\mathbb{Z}_p^\times\ :\ \det g\in(\mathbb{Z}_p^\times)^{k-1},\ (xy)^{(k-1)}=(\det g)^{2-k}\right\} \tag{C.13}$$

with index at most 2. Let $n\coloneqq 2(k-1)+2(k-2)=4k-6$, and set

$$S_n\coloneqq\left\{(g,c)\in\mathrm{GL}_2(\mathbb{Z}_p)\times\mathbb{Z}_p^\times\ :\ c,\det g\in(\mathbb{Z}_p^\times)^n\right\}.$$

Then for any $(g,c)\in S_n$, there exists $\lambda\in\mathbb{Z}_p^\times$ satisfying

$$\lambda^n=(\det g)^{k-2}c^{1-k}.$$

It follows from (C.13) that $(g\lambda^{-1},\lambda,c\lambda)^2$ lies in the image of (C.12); hence $(g,cg)^2$ lies in the image of $(\rho_{f,\mathfrak{p}}\otimes\chi_\mathfrak{p},\rho_{f,\mathfrak{p}}\otimes\chi_\mathfrak{p}^{\mathrm{tw}}):G_L\to\mathrm{GL}_2(E_{0,\mathfrak{p}})\times\mathrm{GL}_2(E_{0,\mathfrak{p}})$ for all $(g,c)\in S_n$. If p is sufficiently large, this immediately implies that admissible elements exist for $\rho_{\pi,\mathfrak{p}}$. ☐

Lemma C.4.8. *Suppose π is not endoscopic, and $\mathrm{BC}(\pi)$ is the (weak) automorphic induction of a Hecke character χ_0 of a quartic field $K\subset\mathbb{C}$. Then there exists a constant n such that, for all but finitely many primes $\mathfrak{p}$ of E_0, the following holds:*

1. *$\rho_{\pi,\mathfrak{p}}(G_\mathbb{Q})$ contains the scalar subgroup $(\mathbb{Z}_p^\times)^n\subset\mathrm{GSp}_4(\mathbb{Z}_p)$.*
2. *If p splits completely in the Galois closure K^c of K, then $\rho_{\pi,\mathfrak{p}}(G_\mathbb{Q})$ contains a conjugate of*

$$\left\{\begin{pmatrix}x&&&\\&y&&\\&&z&\\&&&xz/y\end{pmatrix}\ :\ x,y,z\in(\mathbb{Z}_p^\times)^n\right\}\subset\mathrm{GSp}_4(\mathbb{Z}_p).$$

Proof. Let $\chi\coloneqq\chi_0|\cdot|^{1/2}$. From Theorem 2.2.10(1), we see that the local component χ_v of χ takes algebraic values on $K_v^\times$ for cofinitely many primes v of K; hence χ_∞ is algebraic [117, Théorème 3.1]. Extending E_0 if necessary, for all primes $\mathfrak{p}$ of E_0 we have the $\mathfrak{p}$-adic character $\chi_\mathfrak{p}$ associated to χ, and $\rho_{\pi,\mathfrak{p}}=\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}\chi_\mathfrak{p}$ for all $\mathfrak{p}$. We restrict to those $\mathfrak{p}$ such that $K^c/\mathbb{Q}$ is unramified at p, and $\chi_\mathfrak{p}$ is crystalline at all primes $v|p$. The Hodge-Tate weights of $\chi_\mathfrak{p}$ with respect to the four embeddings $i:K\hookrightarrow\overline{\mathbb{Q}}_p$ are $\{-1,0,1,2\}$ in some order by Theorem 2.2.10(2); hence on the subgroup

$$\mathbb{Z}_p^\times\hookrightarrow(O_K\otimes\mathbb{Z}_p)^\times\hookrightarrow G_K^{\mathrm{ab}},$$

$\chi_\mathfrak{p}$ is given by $z\mapsto z^{-1+0+1+2}=z^2$. In particular, on the subgroup

$$\mathbb{Z}_p^\times\hookrightarrow(O_{K^c}\otimes\mathbb{Z}_p)^\times\hookrightarrow G_{K^c}^{\mathrm{ab}},$$

$\chi_\mathfrak{p}$ is given by $z\mapsto z^{2[K^c:K]}$. The same is true for all $G_\mathbb{Q}$-conjugates of $\chi_\mathfrak{p}$, so the image of $\rho_{\pi,\mathfrak{p}}|_{G_{K^c}}$ contains the scalar subgroup $(\mathbb{Z}_p^\times)^{2[K^c:K]}$, proving (1).

For (2), we decompose

$$\rho_{\pi,\mathfrak{p}}|_{G_{K^c}}=\chi_1\oplus\chi_2\oplus\chi_3\oplus\chi_4, \tag{C.14}$$

where all of the characters χ_j are Galois conjugates of $\chi_\mathfrak{p}|_{G_{K^c}}$ and

$$\chi_1\cdot\chi_2=\chi_3\cdot\chi_4=\chi_{p,\mathrm{cyc}}. \tag{C.15}$$

For each $g\in G_\mathbb{Q}$, $\rho_{\pi,\mathfrak{p}}(G_{K^c})$ contains the image of

$$\begin{pmatrix}g\cdot\chi_1&&&\\&g\cdot\chi_2&&\\&&g\cdot\chi_3&\\&&&g\cdot\chi_4\end{pmatrix}. \tag{C.16}$$

Fix an embedding $i:K^c\hookrightarrow\mathbb{Q}_p$, and for any Hodge-Tate character ρ of G_{K^c}, let $\mathrm{HT}(\rho)$ denote the Hodge-Tate weight with respect to i. Let $I_p\subset G_{K^c}$ be the inertia subgroup for the prime induced by i. In particular, restricting (C.16) to I_p and using that each χ_j is crystalline at primes above p, $\rho_{\pi,\mathfrak{p}}(G_{K^c})$ contains the image of

$$\begin{aligned} \mathbb{Z}_p^\times &\to \mathrm{GL}_4(\mathbb{Z}_p) \\ z &\mapsto \begin{pmatrix} z^{\mathrm{HT}(g\cdot\chi_1)} & & & \\ & z^{\mathrm{HT}(g\cdot\chi_2)} & & \\ & & z^{\mathrm{HT}(g\cdot\chi_3)} & \\ & & & z^{\mathrm{HT}(g\cdot\chi_4)} \end{pmatrix}. \end{aligned}$$

Let

$$L \subset \left\{(x, y, z, w) \in \mathbb{Z}^4 \,:\, x + y = z + w\right\} \tag{C.17}$$

be the sublattice spanned by the vectors $(\mathrm{HT}(g \cdot \chi_1), \mathrm{HT}(g \cdot \chi_2), \mathrm{HT}(g \cdot \chi_3), \mathrm{HT}(g \cdot \chi_4))$ for $g \in G_{\mathbb{Q}}$.

Claim. For a constant $n \geq 1$ independent of $\mathfrak{p}$, the lattice L contains

$$n \cdot \left\{(x, y, z, w) \in \mathbb{Z}^4 \,:\, x + y = z + w\right\}.$$

Note that the claim implies the lemma, because, as long as p is sufficiently large, there exists $z \in (\mathbb{Z}_p^\times)^n$ satisfying (C.9); an element $h \in G_{\mathbb{Q}}$ such that $\rho_{\pi,\mathfrak{p}}(h)$ has eigenvalues $\left\{1, z^3, z, z^2\right\}$ is admissible for $\rho_{\pi,\mathfrak{p}}$, and the claim implies such elements exist.

Now we prove the claim. Let $\mathrm{pr} : \mathbb{Z}^4 \to \mathbb{Z}^3$ be the projection onto the first three factors, and note that it suffices to show $\mathrm{pr}(L)$ contains $n \cdot \mathbb{Z}^3$. Without loss of generality, suppose the Hodge-Tate weights of χ_1, χ_2, χ_3, and χ_4 are 1, 0, 2, and -1 (in order). Because the action of G_{K^c} on the set $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ is transitive, for each $j \in 1, \ldots, 4$ we have some $g_j \in G_{\mathbb{Q}}$ such that $\mathrm{HT}(g_j\chi_j) = 1$. In particular, using (C.15), $\mathrm{pr}(L)$ contains $(1, 0, 2)$; a vector $e = (0, 1, 2)$ or $e = (0, 1, -1)$; and a vector $f = (2, -1, 1)$ or $f = (-1, 2, 1)$. In particular, the set $\{(1, 0, 2), e, f\}$ is always linearly independent; and, since there are only four total possibilities for this set, there exists $n \in \mathbb{Z}$ such that the $\mathbb{Z}$-span of $(1, 0, 2)$, e, and f always contains $n\mathbb{Z}$.

$\square$

Now we are ready to consider assumption (R1) from the main text (see (9.3.1)).

Theorem C.4.9. *Let π be a relevant, non-endoscopic automorphic representation of* $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$*, with strong coefficient field E_0. Then (R1) holds for all but finitely many primes $\mathfrak{p}$ of E_0.*

The theorem is also true in the endoscopic case, but not used in the main text; the proof uses Lemma C.4.13 below.

Proof. This is an immediate consequence of Lemmas C.4.2 through C.4.8.

$\square$

Proposition C.4.10. *Suppose π is not endoscopic, and there exists a prime ℓ such that π_ℓ is of type IIa. Then for all but finitely many primes $\mathfrak{p}$ of E_0, admissible elements exist for $\rho_{\pi,\mathfrak{p}}$.*

Proof. The Weil-Deligne representation $\mathrm{rec}_{\mathrm{GT}}(\pi_\ell)$ is tamely ramified; under the embedding $\mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$ from (1.1.4), it is given by

$$\mathrm{Frob}_\ell = \begin{pmatrix} \pm\ell^{1/2} & & & \\ & \alpha & & \\ & & \pm\ell^{-1/2} & \\ & & & \ell/\alpha \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{C}), \quad N = \begin{pmatrix} & & 1 & \\ & & & \\ & & & \\ & & & \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{C})$$

By the purity assertion in Theorem 2.2.10(1) (for any prime $\mathfrak{p}$ of E_0), we know $|\alpha| = 1$. Extend E_0 if necessary so that $\alpha^2 \in E_0$. Then for all but finitely many primes $\mathfrak{p}$ of E_0, we have:

$$\ell^8 \not\equiv 1 \pmod{\mathfrak{p}},$$
$$\alpha^2\ell \not\equiv \pm 1, \pm\ell^2, \ell^{-2}, \ell^4 \pmod{\mathfrak{p}}.$$

Suppose $\mathfrak{p}$ satisfies the above conditions, and let $\mathrm{Frob}_\ell \in G_{\mathbb{Q}}$ be any lift of Frobenius. By Theorem 2.2.10(1), $\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_\ell^2)$ has eigenvalues $\left\{\ell^2, \alpha^2\ell, 1, \ell^2/\alpha^2\right\}$, hence Frob_ℓ^2 is an admissible element for $\rho_{\pi,\mathfrak{p}}$. ☐

Combining Lemmas C.4.2 through C.4.8 with Proposition C.4.10, we obtain:

Theorem C.4.11. *Let π be a relevant, non-endoscopic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$, with strong coefficient field E_0. There is a set S of rational primes of positive Dirichlet density such that for all $p \in S$ and all $\mathfrak{p}|p$, admissible elements exist for $\rho_{\pi,\mathfrak{p}}$. There exists such an S containing all but finitely many p if π satisfies any of the following:*

- *(i) There exists a prime ℓ such that π_ℓ is of type IIa.*
- *(ii) $\mathrm{BC}(\pi)$ is a symmetric cube lift.*
- *(iii) $\mathrm{BC}(\pi)$ is the automorphic induction of a non-CM automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_K)$ with K real quadratic, and $\pi_0^{\mathrm{tw}} \neq \pi_0^\sigma \otimes \chi$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all quadratic Hecke characters χ of K.*

☐

Finally, we handle the endoscopic case separately.

Proposition C.4.12. *Suppose π is endoscopic, associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ (in any order). Then:*

- *(1) If π_1 does not have CM, then for all but finitely many p and all $\mathfrak{p}|p$, there exist admissible primes for $\rho_{\pi,\mathfrak{p}}$ that are BD-admissible for $\rho_{\pi_1,\mathfrak{p}}$.*
- *(2) If π_1 has CM by a field K and π_2 does not have CM by K, then for all but finitely many p split in K and all $\mathfrak{p}|p$, there exist admissible primes for $\rho_{\pi,\mathfrak{p}}$ that are BD-admissible for $\rho_{\pi_1,\mathfrak{p}}$.*

Proof. Let S be the set of all rational primes in case (1) and all rational primes p split in K in case (2). Then there exists a constant $n \geq 1$ such that, for all but finitely many $p \in S$ and all $\mathfrak{p}|p$, $\rho_{\pi_1,\mathfrak{p}}(G_{\mathbb{Q}})$ contains the diagonal subgroup

$$\left\{\begin{pmatrix} x & \\ & y \end{pmatrix} \,:\, x, y \in (\mathbb{Z}_p^\times)^n\right\}.$$

In the non-CM case this follows from Theorem C.3.2 (with $n = 1$), and in the CM case it follows from either [80, Proposition B.6.3] or a similar argument to Lemma C.4.8. On the other hand, there exists a constant $n \geq 1$ such that, for all but finitely many p and all $\mathfrak{p}|p$, $\rho_{\pi_2,\mathfrak{p}}(G_{\mathbb{Q}})$ contains the scalar subgroup $(\mathbb{Z}_p^\times)^n \subset \mathrm{GL}_2(\mathbb{Z}_p)$.

By Lemma C.4.13 below, after enlarging n if necessary, for all but finitely many $p \in S$ and all $\mathfrak{p}|p$, $(\rho_{\pi_1,\mathfrak{p}} \times \rho_{\pi_2,\mathfrak{p}})(G_{\mathbb{Q}})$ contains

$$\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \,:\, x, y, z \in (\mathbb{Z}_p^\times)^n, z^2 = xy\right\};$$

and this implies the proposition. ☐

Lemma C.4.13. *Suppose π is endoscopic, associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which do not both have CM by the same imaginary quadratic field. Then as $\mathfrak{p}$ varies over primes of E_0, $\mathbb{Q}(\overline{\rho}_{\pi_1,\mathfrak{p}}) \cap \mathbb{Q}(\overline{\rho}_{\pi_2,\mathfrak{p}})$ has bounded degree over $\mathbb{Q}(\mu_p)$.*

Proof. If both π_1 and π_2 are non-CM, then the lemma follows from [70, Theorem 3.2.2], or equivalently from Theorem C.3.6(2) above. Now suppose π_1 is CM and π_2 is not. Since $\mathrm{SL}_2(\mathbb{F}_q)$ is simple for q sufficiently large and $\overline{\rho}_{\pi_1,\mathfrak{p}}$ is dihedral for all $\mathfrak{p}$, it follows from [94, Theorem 3.1] that $\mathbb{Q}(\overline{\rho}_{\pi_1,\mathfrak{p}}) \cap \mathbb{Q}(\overline{\rho}_{\pi_2,\mathfrak{p}}) = \mathbb{Q}(\mu_p)$ for all but finitely many $\mathfrak{p}$.

If π_1 and π_2 are CM with respect to two different imaginary quadratic fields K_1 and K_2, fix elements $\tau_1, \tau_2 \in G_{\mathbb{Q}}$ such that τ_i is a complex conjugation on K_i but acts trivially on K_j, $i \neq j$. The abelian group $H := \mathrm{Gal}(\mathbb{Q}(\overline{\rho}_{\pi_1,\mathfrak{p}})/K_1)$ is a subgroup of $k^\times$ for a quadratic étale algebra k over the residue field of $\mathfrak{p}$; in particular, H is the product of at most two cyclic groups. Let $G := \mathrm{Gal}(K_1K_2(\overline{\rho}_{\pi_1,\mathfrak{p}}) \cap K_1K_2(\overline{\rho}_{\pi_2,\mathfrak{p}})/K_1K_2)$. Because G is a subquotient of H, the conjugation actions on G of both τ_1 and $\tau_1\tau_2$ are by inversion, so the conjugation action of τ_2 is trivial; arguing symmetrically, the conjugation action of τ_1 is also trivial, so G is 2-torsion and generated by at most two elements. We conclude that $|G|$ is uniformly bounded, which implies the lemma. ☐

C.5. **Complements for the second reciprocity law.** In this subsection, we prove some auxiliary results needed in §11. Let us fix a relevant automorphic representation π of GSp_4, and an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ with $p > 3$.

Lemma C.5.1. *Let τ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose archimedean component is discrete series of even weight $k \geq 2$. Let F be a number field such that τ does not have CM by any quadratic imaginary subfield $K \subset F(\rho_{\pi,\iota})$. If $F(\rho_{\pi,\iota}) \cap F(\mathrm{ad}^0 \rho_{\tau,\iota})$ is infinite, then $F(\mathrm{ad}^0 \rho_{\tau,\iota}) \subset F(\rho_{\pi,\iota})$, and moreover one of the following occurs:*

- *(i) π is not endoscopic, and if $g \in G_{\mathbb{Q}}$ is admissible for $\rho_{\pi,\iota}$, then $\rho_{\tau,\iota}(g^2)$ has distinct eigenvalues.*
- *(ii) π is endoscopic associated to a pair (π_1, π_2) of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, and for $j = 1$ or 2, $\pi_j \cong \tau^\sigma \otimes \chi$ for some finite-order Hecke character χ and automorphism $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$.*

Proof. First, we claim that τ does not have CM. Indeed, if τ has CM by a quadratic imaginary field K, then it is easy to check that any infinite subfield of $F(\mathrm{ad}^0 \rho_{\tau,\iota})$ contains K; so if $F(\rho_{\pi,\iota}) \cap F(\mathrm{ad}^0 \rho_{\tau,\iota})$ is infinite then $K \subset F(\rho_{\pi,\iota})$, which contradicts the hypotheses of the lemma.

Since τ does not have CM, $\rho_{\tau,\iota}$ is strongly irreducible by Theorem C.3.2. Hence by Lemma C.2.3, the normal, infinite-index subgroup

$$\mathrm{Gal}\left(F(\mathrm{ad}^0 \rho_{\tau,\iota})/F(\mathrm{ad}^0 \rho_{\tau,\iota}) \cap F(\rho_{\pi,\iota})\right) \trianglelefteq \mathrm{Gal}\left(F(\mathrm{ad}^0 \rho_{\tau,\iota})/F\right)$$

must be trivial; equivalently, we have $F(\mathrm{ad}^0 \rho_{\tau,\iota}) \subset F(\rho_{\pi,\iota})$.

Suppose first that π is not endoscopic, so $V_{\pi,\iota}$ is absolutely irreducible by Lemma 2.2.12. Without loss of generality, we assume that F is Galois and that

$$V_{\pi,\iota}|_{G_F} = \bigoplus_i V_i \tag{C.18}$$

for some strongly irreducible representations V_i, all of the same dimension n (Lemma C.2.5). Let $G = \rho_{\pi,\iota}(G_F)$, and let $H = \mathrm{ad}^0 \rho_{\tau,\iota}(G_F)$; then the inclusion $F(\mathrm{ad}^0 \rho_{\tau,\iota}) \subset F(\rho_{\pi,\iota})$ corresponds to a surjection $G/Z_G \twoheadrightarrow H$ (recall here that H has trivial center by Lemma C.2.2(1)). In particular, $n > 1$. Write $\mathfrak{g} = \mathrm{Lie}(G/Z_G)$ and $\mathfrak{h} = \mathrm{Lie}\, H$, and recall that $\mathfrak{h}$ is simple by Theorem C.3.2 and Corollary C.1.2(2). We have a surjection

$$\mathfrak{g} \twoheadrightarrow \mathfrak{h} \tag{C.19}$$

which identifies $\mathfrak{h}$ with a simple factor of $\mathfrak{g}$.

Suppose first that $n = 2$. Then the decomposition (C.18) has exactly two factors, and $\mathfrak{g}$ is either simple, or isomorphic to $\mathfrak{g}_0 \times \mathfrak{g}_0$, with $\mathfrak{g}_0$ simple and the factors interchanged by the action of $G_{\mathbb{Q}}$. Since (C.19) is $G_{\mathbb{Q}}$-equivariant, $\mathfrak{g}$ is simple, and (C.19) is an isomorphism. If $\rho_{\tau,\iota}(g^2)$ has only one eigenvalue for some $g \in G_{\mathbb{Q}}$, then g^2 acts unipotently on $\mathfrak{h}$, hence also on $\mathfrak{g}$. Since g^2 is a square, it preserves the decomposition

(C.18), so by Lemma C.2.2(3), we see that g^2 has at most two eigenvalues on $V_{\pi,\iota}$, which contradicts g being admissible.

We are now reduced to the case $n = 4$, i.e. $\mathrm{BC}(\pi)$ is not an automorphic induction. Hence $\mathfrak{g}$ is simple by Proposition C.2.4(1) and Corollary C.1.2(2), and again (C.19) is an isomorphism. If $\rho_{\tau,\iota}(g^2)$ had only one eigenvalue, then $g^2 \in G_{\mathbb{Q}}$ would act unipotently on $\mathfrak{h}$, hence also on $\mathfrak{g}$; but by Lemma C.2.2(3), this contradicts the admissibility of g.

It remains to consider the case when π is endoscopic associated to a pair (π_1, π_2) of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. We let $G_\pi = \rho_{\pi,\iota}(G_{\mathbb{Q}})$, $G_{\pi_j} = \rho_{\pi_j,\iota}(G_{\mathbb{Q}})$, $\mathfrak{g}_\pi = \mathrm{Lie}(G_\pi/Z_{G_\pi})$, $\mathfrak{g}_{\pi_j} = \mathrm{Lie}(G_{\pi_j}/Z_{G_{\pi_j}})$, for $j = 1, 2$. The assumption $F(\mathrm{ad}^0 \rho_{\tau,\iota}) \subset F(\rho_{\pi,\iota})$ implies that $G_{\mathbb{Q}}$ does not have open image in the product $G_\pi/Z_{G_\pi} \times H$. By Goursat's Lemma, the Lie algebra of the image of $G_{\mathbb{Q}}$ is the graph of an isomorphism between simple factors of $\mathfrak{g}_\pi \subset \mathfrak{g}_{\pi_1} \oplus \mathfrak{g}_{\pi_2}$ and $\mathfrak{h}$. Since τ is non-CM, we conclude that for $j = 1$ or 2, π_j is non-CM and $G_{\mathbb{Q}}$ has non-open image in $G_{\pi_j}/Z_{\pi_j} \times H$. Hence by [70, Proposition 3.3.2], there exists an automorphism $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ and a Hecke character χ such that $\tau \cong \pi_j^\sigma \otimes \chi$, for $j = 1$ or 2; this concludes the proof of the lemma.

☐

For the rest of the section, we fix a strong coefficient field E_0 for π and let $\mathfrak{p}$ be the prime of E_0 induced by ι.

Lemma C.5.2. *Let τ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose archimedean component is discrete series of even weight $k \geq 2$. If τ does not have CM by any quadratic imaginary subfield $K \subset \mathbb{Q}(\rho_{\pi,\iota})$, then for any number field F and any $O_{\mathfrak{p}}$-stable lattice $T_\pi \subset V_{\pi,\mathfrak{p}}$, $H^1(\mathrm{Gal}(F(\rho_{\pi,\iota}, \mathrm{ad}^0 \rho_{\tau,\iota})/\mathbb{Q}), T_\pi)$ is finite.*

Proof. By inflation-restriction, we may assume without loss of generality that $F = \mathbb{Q}$. Applying Corollary C.2.8 and inflation-restriction again, to prove the lemma it suffices to show

$$H^1(\mathrm{Gal}(\mathbb{Q}(\rho_{\pi,\iota}, \mathrm{ad}^0 \rho_{\tau,\iota})/\mathbb{Q}(\rho_{\pi,\iota}), T_\pi) = \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{Gal}(\mathbb{Q}(\rho_{\pi,\iota}, \mathrm{ad}^0 \rho_{\tau,\iota})/\mathbb{Q}(\rho_{\pi,\iota})), T_\pi) = 0. \tag{C.20}$$

By Lemma C.5.1, we may assume without loss of generality that $\mathbb{Q}(\rho_{\pi,\iota}) \cap \mathbb{Q}(\mathrm{ad}^0 \rho_{\tau,\iota})$ is finite. Since $\chi_{p,\mathrm{cyc}}$ has infinite order, there exists $g \in G_{\mathbb{Q}}$ such that $\mathrm{ad}^0 \rho_{\tau,\iota}(g) = 1$ and $\chi_{p,\mathrm{cyc}}(g)$ has infinite order, meaning in particular that $\rho_{\pi,\iota}(g) \neq 1$. Then g acts trivially by conjugation on $\mathrm{Gal}\left(\mathbb{Q}(\rho_{\pi,\iota}, \mathrm{ad}^0 \rho_{\tau,\iota})/\mathbb{Q}(\rho_{\pi,\iota})\right) \hookrightarrow \mathrm{ad}^0 \rho_{\tau,\iota}(G_{\mathbb{Q}})$. In particular, any $G_{\mathbb{Q}}$-invariant homomorphism $h : \mathrm{Gal}\left(\mathbb{Q}(\rho_{\pi,\iota}, \mathrm{ad}^0 \rho_{\tau,\iota})/\mathbb{Q}(\rho_{\pi,\iota})\right) \to T_\pi$ has image contained in $T_\pi^{g=1} \subsetneq T_\pi$. If π is non-endoscopic, this shows $h = 0$ by Lemma 2.2.12; if π is endoscopic associated to (π_1, π_2), the same argument applies because we cannot have $\rho_{\pi_1,\iota}(g) = 1$ or $\rho_{\pi_2,\iota}(g) = 1$ under the assumption that $\chi_{p,\mathrm{cyc}}(g)$ has infinite order. This shows (C.20). ☐

Proposition C.5.3. *Let π, ι, E_0, and $\mathfrak{p}$ be as above with π non-endoscopic, and suppose admissible primes exist for $\rho_\pi = \rho_{\pi,\mathfrak{p}}$. Suppose given the following data:*

- ○ *A quadratic field $F \not\subset \mathbb{Q}(\rho_\pi)$.*
- ○ *A cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose archimedean component is discrete series of even weight $k \geq 2$, such that τ does not have CM by any quadratic field $K \subset F(\rho_\pi)$.*
- ○ *A $G_{\mathbb{Q}}$-stable $O_{\mathfrak{p}}$-lattice $T_\pi \subset V_{\pi,\mathfrak{p}}$, and a non-torsion cocycle $c \in H^1(\mathbb{Q}, T_\pi)$.*

Then there exists an element $g \in G_{\mathbb{Q}}$ such that:

1. *g is is admissible for ρ_π and has nontrivial image in $\mathrm{Gal}(F/\mathbb{Q})$.*
2. *$\rho_{\tau,\iota}(g^2)$ has distinct eigenvalues.*
3. *$c(g)$ has nonzero component in the 1-eigenspace for g.*

Note the last condition is independent of the choice of cocycle representative for c.

Proof. First choose g satisfying (1), and with the additional property that g has trivial image in $\mathrm{Gal}(K/\mathbb{Q})$ if τ has CM by a quadratic field K. (This choice is possible because we have $F \not\subset \mathbb{Q}(\rho_\pi)$ and $K \not\subset F(\rho_\pi)$.)

Claim. There exists $h \in G_{F(\rho_\pi)}$ such that hg satisfies (2).

Proof of claim. If $F(\rho_\pi) \cap F(\mathrm{ad}^0\, \rho_{\tau,\iota})$ is infinite then taking $h = 1$ suffices by Lemma C.5.1, so we may assume without loss of generality that $F(\rho_\pi) \cap F(\mathrm{ad}^0\, \rho_{\tau,\iota})$ is finite. If τ is non-CM, then because $F(\rho_\pi) \cap F(\mathrm{ad}^0\, \rho_{\tau,\iota})$ is finite, Theorem C.3.2 implies that the image of $\rho_{\tau,\iota}|_{G_{F(\rho_\pi)}}$ contains a compact open subgroup of $\left\{x \in D^\times : \mathrm{Nm}(x) \in \mathbb{Q}_p^\times\right\} \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, for a quaternion algebra D over a finite extension E of $\mathbb{Q}_p$. Since $x\rho_{\tau,\iota}(g)x\rho_{\tau,\iota}(g)$ having distinct eigenvalues is an open condition on $x \in D^\times$, the claim follows when τ is non-CM.

If on the other hand τ has CM by an imaginary quadratic field K, then because $\rho_{\tau,\iota}|_{G_{\mathbb{Q}_p}}$ has distinct Hodge-Tate weights, there exists $h_0 \in G_K$ such that $\rho_{\tau,\iota}(h_0)$ has eigenvalues whose ratio is of infinite order. After replacing h_0 with a finite power, it acts trivially on $F(\rho_\pi) \cap F(\mathrm{ad}^0\, \rho_{\tau,\iota})$; thus there exists $h \in G_{K\cdot F(\rho_\pi)}$ such that $\rho_{\tau,\iota}(h^2)$ has distinct eigenvalues. Since g has trivial image in $\mathrm{Gal}(K/\mathbb{Q})$, $\rho_{\tau,\iota}(g)$ and $\rho_{\tau,\iota}(h)$ commute; in particular, if $\rho_{\tau,\iota}(g^2)$ is scalar, then $\rho_{\tau,\iota}(hghg) = \rho_{\tau,\iota}(h^2)\rho_{\tau,\iota}(g^2)$ has distinct eigenvalues. Hence either g or hg satisfies (2), which shows the claim. ☐

Replacing g with hg as in the claim, we may now assume g satisfies both (1) and (2). By Lemma C.5.2 and inflation-restriction, c has nonzero image in

$$H^1(F(\rho_\pi, \mathrm{ad}^0\, \rho_{\tau,\iota}), T_\pi)^{G_{\mathbb{Q}}} = \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/F(\rho_\pi, \mathrm{ad}^0\, \rho_{\tau,\iota})), T_\pi),$$

and because $V_{\pi,\mathfrak{p}}$ is absolutely irreducible, there exists $h \in G_F$ such that $\mathrm{ad}^0\, \rho_{\tau,\iota}(h) = \rho_\pi(h) = 1$ and $c(h)$ has nonzero component in the 1-eigenspace for g. Then either g or hg satisfies (1), (2), and (3), which proves the proposition.

☐

Finally, we have the endoscopic analogue of Proposition C.5.3.

Proposition C.5.4. *Let π, ι, E_0, and $\mathfrak{p}$ be as above, with π endoscopic associated to pair (π_1, π_2) of automorphic representations of* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$*; and assume that E_0 is a common strong coefficient field of π_1 and π_2. Let $j = 1$ or 2, and suppose there exist admissible primes for $\rho_{\pi,\mathfrak{p}}$ which are BD-admissible for $\rho_{\pi_j} = \rho_{\pi_j,\mathfrak{p}}$.*

Suppose given the following data:

- *A quadratic field $F \not\subset \mathbb{Q}(\rho_\pi)$.*
- *A cuspidal automorphic representation τ of* GL_2 *whose archimedean component is discrete series of weight at least 2, such that τ does not have CM by any quadratic field $K \subset F(\rho_\pi)$.*
- *A $G_{\mathbb{Q}}$-stable $O_{\mathfrak{p}}$ lattice $T_{\pi_j} \subset V_{\pi_j,\mathfrak{p}}$, and a non-torsion cocycle $c \in H^1(\mathbb{Q}, T_{\pi_j})$.*

Then there exists an element $g \in G_{\mathbb{Q}}$ such that:

1. *g is is admissible for ρ_π and BD-admissible for ρ_{π_j}, and has nontrivial image in* $\mathrm{Gal}(F/\mathbb{Q})$.
2. *$\rho_{\tau,\iota}(g^2)$ has distinct eigenvalues.*
3. *$c(g)$ has nonzero component in the 1-eigenspace for g.*

Proof. Without loss of generality, suppose $j = 1$. Clearly there exists $g \in G_{\mathbb{Q}}$ satisfying (1), such that, if τ has CM by an imaginary quadratic field K, g has trivial image in $\mathrm{Gal}(K/\mathbb{Q})$. We next claim:

Claim. There exists $g \in G_{\mathbb{Q}}$ satisfying (1) and (2).

Proof. If $F(\rho_\pi) \cap F(\mathrm{ad}^0\, \rho_{\tau,\iota})$ is finite, then we conclude using the same argument as for the claim in the proof of Proposition C.5.3. By Lemma C.5.1, we may therefore assume that, for $i = 1$ or 2, $\pi_i \cong \tau^\sigma \otimes \chi$ for some finite-order Hecke character χ and automorphism $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$. In this case, τ is necessarily non-CM (because its CM field would be contained in $\mathbb{Q}(\rho_{\pi_i})$), so π_1 and π_2 cannot both be CM; and it suffices to show there exists $g \in G_{\mathbb{Q}}$ satisfying (1), such that $\rho_{\pi_2}(g^2)$ has distinct eigenvalues. Hence it suffices to show that $\mathbb{Q}(\rho_{\pi_1}) \cap \mathbb{Q}(\rho_{\pi_2})$ is finite over $\mathbb{Q}(\mu_{p^\infty})$; and this follows from an argument very similar to Lemma C.4.13, using that π_1 and π_2 are not both CM. ☐

Now take g as in the claim. By Lemma C.5.2, c has nonzero image in

$$H^1(F(\rho_\pi, \mathrm{ad}^0\, \rho_{\tau,\iota}), T_{\pi_1}).$$

Arguing as in Proposition C.5.3 and using the absolute irreducibility of ρ_{π_1}, the proposition follows. ☐

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